Daniel Tubbenhauer
daniel.tubbenhauer@math.uzh.ch
Universität Zürich

Address
Institut für Mathematik
Winterthurerstrasse 190
Campus Irchel, Office Y27J32
CH-8057 Zürich, Switzerland

web www.dtubbenhauer.com
mobile +41 (0) 76 701 88 16
phone +41 (0) 44 635 58 44

My short- and long-term research goals

Keywords

Currently my research is focused on three (long-term) projects:

▷ 2-representation theory ("the representation theory of the 21$^{\text{th}}$ century") – a subject which is still in its infant years.

- **Ingredients.** (Modular) representation theory, categorical algebra, (higher) category theory, group and semigroup theory.
- **Keywords.** 2-representations, Hecke categories or Soergel bimodules, Kazhdan–Lusztig theory, tensor & fusion & modular categories, classical Lie theory.
- **My latest results.** [M^{3}T18], [M^{3}TZ19], [M^{3}TZ20].
- **Some recent collaborators.** M. Mackaay, V. Mazorchuk, V. Miemietz and X. Zhang.

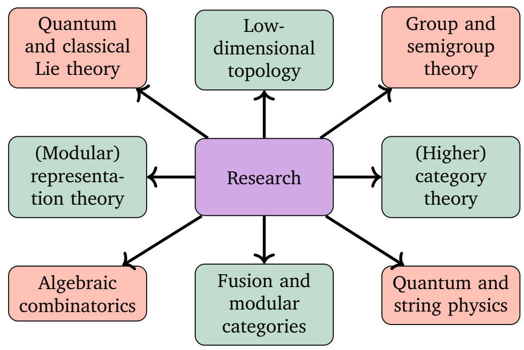
▷ Representation theory of Lie algebras or finite-dimensional algebras, in particular, their diagrammatic presentations and properties such as cellularity.

- **Ingredients.** Classical and quantum algebra, (modular) representation theory, low-dimensional topology, algebraic combinatorics.
- **Keywords.** Characteristic p, diagram algebras, Schur–Weyl dualities, quasi-hereditary and cellular algebras, quivers.
- **My latest results.** [ET18], [TW19], [TW20].
- **Some recent collaborators.** M. Ehrig and P. Wedrich.

▷ Knot homologies, topological quantum field theories, Lie theory and geometry.

- **Ingredients.** Low-dimensional topology, representation theory, quantum Lie theory, quantum and string physics, homological algebra.
- **Keywords.** Link homologies, braid groups and Hecke algebras, quantum groups, (singular) TQFTs, cobordisms and foams, category $\mathcal{O}$.
- **My latest results.** [RT19].
- **Some recent collaborators.** D.E.V. Rose.

Mnemonic



A word about 2-representations

What?

Groundbreaking ideas of Chuang–Rouquier and Khovanov–Lauda on categorifications of quantum groups and their representations opened a completely new field of research, often called 2-*representation theory*.

However, their works were mostly example-based and a general theory of 2-representations was missing. Working in this direction, Mazorchuk–Miemietz came up with a framework which plays the 2-categorical analog of the theory of 'representation theory of finite-dimensional algebras' and generalizes ideas from (semi)group theory. Let us call it *finitary 2-representation theory*. As an example, while representation theory studies the representations of Coxeter groups or Hecke algebras, finitary 2-representation theory studies the 2-representations of the categorification of these, the 2-representations Soergel bimodules.

Why?

Instead of studying algebras acting on vector spaces, 2-representation theory studies the categorical actions of 2-categories such that one recovers the classical picture on the decategorified level:

$$\begin{tikzcd} \text{2-cat} \arrow[r, "\text{categorical action}"] \arrow[d, "\text{decat}"'] & \mathcal{E}\mathrm{nd}(\text{cat}) \arrow[d, "\text{decat}"] \\ \text{alg} \arrow[r, "\text{classical action}"'] & \mathrm{End}(\text{vs}). \end{tikzcd}$$

The categorical structure is usually richer, which has several implications. Here are a few:

- ▷ It is easier to see connections to other fields – *e.g.* while classical representation theory appears crucially in quantum and string theory via $3d$ Chern–Simons–Witten theory, 2-representation theory might play the same role for its $4d$ counterpart.
- ▷ Question on the decategorified level can often be proven having more structure at hand – the proof of the Kazhdan–Lusztig conjecture being a good example.
- ▷ Several classical questions in *e.g.* modular representation theory are stated in terms of functors action on categories – which is part of what 2-representation theory studies.

My motivation.

The subject of finitary 2-representation theory is rapidity growing with several new results published every year. Nevertheless, it is still widely open at the moment and a lot of natural and interesting questions remain to be studied. It also can be seen as a non-semisimple generalization of the study of fusion categories which arose in the 90s from low-dimensional topology, as made precise in recent work, see *e.g.* [M^{3}T18], [M^{3}TZ19].

Since I believe in the future usefulness of this approach (think *e.g.* about the impact of representation theory on other fields of mathematics – the same should be true on the 2-categorical level), I want continue my research on 2-representations of Soergel bimodules and related 2-categories.

My latest results.

In recent years it became evident that finitary 2-representation theory is a non-semisimple, non-abelian version of the study of fusion categories. The first observation in this direction was made in [MT16] – connecting the 2-representation theory of dihedral Soergel bimodules to the work of Kirillov–Ostrik on "finite subgroups" of quantum $\mathfrak{sl}_2$ at roots of unity and the quantum analog of the McKay correspondence. This connection is made rigorous in joint work with Mackaay–Mazorchuk–Miemietz [M^{3}T19] using quantum Satake and (co)algebra 1-morphisms, and later [M^{3}T18] generalized to higher ranks.

In the latest joint work (including X. Zhang) [M^{3}TZ19] we use this connection to state an explicit conjecture how the 2-representation theory of Soergel bimodules (non-abelian, left aside semisimple) is governed by the 2-representation theory of certain fusion categories (semisimple) – basically solving the classification problem for Soergel bimodules. Let me add that we prove our conjecture in a none negligible number of cases and we have plenty of numerical evidence to support our conjecture.

Next steps?

The following are explicit questions on which I work jointly with (a subset of) Mackaay, Mazorchuk, Miemietz and Zhang.

The main focus at the moment is to develop machinery to prove our conjecture from [M^{3}TZ19]. We have some precise ideas how to prove it for classical Weyl types – which would leave only a hand full of cases open. Moreover, since the ideas from the joint work with Andersen heavily draw from a construction of Khovanov–Seidel (which is related to categorical actions of braid groups), one might expect connections from [MT16] to braid groups. Another path we are exploring is that we hope that the construction presented in [MT16] generalizes to other *e.g.* (infinite) Coxeter types. Finally, in [M^{3}T18] we defined a new algebra and its categorification and even some basic questions of this are still open.

A particular project.

As stated above, the 2-representation theory of Soergel bimodules (the categorification of the representation theory of Hecke algebras) could be (almost completely) solved by proving the conjecture in [M^{3}TZ19]. This is very high on my list of projects and we hope to get some results soon.

A word about diagrammatic representation theory

What?

Consider the following question: Given *e.g.* some Lie algebra, can one give a generator-relation presentation for the category of its finite-dimensional representations, or for some well-behaved subcategory?

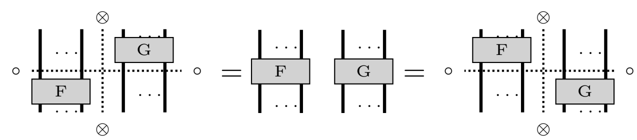
Maybe the best-known instance of this is the case of the monoidal category generated by the vector representation of $\mathfrak{sl}_2$. Its generator-relation presentation is known as the Temperley–Lieb category and goes back to work of Rumer–Teller–Weyl and Temperley–Lieb (the latter in the quantum setting). Note that the Temperley–Lieb category is given diagrammatically which eases to work with it, and the relation from it to representations of $\mathfrak{sl}_2$ is given by the classical Schur–Weyl duality.

Experience tells us that having diagrammatic presentations is very helpful in studying the original questions in representation theory, invariant theory *etc.*

Why?

A general philosophy in category theory is that the morphisms are more important than the objects. The starting observation now is that several formulas in categories have natural diagrammatic counterparts:

$(\mathrm{id}\otimes \mathrm{G})(\mathrm{F}\otimes \mathrm{id}) =$



$= (\mathrm{F}\otimes \mathrm{id})(\mathrm{id}\otimes \mathrm{G}).$

However, a lot of work in classical representation theory is concerned with the objects, *a.k.a.* representations, only. The categorical point of view shows new aspects as *e.g.*:

- ▷ As before, it is easier to see connections to other fields – *e.g.* the relations of representation theory to low-dimensional topology become visible in the literal sense.
- ▷ New aspects of classical theories can be discovered – *e.g.* explicit presentations reveal structure in hitherto unseen detail.
- ▷ Having the category rather then the objects gives rise to the potential to generalize the original constructions – Deligne categories being key examples of such generalizations.

My motivation.

Apart from some well-behaved cases, finding suitable generators and relations is a very difficult task. However, the known examples are very interesting, provide some nice algebras (as endomorphism algebras of the studied categories) and related to many parts of modern mathematics and physics. Most prominent example of such algebras are Temperley–Lieb algebras, Brauer algebras, Hecke algebras, arc algebras, web algebras, KLR algebras and many others.

Note that all of these are given diagrammatically which comes with the upshot that connections to low-dimensional topology become evident. For example, the Temperley–Lieb algebras give a way to define and compute the celebrated Jones polynomial of a knot.

My latest results.

One of the current main applications of diagrammatic methods is in the notoriously hard field of modular representation theory. Various breakthroughs in the past years have be made by the use of diagrammatic machinery. In joint work with Wedrich [TW19], we use the diagrammatics of the Temperley–Lieb category in prime characteristic to nail down the quiver for tilting modules of $\mathrm{SL}_2(\mathbb{K})$ for an algebraically closed field $\mathbb{K}$ of prime characteristic. (Note that the diagrammatics allows us to study tilting modules as a category and not just on the level of the objects=tilting modules.) The result is a fractal-like structure of which we think as being of independent interest – left aside its potential use in representation theory.

In [ET17] we go into a different direction and study the representation theory of such diagram algebras. More precisely, we generalize the notion of cellular algebras to what we call relative cellular algebras and develop a theory of such algebras. We also give several examples which are relative cellular, but not cellular – most of them connected to modular representation theory and some diagrammatic in nature. These examples motivated us to go further and ask for properties of certain quiver algebras [ET18].

Next steps?

The following are explicit questions which remain open.

The representation theory of $\mathrm{SL}_2(\mathbb{K})$ – at least as the modules are concerned – is, of course, well-understood. However, a generalization of [TW19] to higher ranks would be a very good results as already $\mathrm{SL}_3(\mathbb{K})$ is not really understood in prime characteristic.

Secondly, in [ET17] most of our example are related to characteristic p representation theory. It is a striking question whether this is just a coincidence or whether there is some deeper connection between the theories.

Finally, the results of [ET18] should be applicable to the situation studied in [M^{3}T18].

A particular project.

We are currently investigating how [TW19] can be generalized to other groups using the web technology developed in *e.g.* [TVW17].

A word about homological knot theory

What?

In his pioneering work, Khovanov introduced what is nowadays called Khovanov homology – his celebrated categorification of the Jones polynomial – which is a homological invariant of links.

Studying link homologies has become a big industry after Khovanov's breakthrough, and many link homologies are known by now, coming and connecting various fields of modern mathematics. Most notably for my purpose, the categorification of the HOMFLYPT polynomial, called HOMFLYPT or triply-graded homology.

Why?

Homology theories are ubiquitous in modern mathematics, ranging from singular homology of topological spaces to knot homologies. These homological invariants take values in, say, isomorphism classes of vector spaces, instead of in numbers as *e.g.* Betti numbers do. One main point is that these homology theories usually extend to functors, *e.g.*

$$\left\{\begin{matrix}\text{topolocial spaces}\\ \text{continuous maps}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{vector spaces}\\ \text{linear maps}\end{matrix}\right\}.$$

So, additionally to being stronger invariants, they also provide information about how certain structure are related. Here are a few upshots of these homology theories à la Khovanov:

- ▷ Again, it is easier to see connections to other fields – *e.g.* several conjectures (already proven or still open) from physics are related to versions of Khovanov's link homology.

- ▷ Open question for the polynomials are proven for the homologies – *e.g.* Khovanov homology detects the unknot, which is only conjecturally true for the Jones polynomial.

- ▷ The analog of the continuous maps for topological spaces are cobordisms between knots which live in 4-space – thus, link homologies have direct connections to $4d$ topology.

My motivation.

For me the important point about link homologies is not that they are strong link invariants, but rather that they are functorial (see *e.g.* [ETW18]) – they provide information about cobordisms between links. These ideas have led to several new results in smooth four-dimensional topology, *e.g.* a combinatorial proof of the Milnor conjecture regarding slice torus knots by Rasmussen and purely combinatorial constructions of exotic structures in four space by Gompf–Rasmussen.

Another point I would like to make is that the study of these homologies connects many fields of modern mathematics and physics by providing a common ground to talk. This is, in my humble opinion, a very important feature of the HOMFLYPT homology.

My latest results.

As stated above, Khovanov's categorification of the HOMFLYPT polynomial has been generalized to many other setups and serves as a bridge between various fields of modern mathematics and physics. But, to the best of our knowledge, almost all variants of Khovanov's invariant, as well as almost all variants of link homologies in general, stay in type A, meaning for us that they are related to the classical braid group.

A main motivation for me is to generalize these HOMFLYPT invariants to other settings involving different braid groups. A first step in this direction is joint work with Rose [RT19], defining a HOMFLYPT invariant for links in a handlebody which is functorial on handlebody braids (almost by birth).

Next steps?

The connection of the classical theory of Artin–Tits braid groups to low-dimensional topology remains mysterious. It would be a very important step in understanding what is going on if one could make a precise connection, *e.g.* one a categorical level by defining link homologies (in what setting remains to be determined) sensitive to different braidings.

A particular project.

The main statement of the paper [RT19] defines a triply-graded invariant of (colored) links in a genus g handlebody. Our ingredients are singular Soergel bimodules of type A and it is a nice question what one would get using different types. This is work in progress, and we think these link homologies might turn out to be very interesting.

My research philosophy

In order to obtain good research results, one should get deep and broad – that is something most mathematicians would agree on. However, as this is impossible to do simultaneously, I try to "zigzag my way" to this goal. Hereby the quote of Raul Bott "There are two ways to do mathematics. The first is to be smarter than everybody else. The second way is to be stupider than everybody else – but persistent." is my guiding principle.

[ET18] M. Ehrig, D. Tubbenhauer. Algebraic properties of zigzag algebras. *Comm. Algebra* 48 (2020), no. 1, 11–36. arXiv:1807.11173.

[ET17] M. Ehrig, D. Tubbenhauer. Relative cellular algebras. To appear in *Transform. Groups*. arXiv:1710.02851.

[ETW18] M. Ehrig, D. Tubbenhauer, P. Wedrich. Functoriality of colored link homologies. *Proc. Lond.*

Math. Soc. (3) 117 (2018), no. 5, 996–1040. arXiv:1703.06691.

[MT16] M. Mackaay, D. Tubbenhauer. Two-color Soergel calculus and simple transitive 2-representations. *Canad. J. Math.* 71 (2019), no. 6, 1523–1566. arXiv:1609.00962.

[M^{3}T19] M. Mackaay, V. Mazorchuk, V. Miemietz, D. Tubbenhauer. Simple transitive 2-representations via (co)algebra 1-morphisms. *Indiana Univ. Math. J.* 68 (2019), no. 1, 1–33. arXiv:1612.06325.

[M^{3}T18] M. Mackaay, V. Mazorchuk, V. Miemietz, D. Tubbenhauer. Trihedral Soergel bimodules. *Fund. Math.* 248 (2020), no. 3, 219–300. arXiv:1804.08920.

[M^{3}TZ19] M. Mackaay, V. Mazorchuk, V. Miemietz, D. Tubbenhauer, X. Zhang. 2-representations of Soergel bimodules. arXiv:1906.11468.

[M^{3}TZ20] M. Mackaay, V. Mazorchuk, V. Miemietz, D. Tubbenhauer, X. Zhang. Finitary birepresentations of finitary bicategories. arXiv:2008.01658.

[RT19] D.E.V. Rose, D. Tubbenhauer. HOMFLYPT homology for links in handlebodies via type A Soergel bimodules. To appear in *Quantum Topol.* arXiv:1908.06878.

[TVW17] D. Tubbenhauer, P. Vaz, P. Wedrich. Super q-Howe duality and web categories. *Algebr. Geom. Topol.* 17-6 (2017), 3703–3749. arXiv:1504.05069.

[TW19] D. Tubbenhauer, P. Wedrich. Quivers for SL_2 tilting modules. arXiv:1907.11560.

[TW20] D. Tubbenhauer, P. Wedrich. The center of SL_2 tilting modules. arXiv:2004.10146.

Daniel Tubbenhauer; November 28, 2020