What are...subrings and ideals?

Or: Two valid substructures

Substructures vs. quotients

	substructure	good quotients
sets	subset	congruence
vector spaces	linear subspace	linear subspace
groups	subgroup	normal subgroup
rings	subring	ideals
categories	subcategory	congruence

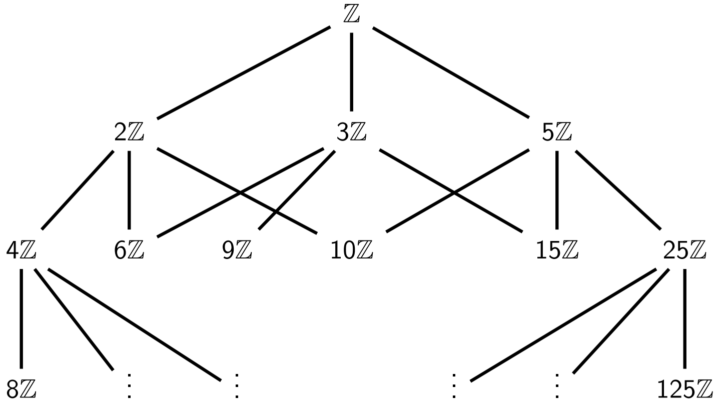
(a) For rings substructures do not give good quotients

(b) Substructures = subring Substructures

(c) Ideals $\Leftrightarrow$ good quotients Quotients

Ideals generalize divisibility

- 0 is divisible by 8 Additive unit
- If $a, b \in \mathbb{Z}$ are divisible by 8, then so is $a + b$ Internal addition
- 1 is not divisible by 8 No multiplicative unit
- If $a \in \mathbb{Z}$ is divisible by 8, then so is ar for all $r \in \mathbb{Z}$ External multiplication



Ideals generalize row and column operations

- A matrix times a row matrix is a row matrix:

$$\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 0 & 2 & 0\\ 0 & 3 & 0\end{pmatrix} = \begin{pmatrix}0 & 14 & 0\\ 0 & 32 & 0\\ 0 & 50 & 0\end{pmatrix}$$

External left multiplication

- A column matrix times a matrix is a column matrix:

$$\begin{pmatrix}0 & 0 & 0\\ 1 & 2 & 3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 30 & 36 & 42\\ 0 & 0 & 0\end{pmatrix}$$

External right multiplication

This is not two-sided, *e.g.*:

$$\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 1 & 2 & 3\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}2 & 4 & 6\\ 5 & 10 & 15\\ 8 & 16 & 24\end{pmatrix}$$

For completeness: Two formal definitions

(a) A subring $S \subset R$ is a subset of a ring R that forms a ring under the induced addition and multiplication

(b) A two-sided ideal $I \subset R$ is a subset of a ring R that forms a subgroup under the induced addition and is closed under external multiplication $ri \in I$ and $ir \in I$ for all $i \in I$ and $r \in R$

(c) A left ideal $I \subset R$ is a subset of a ring R that forms a subgroup under the induced addition and is closed under external left multiplication $ri \in I$ for all $i \in I$ and $r \in R$

(d) A right ideal $I \subset R$ is a subset of a ring R that forms a subgroup under the induced addition and is closed under external right multiplication $ir \in I$ for all $i \in I$ and $r \in R$

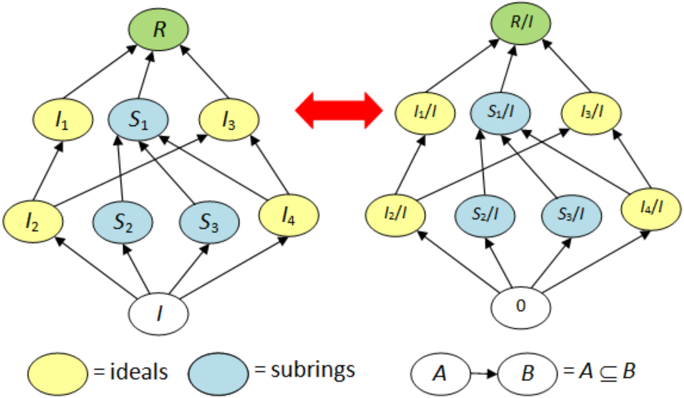
- $\mathrm{M}_{2\times 2}(\mathbb{C})$ has infinitely many non-trivial subrings
- $\mathrm{M}_{2\times 2}(\mathbb{C})$ has no non-trivial two-sided ideals

Subrings and ideals under quotients

Theorem. There are 1:1 correspondences

$$\{S \mid I \subset S \subset R \text{ subring}\} \overset{1:1}{\longleftrightarrow} \{T \mid T \subset R/I \text{ subring}\}$$

$$\{J \mid I \subset J \subset R \text{ ideal}\} \overset{1:1}{\longleftrightarrow} \{K \mid K \subset R/I \text{ ideal}\}$$



preserving the lattice structure (a.k.a. the inclusion)

Thank you for your attention!

I hope that was of some help.