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My short- and long-term research goals

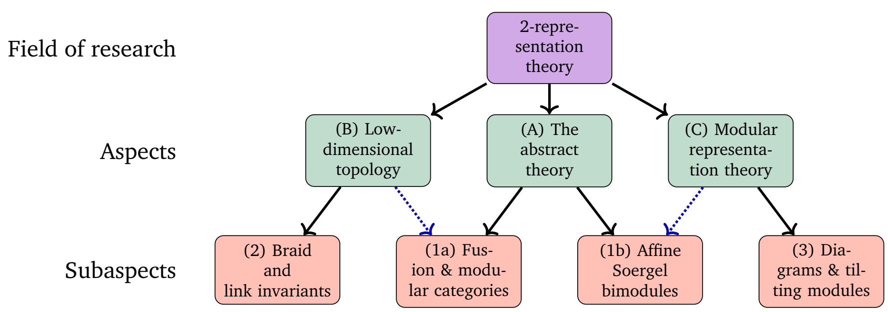
Aims and background

Over the last 20 years we witnessed history of mathematics in its making with Khovanov's discovery of his celebrated categorification of the Jones polynomial [Kh00]. In 2020 google scholar lists more than 1200 citations to [Kh00], while Scopus/MathSciNet list about 500 citations, all phenomenal numbers for mathematics, including citations beyond mathematics from fields such as molecular chemistry. This discovery was transformative, and since then it has become clear that functorial actions provide the right language for understanding Khovanov's work, and its generalizations, and these actions have now been axiomatized into the emerging field of 2-representation theory. (See *e.g.* [CR08], [EGNO15] or [Ma17] for various flavors of 2-representation theory.)

This new field is at heart of an explosion of new discoveries across a range of fields including algebraic geometry, combinatorics, classical and modular representation theory, and low dimensional topology and it is expected that there will be future applications in physics and chemistry.

My research is focused on three aspects involving 2-representation theory:

(A) The abstract theory: Allow infinite 2-categories and work in finite characteristic.
(B) Low-dimensional topology: link homologies and 2-representations of braid groups.
(C) Modular representation theory: 2-representations of tilting modules.



Background.

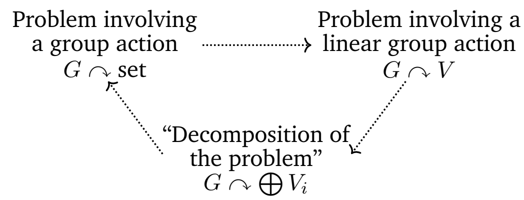
The study of group actions is of critical importance in mathematics and related fields such as physics and chemistry. Its significance can hardly be overestimated.

The approach of Frobenius ∼1895, Burnside ∼1900 and many others, nowadays called **representation theory** (or I would say the representation theory of the 20th century), is to linearly approximate such actions. For example, let G be a group or a ring or an algebra *etc.* Representation theory is the study of linear group actions

$$G \longrightarrow \mathrm{End}(V),\ g \mapsto M(g) \quad \text{or} \quad G \curvearrowright V.$$

That is, representation theory assigns to each group element a matrix $M(g)$ acting on a vector space V – its linear shadow. The representation theory approach is that classifying linear G-actions has, in contrast to arbitrary group actions, a satisfactory answer for many groups.

The basic building blocks V_i of such actions tell us a lot about the problem we started with. (The strategy of representation theorists is summarized on the right.) In fact, experience tells us that the collection of such linear shadows is an interesting structure in its own right and maybe even more worthwhile to study than G itself.

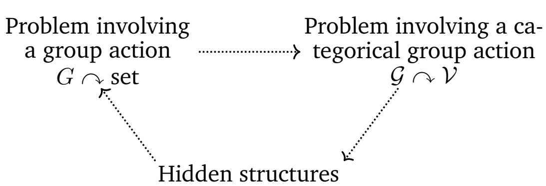


Developing over the past century (and still in development), Frobenius and Burnside's theory is pervasive across many fields of mathematics. The success of representation theory has led to numerous generalizations and applications, *e.g.* in the aforementioned molecular chemistry or quantum physics, but also in engineering such as robotics. (How do you figure out how robots move before building them? Indeed, using representation theory.)

Instead of studying groups, rings or algebras acting on vector spaces, **2-representation theory** studies the categorical actions of these. Or, more generally, actions of (2-)categories $\mathcal{G}$, such that one recovers the classical picture on the decategorified level. (Decategorification is the reverse of categorification and turns an n-category into an (n-1)-category, *e.g.* a category into a set.)

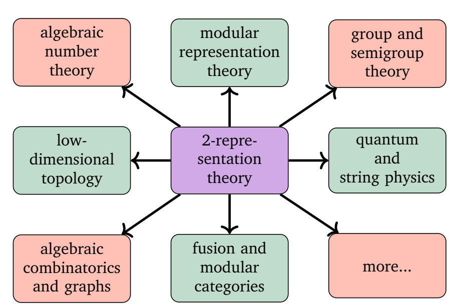
$$\begin{array}{ccc} \mathcal{G} & \xrightarrow[g\mapsto\mathcal{M}(g)]{\text{categorical action}} & \mathcal{E}\mathrm{nd}(\mathcal{V}) \\ \text{decat}\downarrow & & \downarrow\text{decat} \\ G & \xrightarrow[\text{classical action}]{g\mapsto M(g)} & \mathrm{End}(V)\end{array} \quad\text{or}\quad \begin{array}{c}\mathcal{G}\curvearrowright\mathcal{V}\\ \downarrow\text{decat}\\ G\curvearrowright V\end{array}.$$

In other words, 2-representation theory assigns to each group element a functor $\mathcal{M}(g)$ acting on a category $\mathcal{V}$ – its categorical shadow.



The categorical structure is usually richer, and the 2-representation theoretical approach can be summarized by the diagram on the left. That is, starting with a group action in the wild, 2-representation theory turns it into a question involving richer categorical structures, which then reveal hidden symmetries within the original formulation.

2-representation theory links diverse fields, as sketched on the right. For my research the most important incarnations of 2-representation theory are in the green boxes on the right. Starting from the bottom and going clockwise, the relevant connections are *e.g.* via [M³TZ19], Khovanov homology [Kh00], the Riche–Williamson program [RW18] and Chern–Simons(–Witten) theory [Wi12].



2-representation theory – in general and, specifically, as in my research – is having a strong impact on these fields because it provides richer structures and the tools to analyze them, *e.g.*:

▷ Categorifications of Hecke algebras through Soergel bimodules and their 2-representations are of fundamental importance in modern Lie theory and low-dimensional topology [Kh07]. For example, they have led to new results in the representation theory of Lie algebras [MS08].

▷ There are also remarkable connections between Soergel bimodules and their 2-representations with modern geometry. For example, see the groundbreaking work [Wi17].

- ▷ Pioneering ideas of Chuang–Rouquier [CR08] and Khovanov–Lauda [KL10] opened, on the one hand, a new field of research, 2-representation theory of Lie algebras. On the other hand, their ideas solved longstanding open problems, *e.g.* Broué's abelian defect group conjecture.
- ▷ It is easier to see connections to other fields. For example, while classical representation theory appears crucially in quantum or string theory via 3d Chern–Simons theory, 2-representation theory is expected to play the same role for its 4d counterpart [Wi12].
- ▷ Questions on the de- or categorified level can be proven with more structure; the proof of the Kazhdan–Lusztig conjecture [EW14] or that Khovanov homology detects the unknot [KM11] being examples. The categorical structures are also usually richer, *e.g.* Khovanov's link homology is functorial [ETWe18] (the proof relies on 2-representations).
- ▷ Several classical questions in *e.g.* modular representation theory are stated in terms of functors acting on categories, which is part of what 2-representation theory studies. For example, the Lascoux–Leclerc–Thibon conjecture was proven by using functorial action of an affine Lie algebra on categories of representations of affine Hecke algebras [Ar96].

So 2-representations play a central role in our way as we understand actions today. **One could call 2-representation theory the representation theory of the 21th century**, with expected wide-ranging applications in mathematics and beyond. So rephrasing the first sentence of this section:

> The study of 2-representations is going to be of critical importance in mathematics and related field such as physics and chemistry. Its future significance can hardly be overestimated.

However, in some sense we are at the same stage Frobenius and Burnside were 120 years ago: we have enough examples to see that our theory is rich and we have a satisfactory theory in specific cases, but we are lacking a general theory and the full range of examples to utilize the full power of 2-representation theory. The main aims of this proposal address these two obstacles in 2-representation theory: we will develop the general theory and study interesting new examples of 2-representations.

Details for potential projects.

My research is about 2-representation theory and its applications in categorification, modular representation theory, low-dimensional topology, mathematical physics and related fields. To develop the vibrant field of 2-representation theory, to strengthen its impact and to find novel applications is the objective of my research. More precisely, my research is focused on three aspects of 2-representation theory, all of which are part of the research envisioned in the present application:

- ▷ 2-representation theory ("the representation theory of the 21^{th} century"). `Ingredients.` (Modular) representation theory, categorical algebra, (higher) category theory, group and semigroup theory. `My latest results.` [M^3T18], [M^3TZ19], [M^3TZ20].
- ▷ Representation theory of Lie or finite-dimensional algebras, especially, their diagrammatic presentations and properties such as cellularity. `Ingredients.` Classical and quantum algebra, (modular) representation theory, low-dimensional topology, combinatorics. `My latest results.` [ET18], [TW19], [TW20], [MT21].
- ▷ Knot homologies, topological quantum field theories, Lie theory and geometry. `Ingredients.` Low-dimensional topology, representation theory, quantum Lie theory, quantum or string physics, homological algebra. `My latest results.` [RT19].

(A). The analog of simple modules in 2-representation theory are simple transitive 2-representations, and a question of fundamental importance is to ask for a classification of these. One of the crucial new and exciting developments in the field was the observation that the classification of simple transitive 2-representations, in many cases, can be reduced to the study of fusion categories, while still being the richer structure as finitary 2-representation theory is non-semisimple, non-abelian. The first observation in this direction was made in [MT16], and this is made rigorous in [M^{3}T19] using quantum Satake and (co)algebra 1-morphisms.

In recent work, which will be available soon and which is based on [M^{3}TZ19], we solved the classification problem of simple transitive 2-representations of Soergel bimodules (non-semisimple, non-abelian) for finite Weyl groups in characteristic 0 by solving the analog classification question for certain fusion categories (semisimple). Thus:

Problem. Study the 2-representation theory of Soergel bimodules and related 2-categories in characteristic p. Explore the connections to fusion categories with an eye on applications.

`Subprogram 1.` A focus of this aspect is to develop machinery to study 2-representations of Soergel bimodules similarly to [M^{3}TZ19], but in characteristic p. We have precise ideas how to attack this in type A, using the technology of H-reduction [M^{3}TZ20]. The other types need additional work as the cell theory in finite characteristic is significantly different from characteristic 0. To this end, new ideas and approaches will be needed, replacing or adapting the H-reduction, and will have impact beyond the theory.

`Subprogram 2.` The fusion categories arising from exceptional Coxeter groups are exotic examples of such categories – not fitting in the general philosophy that almost all fusion categories are of the form $\mathcal{V}\mathrm{ect}(G)$, $\mathcal{R}\mathrm{ep}(G)$ or $\mathcal{R}\mathrm{ep}^{ss}(U_q(\mathfrak{g}))$. Usually these exceptional examples tell a lot about the the general theory, and having more of them is desirable. A source of these potential examples is the application of the arguments from [M^{3}TZ19] and [M^{3}T18] where we expect several such examples to turn up.

`Subprogram 3.` In the dihedral case the fusion categories obtained from Soergel bimodules are modular, which means they give rise to 3-manifold invariants by the Witten–Reshetikhin–Turaev approach and its siblings. As they arise as the semisimple part of the bigger, non-semisimple 2-category of Soergel bimodules, we expect Soergel bimodules to give richer invariants. These will be related to invariants studied under the slogan of modified traces, *cf.* [GPMT09], and will reveal hidden structures in topology.

(B). Homology theories are ubiquitous in modern mathematics, ranging from singular homology of topological spaces to knot homologies. These homological invariants take values in, say, isomorphism classes of vector spaces instead of in numbers as *e.g.* Betti numbers do. One main point is that these homology theories usually extend to functors and provide information about how certain structures are related. In his pioneering work, Khovanov introduced what is nowadays called Khovanov homology [Kh00] – his celebrated categorification of the Jones polynomial – which is a homological invariant of links. Studying link homologies has become a big industry after Khovanov's breakthrough, and many link homologies are known by now, coming from and connecting various fields, from mathematics to physics. The most important example of such homologies is the categorification of the HOMFLYPT polynomial [Kh07], called HOMFLYPT or triply-graded homology.

Almost all variants of Khovanov's invariant stay in type A, meaning for us that they are related to the classical braid group. For example, Khovanov's construction of triply-graded homology uses Soergel bimodules of type A. Thus, an exciting problem is:

Problem. Construct link homologies and categorical braid group actions outside of type A by using 2-representation theory.

`Subprogram 1.` A main motivation for me is to generalize these HOMFLYPT invariants to different braid groups. A first step in this direction is [RT19], defining a HOMFLYPT invariant for links in handlebodies using type A Soergel bimodules, which is functorial on handlebody braids – a fact which I can only prove using 2-representations. To develop this by considering other types of Soergel bimodules is my aim, and these homologies will turn out to be very interesting.

`Subprogram 2.` An ingredient in the construction of triply-graded homology is the Rouquier complex, *cf.* [Kh07], which categorifies the representation of braid groups on Hecke algebras. The categorification has more structure: using cell 2-representations one can show that the Rouquier complex gives a faithful braid group action, see *e.g.* [Je17], while this is still open for the algebras. 2-representations will allow me to extend these ideas to affine braid groups, and beyond.

(C). Given *e.g.* some Lie algebra, can one give a generator-relation presentation for the category of its finite-dimensional representations, or for some well-behaved subcategory? Maybe the best-known instance of this is the case of the monoidal category generated by the vector representation of SL_2. Its generator-relation presentation is known as the Temperley–Lieb category and goes back to work of Rumer–Teller–Weyl and Temperley–Lieb. Note that the Temperley–Lieb category is given diagrammatically which makes it easier to work with, and its relation to representations of SL_2 is given by the classical Schur–Weyl duality. Experience tells us that having diagrammatic presentations is very helpful in studying questions in representation theory, invariant theory *etc.* The philosophy is that morphisms are more important than objects. However, a lot of work in classical representation theory is concerned with objects only. This is where 2-representation theory enters the game.

Currently one of the main applications of diagrammatic methods is in the notoriously hard field of modular representation theory. Various breakthroughs in the past years have be made by the use of diagrammatic machinery *e.g.* [RW18]. Hence:

Problem. Apply 2-representations of affine type Soergel bimodules and Temperley–Lieb-like categories to modular representation theory.

`Subprogram 1.` In [TW19], we use the Temperley–Lieb category to nail down the quiver for tilting modules of $\mathrm{SL}_2(\mathbb{K})$ for an algebraically closed field $\mathbb{K}$ of prime characteristic. A generalization of [TW19] to higher ranks would be a breakthrough as already $\mathrm{SL}_3(\mathbb{K})$ is not understood at all. The easier quantum group case is work in progress and results will follow soon.

`Subprogram 2.` Among other things, Tilting modules provide the decomposition numbers for symmetric groups, and [TW19], extended to higher ranks, further gives geometric objects: fractal-graphs. A general philosophy, *cf.* [CEF15], is to study such data in the setting of stability. A task is to look at geometric stability, which will lead to connections to number theory.

References

[AT17] H.H. Andersen, D. Tubbenhauer. Diagram categories for $\mathbf{U}_q$-tilting modules at roots of unity. *Transform. Groups.* 22 (2017), no. 1, 29–89.

[Ar96] S. Ariki. On the decomposition numbers of the Hecke algebra of G(m,1,n). *J. Math. Kyoto Univ.* 36 (1996), 789–808.

[CR08] J. Chuang, R. Rouquier. Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification. *Ann. of Math.* (2) 167 (2008), no. 1, 245–298.

[CEF15] T. Church, J. Ellenberg, B. Farb. FI-modules and stability for representations of symmetric groups. *Duke Math. J.* 164 (2015), no. 9, 1833–1910.

[ET18] M. Ehrig, D. Tubbenhauer. Algebraic properties of zigzag algebras. *Comm. Algebra* 48 (2020), no. 1, 11–36.

[ETWe18] M. Ehrig, D. Tubbenhauer, P. Wedrich. Functoriality of colored link homologies. *Proc. Lond. Math. Soc.* (3) 117 (2018), no. 5, 996–1040.

[EW14] B. Elias, G. Williamson. The Hodge theory of Soergel bimodules. *Ann. of Math.* (2) 180 (2014), 1089–1136.

[EGNO15] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik. Tensor categories. *Mathematical Surveys and Monographs*, 205. American Mathematical Society, Providence, RI, 2015. xvi+343 pp.

[EO04] P. Etingof, V. Ostrik. Module categories over representations of $\mathrm{SL}_q(2)$ and graphs. *Math. Res. Lett.* 11 (2004), no. 1, 103–114.

[GPMT09] N. Geer, B. Patureau-Mirand, V. Turaev. Modified quantum dimensions and re-normalized link invariants. *Compos. Math.* 145 (2009), no. 1, 196–212.

[Je17] L. Jensen. The 2-braid group and Garside normal form. *Math. Z.* 286 (2017), no. 1-2, 491–520.

[Kh00] M. Khovanov. A categorification of the Jones polynomial. *Duke Math. J.* 101 (2000), 359–426.

[Kh07] M. Khovanov. Triply-graded link homology and Hochschild homology of Soergel bimodules. *Internat. J. Math.* 18 (2007), no. 8, 869–885.

[KL10] M. Khovanov, A. Lauda. A categorification of quantum sl(n). *Quantum Topol.* 1 (2010), 1–92.

[KM11] P. Kronheimer, T. Mrowka. Khovanov homology is an unknot-detector. *Publ. Math. Inst. Hautes Études Sci. No.* 113 (2011), 97–208.

[M^{3}T19] M. Mackaay, V. Mazorchuk, V. Miemietz, D. Tubbenhauer. Simple transitive 2-representations via (co)algebra 1-morphisms. *Indiana Univ. Math. J.* 68 (2019), no. 1, 1–33.

[M^{3}T18] M. Mackaay, V. Mazorchuk, V. Miemietz, D. Tubbenhauer. Trihedral Soergel bimodules. *Fund. Math.* 248 (2020), no. 3, 219–300.

[M^{3}TZ19] M. Mackaay, V. Mazorchuk, V. Miemietz, D. Tubbenhauer, X. Zhang. Simple transitive 2-representations of Soergel bimodules for finite Coxeter types. `arXiv:1906.11468`.

[M^{3}TZ20] M. Mackaay, V. Mazorchuk, V. Miemietz, D. Tubbenhauer, X. Zhang. Finitary birepresentations of finitary bicategories. *Forum Math.* 33 (2021), no. 5, 1261–1320

[MT16] M. Mackaay, D. Tubbenhauer. Two-color Soergel calculus and simple transitive 2-representations. *Canad. J. Math.* 71 (2019), no. 6, 1523–1566.

[MT21] A. Mathas, D. Tubbenhauer Subdivision and cellularity for weighted KLRW algebras. `arXiv:2111.12949`.

[MN20] C. Manolescu, I. Neithalath. Skein lasagna modules for 2-handlebodies. `arXiv:2009.08520`.

[Ma17] V. Mazorchuk. Classification problems in 2-representation theory. *São Paulo J. Math. Sci.* 11 (2017), no. 1, 1–22.

[MS08] V. Mazorchuk, C. Stroppel. Categorification of (induced) cell modules and the rough structure of generalised Verma modules. *Adv. Math.* 219 (2008), no. 4, 1363–1426.

[RW18] S. Riche, G. Williamson. Tilting modules and the p-canonical basis. *Astérisque* 2018, 184 pp.

[RT19] D.E.V. Rose, D. Tubbenhauer. HOMFLYPT homology for links in handlebodies via type A Soergel bimodules. *Quantum Topol.* 12 (2021), no. 2, 373–410.

[TW19] D. Tubbenhauer, P. Wedrich. Quivers for SL_2 tilting modules. *Represent. Theory.* 25 (2021), 440–480.

[TW20] D. Tubbenhauer, P. Wedrich. The center of SL_2 tilting modules. *Glasg. Math. J.* 64 (2022), no. 1, 165–184.

[Wi17] G. Williamson. Schubert calculus and torsion explosion. *J. Amer. Math. Soc.* 30 (2017), 1023–1046.

[Wi12] E. Witten. Fivebranes and knots. *Quantum Topol.* 3 (2012), no. 1, 1–137.

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