RESEARCH STATEMENT

DANIEL TUBBENHAUER

Disclaimer. My intention is not to give rigorous mathematical definitions or statements, but rather to give an informal overview about my research. I hope the reader will forgive me my sloppy formulations.

Contents

In short

My main research interest is 2-categorical representation theory (of e.g. Coxeter groups), categorification (of e.g. quantum groups) and applications in representation theory, low-dimensional topology and algebraic geometry. In particular, I am interested in algebraic, combinatorial and diagrammatic aspects of categorification. I am also interested in related topics as for example representation theoretical questions about Hecke/Brauer algebras or Lie groups and modular representation theory.

My research interest at the moment basically splits into a topologically motivated part concerning algebraic constructions of link homologies and their functoriality, see Section 2B, as well as a representation theory motivated part concerning questions coming from invariant theory, see Section 2A, from the theory of finite-dimensional algebras and their representations, see Section 2C, respectively from 2-representations of Coxeter groups, see Section 2D.

1. General overview

Before going into details of my current research, let me try to motivate the basic questions which play a major role in all of my research interests.

1A. **Categorification.**

1A.1. *Seeking "higher" structure.* The notion *categorification* was introduced by Crane [Cra95] based on an earlier joint work with Frenkel [CF94].

But the concept of categorification has a much longer history than the word itself. Forced to explain the concept in one sentence, I would choose

Interesting integers are shadows of richer structures in categories.

The basic idea is as follows. Take a set-based structure S and try to find a category-based structure $\mathbf{C}$ such that S is just a shadow of the category $\mathbf{C}$. If the category $\mathbf{C}$ is chosen in a "good" way, then one has an explanation of facts about the structure S in a categorical language. That is, certain facts in S can be explained as special instances of natural constructions.

Experience tells us that the categorical structure does not only explain properties of the set-based structure, but is usually much richer and more interesting itself.

Remark

In principal, one can perform such a *categorification process* on any level, e.g. one can categorify an n-category like structure into an $n+1$-category like structure. Without going into any details about higher categories, the slogan for me is that a set is a collection of number like structures with the set of natural numbers as a prototypical example; a category is a collection of set like structures with the category of sets as a prototypical example; a 2-category is a collection of category like structures with the 2-category of categories as a prototypical example; a 3-category is a collection of 2-category like structures etc.

1A.2. *Decategorification.* Categorification comes with an "inverse" called *decategorification*; and categorification can be seen as remembering or inventing information while decategorification is more like forgetting or identifying structure, which is, of course, easier.

Remark

One has to specify what is meant by decategorification before saying "We have categorified XY". However, I will be a bit sloppy in what follows.

A basic example: the category of $\mathbb{K}$-vector spaces $\mathbb{K}\mathbf{Vect}$ (over some field $\mathbb{K}$) categorifies $\mathbb{Z}_{\geq 0}$; the decategorification is given by taking dimensions.

Another example is to take (a suitable version of) the Grothendieck group $[\cdot]_{\mathbb{K}} = K_0(\cdot)\otimes_{\mathbb{Z}}\mathbb{K}$, and we have $K_0(\mathbb{K}\mathbf{Vect}) \cong \mathbb{Z}$ and $[\mathbb{K}\mathbf{Vect}]_{\mathbb{K}} \cong \mathbb{K}$.

In fact, if we think of the category $\mathbb{K}\mathbf{Vect}$ as being a set-based structure, then we might want to categorify this further by considering the 2-category $\mathbb{K}\mathfrak{Cat}$ of $\mathbb{K}$-linear categories, $\mathbb{K}$-linear functors and $\mathbb{K}$-linear natural transformations. Taking an appropriate 2-categorical Grothendieck group recovers $\mathbb{K}\mathbf{Vect}$.

1A.3. *One diagram is worth a thousand words.* Each step of a categorification process should reveal more structure. An illustration for the example from above is the following (omitting the $\mathbb{K}$):

$$
\begin{tikzcd}
\text{categories} \arrow[d, "\text{“categorifies”}"] & \text{functors} \arrow[l, dotted] \arrow[d, dashed, "\text{“categorifies”}"] & \text{nat. trafos} \arrow[l, dotted] \\
\text{vector spaces} \arrow[d, "\text{“categorifies”}"] & \text{linear maps} \arrow[l, dotted] & \\
\text{numbers} & &
\end{tikzcd}
$$

Here we first categorify numbers into $\mathbb{K}$-vector spaces. The new information available are now $\mathbb{K}$-linear maps between $\mathbb{K}$-vector spaces. (Thus, we have the whole power of linear algebra at hand.) There is no reason to stop: we can categorify $\mathbb{K}$-vector spaces into $\mathbb{K}$-categories, $\mathbb{K}$-linear maps into $\mathbb{K}$-linear functors. Again, we see a new layer of information, namely the natural transformations between these functors.

1A.4. *Examples of categorification.* The following list of example is already long, but biased and far from being complete. Much more can be found in the work of Baez–Dolan [BD98], [BD01] for examples which are related to more combinatorial parts of categorification, or Crane–Yetter [CY98], Khovanov–Mazorchuk–Stroppel [KMS09], Mazorchuk [Maz12] or Savage [Sav14] for examples from algebraic categorification.

- ▷ In some sense the “most classical, but quite recent” example is Khovanov’s categorification of the Jones (or $\mathfrak{sl}_2$) polynomial [Kho00].
- ▷ Khovanov’s construction can be extended to a categorification of the Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomial and the HOMFLY–PT polynomial, e.g. see [KR08]. Moreover, some applications of Khovanov’s categorification are:
  - ▶ It is functorial, e.g. see [Cap08], [CMW09], [Bla10] or [ETW17]: it “knows” about link cobordisms. Since such cobordisms are cobordisms embedded in four-space, this gives a way to get information about smooth structures in dimension four. (And smooth, four-dimensional topology is hell!)
  - ▶ Kronheimer–Mrowka showed [KM11], by comparing Khovanov homology to Knot Floer homology, that Khovanov homology detects the unknot. This is still an open question for the Jones polynomial.
  - ▶ Rasmussen obtained his famous invariant by comparing Khovanov homology to a variation of it. He used his invariant to give a combinatorial proof of the Milnor conjecture [Ras10]. Note that he also gives a way to construct exotic structures on $\mathbb{R}^4$ from his approach [Ras05].
  - ▶ There is a variant of Khovanov homology, called odd Khovanov homology, see [ORS13], which differs over $\mathbb{Q}$ and can not be seen on the level of polynomials.
  - ▶ There is a variant that categorifies the HOMFLY–PT polynomial. This categorification is a rich structure itself and has a lot of connections to various parts of mathematics and related fields, see e.g. [GORS14] and the references therein.
  - ▶ Not the main point but: it is strictly stronger than the Jones polynomial.
- ▷ Other notable categorifications related to low-dimensional topology are:
  - ▶ Floer homology can be seen as a categorification of the Casson invariant of a manifold. Floer homology is “better” than the Casson invariant, e.g. it is

possible to construct a 3+1 dimensional TQFT which for closed four-dimensional manifolds gives Donaldson's invariants, see for example [Wit12].

- ▶ Knot Floer homology can be seen as a categorification of the classical knot invariant of Alexander–Conway, see for example [OS04].
- ▶ The approach to categorify the Reshetikhin–Turaev $\mathfrak{g}$-polynomial for arbitrary simple Lie algebra $\mathfrak{g}$ by Webster [Web17].

▷ The notion categorification is from the interplay of low-dimensional topology and representation theory. Hence, there are also several examples coming from representation theory as e.g.:

- ▶ Ariki [Ari96] gave a remarkable categorification of all finite-dimensional, irreducible representation of $\mathfrak{sl}_n$, for all n, as well as a categorification of integrable, irreducible representations of the affine version $\widehat{\mathfrak{sl}}_n$. In short, he identified the Grothendieck group of blocks of so-called Ariki–Koike cyclotomic Hecke algebras with weight spaces of such representations in such a way that direct summands of induction and restriction functors between cyclotomic Hecke algebras for $m, m+1$ act on K_0 as the E_i, F_i of $\mathfrak{sl}_n$.
- ▶ Chuang–Rouquier [CR08] masterfully used the categorification of good old $\mathfrak{sl}_2$ to solve an open problem in modular representation theory of the symmetric group using a completely new approach.
- ▶ Khovanov–Lauda [KL10], and, independently, Rouquier [Rou08] have categorified all quantum Kac–Moody algebras with their canonical bases.
- ▶ Khovanov–Qi [KQ15] and Elias–Qi [EQ16] have an approach how to categorify at roots of unity. Their categorification of quantum $\mathfrak{sl}_2$ for the quantum parameter q being a (certain type of) root of unity can be (the future will prove me right or wrong) the first step to categorify the Witten–Reshetikhin–Turaev invariants of 3-manifolds.
- ▶ The so-called category of Soergel bimodules can be seen in the same vein as a categorification of the Hecke algebras (for Coxeter groups) in the sense that the split Grothendieck group gives the Hecke algebras. We note that Soergel's construction shows that Kazhdan–Lusztig bases have positive integrality properties, see [Soe90] and related publications. Indeed, this approach was masterfully carried out by Elias–Williamson who finally proved the Kazhdan–Lusztig basis conjecture for all Coxeter types [EW14].

▷ Categorifications are also studied in physics, e.g.:

- ▶ In conformal field theory (CFT) researchers study fusion algebras, e.g. the Verlinde algebra. Examples of categorifications of such algebras are known, e.g. using categories connected to the representation theory of quantum groups at roots of unity [EGNO15], [Kho16], and contain more information than these algebras, e.g. the R-matrix and the quantum $6j$-symbols.
- ▶ The Witten genus of certain moduli spaces can be seen as an element of $\mathbb{Z}[[q]]$. It can be realized using elliptic cohomology, see e.g. [AHS01].

1B. **Higher representation theory.**

1B.1. *The classical question.*

Groups, as men, will be known by their actions. – Guillermo Moreno

The study of group actions is of fundamental importance in mathematics and related field. Sadly, it is also very hard. Luckily, Frobenius (∼1895 onwards), Burnside (∼1900 onwards) and many others gave us the following.

Let $\mathbb{K}[G]$ be the group ring of a (finite) group G. Representation theory is the study of $\mathbb{K}$-linear group actions:

$$\mathrm{R}\colon \mathbb{K}[G] \longrightarrow \mathrm{End}(V), \quad \mathrm{R}(g) = \text{a “matrix” in } \mathrm{End}(V),$$

with V being some $\mathbb{K}$-vector space. We call V a G-module or a G-representation.

Representation theory approach: the analogous $\mathbb{K}$-linear problem of classifying G-modules has a satisfactory answer for many groups.

$$\begin{tikzcd}[column sep=small]
\begin{array}{c}\text{Problem involving}\\ \text{a group action}\\ G \circlearrowright X\end{array} \arrow[rr, dotted] & & \begin{array}{c}\text{Problem involving a}\\ \mathbb{K}\text{-linear group action}\\ \mathbb{K}[G] \circlearrowright \mathbb{K}[X]\end{array} \arrow[dl, dotted] \\
& \begin{array}{c}\text{“Decompostion of}\\ \text{the problem”}\\ \mathbb{K}[G] \circlearrowright \bigoplus V_i\end{array} \arrow[ul, dotted] &
\end{tikzcd}$$

Thus, given a group G (or a ring, an algebra etc.), a classical and interesting question is:

Can we describe the symmetries G can act on,
i.e. its representation theory?

1B.2. *Categorified representation theory.*

The related, categorical question that arises is whether we can categorify the classical notions. That is:

Can we describe the symmetries a category $\mathbf{C}$ can act on,
i.e. its representation theory?

Moreover, the next question would be if we can categorify this again:

Can we describe the symmetries a 2-category $\mathfrak{C}$ can act on,
i.e. its 2-representation theory?

I will give a short introduction to the basic ideas. Much more details can, for example, be found in Rouquier's paper [Rou08]. Another also very nice introduction is the book of Mazorchuk [Maz12].

Let A be some (group) algebra, V be an A-module and $\mathbf{V}$ be a (suitable) category. Let $2\mathbf{End}(\mathbf{V})$ denotes its associated 2-category of endofunctors. Then:

$$\text{Classical} \rightsquigarrow \text{Higher}$$

$$a \mapsto \mathrm{R}(a) \in \mathrm{End}(V) \rightsquigarrow a \mapsto \boldsymbol{\mathcal{R}}(a) \in 2\mathbf{End}(\mathbf{V})$$

$$(\mathrm{R}(a_1) \cdot \mathrm{R}(a_2))(v) = \mathrm{R}(a_1a_2)(v) \rightsquigarrow (\boldsymbol{\mathcal{R}}(a_1) \circ \boldsymbol{\mathcal{R}}(a_2))(X) \cong \boldsymbol{\mathcal{R}}(a_1a_2)(X)$$

So, equalities between matrices lift to isomorphisms of functors, and on the level of the Grothendieck group the functors will descent to the matrices which they categorify.

A *(weak) categorification of the* A*-module* V should be thought of as a categorical action of A on $\mathbf{V}$ with an isomorphism ψ such that

$$\begin{tikzcd}
{[\mathbf{V}]_{\mathbb{K}}} \arrow[r, "{[\boldsymbol{\mathcal{R}}_a]}"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\circlearrowleft"] & {[\mathbf{V}]_{\mathbb{K}}} \arrow[d, "\psi"] \\
V \arrow[r, "\mathrm{R}(a)"'] & V
\end{tikzcd}$$

commutes. Note that such a categorification again "knows" more, i.e. it "knows the relations" between the acting matrices instead of just the acting matrices. But the higher structure is not fixed in such a categorification.

Remark

In a lot of cases there is also a "honest higher layer". Instead of having the algebra A acting on the $\mathbb{K}$-vector space V, we want it to act on the $\mathbb{K}$-linear category $\mathbf{V}$. But in the usual spirit of categorification this picture should upgrade:

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "\text{full-grown 2-action}"] \arrow[d, "{[\cdot]_{\mathbb{K}}}"'] & \mathbf{2End}(\mathbf{V}) \arrow[d, "{[\cdot]_{\mathbb{K}}}"] \\
\mathrm{A} \arrow[r, "\text{classical action}"'] \arrow[ur, dotted, "\text{categorical action}"] & \mathrm{End}(V)
\end{tikzcd}$$

Hereby we view the uncategorified action as a functor from A to $\mathrm{End}(V)$ (both seen e.g. as categories with one object). Then there should be a categorification $\mathfrak{A}$ of A such that we have a diagram as above, with $\mathfrak{A}$ acting via a 2-functor (which in particular fixes the higher structure).

This is sometimes called *(strong)* 2*-action of* A and works surprisingly often.

1B.3. *Examples of categorified representations.* The following list is again biased. As before, the two sources [KMS09] and [Maz12] give several other examples.

- ▷ In some sense one of the most classical example of categorified representations is provided by (versions of) the BGG category $\mathcal{O}$.
  - ▶ For instance, the regular representation of the group ring $\mathbb{C}[S_n]$ of the symmetric group S_n has a categorification given by projective functors acting on a regular block of the BGG category $\mathcal{O}$ for $\mathfrak{sl}_n$, see e.g. [BG80] (where the authors, of course, do not use the word categorification).
  - ▶ Integral Specht modules of S_n can be categorified in a quite similar fashion, see e.g. [KMS08].
  - ▶ Similarly for categorifications of the associated Hecke algebra.
  - ▶ Other constructions in this spirit are known, see [Maz12] for an overview.
- ▷ Another nowadays classical categorification is connected to categorification of the tensor product of the vector representation of (quantum) $\mathfrak{sl}_2$:

- Bernstein–Frenkel–Khovanov [BFK99] categorified the n-fold tensor product of $\mathfrak{sl}_2$ by using certain induction/restriction functors coming from cohomology rings of Grassmannians and flag varieties.
- Frenkel–Khovanov–Stroppel extended this to the graded setup, and also include a categorification of the highest weight summand of the n-fold tensor product, see [FKS06].
- Recently Naisse–Vaz [NV16] extended this into a categorification of the Verma modules for $\mathfrak{sl}_2$.

▷ As already mentioned above:

- Based on joint work with Chuang [CR08], Rouquier [Rou08] (and, independently, Khovanov–Lauda [KL10]) gave a categorification of all simple modules of $\mathfrak{g}$, for $\mathfrak{g}$ being a simple Lie algebra.
- Very much in this spirit, Webster [Web17] proposed a categorification of tensor products of simple $\mathfrak{g}$-modules.
- Categorification in this spirit are a "big industry" nowadays (and it is not possible to summarize it briefly) and its very hard to overestimate the influence of the approach of Chuang–Rouquier and Khovanov–Lauda.

▷ Shortly after the groundbreaking work of (Chuang and) Rouquier, Mazorchuk–Miemietz (and their coauthors) started a systematic study of categorifications of finite-dimensional modules of finite-dimensional algebras (cf. [MM16a] or [MM16b] and the references therein):

- In [MM16b] they defined an appropriate 2-categorical analogue of the simple representations of finite-dimensional algebras.
- Several examples are know, e.g. a well-studied class of examples is given by so-called cell representations of categorified Coxeter groups, see [MM11].
- At the moment, even in the case of Coxeter groups, not much is known, see e.g. [KMMZ16], [MT16], [MMMT16] or [MMMZ18].

2. Recently finished projects

This section is intended to describe some of my newest research projects.

2A. **Webs calculi in representation theory.**

Paper [ST17]

A. Sartori and D. Tubbenhauer, *Webs and q-Howe dualities in types* BCD, to appear in Trans. Amer. Math. Soc., https://arxiv.org/abs/1701.02932.

Abstract. We define web categories describing intertwiners for the orthogonal and symplectic Lie algebras, and, in the quantized setup, for certain orthogonal and symplectic coideal subalgebras. They generalize the Brauer category, and allow us to prove quantum versions of some classical type BCD Howe dualities.

2A.1. *Diagrammatic representation theory.* Consider the following question:

> Given some Lie algebra $\mathfrak{g}$, can one give a generator-relation presentation for the category of its finite-dimensional representations, or for some well-behaved subcategory?

Maybe the best-known instance of this is the case of the monoidal category generated by the vector representation of $\mathfrak{sl}_2$, or by the corresponding representation of its quantized enveloping algebra of $\mathfrak{sl}_2$. Its generator-relation presentation is known as the Temperley–Lieb category and goes back to work of Rumer–Teller–Weyl and Temperley–Lieb (the latter in the quantum setting).

2A.2. *Web calculi and representation theory.* In pioneering work, Kuperberg [Kup96] extended this to all rank 2 Lie algebras and their quantum enveloping algebras. However, it was not clear for quite some time how to extend Kuperberg's constructions further (although some partial results were obtained). Then, in seminal work [CKM14], Cautis–Kamnitzer–Morrison gave a generator-relation presentation of the monoidal category generated by (quantum) exterior powers of the vector representation of quantum $\mathfrak{gl}_n$.

Their crucial observation was that a classical tool from representation and invariant theory, known as skew Howe duality [How95, How89], can be quantized and used as a device to describe intertwiners of quantum $\mathfrak{gl}_n$.

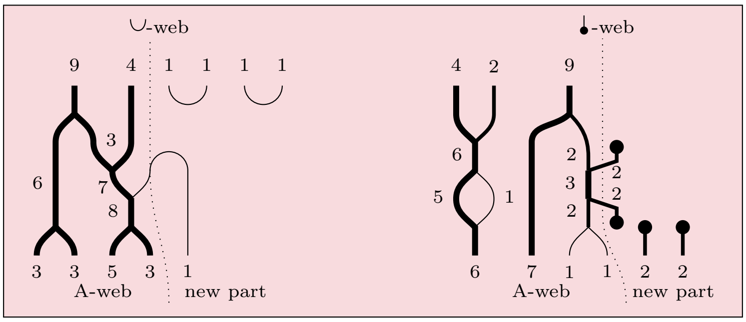
In fact, as explained in [CKM14], they allow a nice diagrammatic interpretation via so-called A-webs.

2A.3. *Other classical groups.* The results from [CKM14] were then extended to various other instances. But, to the best of our knowledge, all generalizations so far stay in type A.

The idea which started the paper [ST17] was to extend Cautis–Kamnitzer–Morrison's approach to types BCD. However, the main obstacle immediately arises: while the quantization of skew Howe duality is fairly straightforward in type A, it is very unclear how this should be done in the other classical cases.

One of the main ideas to overcome this is to use non-monoidal web calculi. That is, our main tool are certain diagrams made out of trivalent graphs with edge labels from $\mathbb{Z}_{\geq 0}$, which we call A-, $\cup$- and $\downarrow$-webs.

The A-webs where introduced in [CKM14] and assemble into a monoidal category. The $\cup$- and $\downarrow$-webs were introduced in [ST17]. Here is a picture:



Using these, we are not just able to extend the web calculi to types BCD, generalizing Brauer's classical diagram calculus [Bra37], but we also quantize Howe's dualities [How95, How89] for orthogonal and sympletic groups.

2B. **Link homologies.**

Paper [ETW17]

M. Ehrig, D. Tubbenhauer and P. Wedrich, *Functoriality of colored link homologies*, https://arxiv.org/abs/1703.06691.

Abstract. We prove that the bigraded, colored Khovanov–Rozansky type A link and tangle invariants are functorial with respect to link and tangle cobordisms.

2B.1. *Link and tangle invariants.* As already mentioned in Section 1A.4, Khovanov [Kho00] introduced a link homology theory categorifying the Jones polynomial. His construction is one of the main examples of categorification has has attracted a lot of attention from various fields of mathematics and physics. Building on Khovanov's categorification of the Jones polynomial, Khovanov–Rozansky [KR08] introduced a link homology theory categorifying the type A Reshetikhin–Turaev invariant. Their homology theory associates bigraded vector spaces to link diagrams, two of which are isomorphic whenever the diagrams differ only by Reidemeister moves. In the original formulation, the link invariant, thus, takes values in isomorphism classes of bigraded vector spaces.

2B.2. *Functoriality.* As mentioned in Section 1A.4, one of the main features of Khovanov's link homology are its functorial properties. To elaborate a bit:

The first question posed by this construction is whether there is a natural choice of Reidemeister move isomorphisms, such that any isotopy of links in $\mathbb{R}^3$ gives rise to an explicit isomorphism between the Khovanov–Rozansky invariants, which only depends on the isotopy class of the isotopy. A positive answer to this question provides a functor:

$$\boxed{\left\{\begin{matrix}\text{link embeddings in } \mathbb{R}^3 \\ \text{isotopies modulo isotopy}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{bigraded vector spaces} \\ \text{isomorphisms}\end{matrix}\right\}}$$

The second question building on the first, is whether this functor can be extended to a functor:

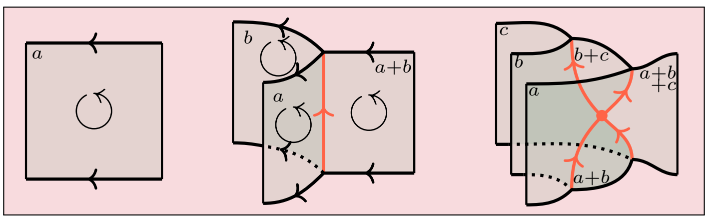
$$\boxed{\left\{\begin{matrix}\text{link embeddings in } \mathbb{R}^3 \\ \text{link cobordisms in } \mathbb{R}^3 \times [0,1] \text{ modulo isotopy}\end{matrix}\right\} \to \left\{\begin{matrix}\text{bigraded vector spaces} \\ \text{homogeneous linear maps}\end{matrix}\right\}}$$

2B.3. *Functoriality of colored link homologies.* The goal of the paper [ETW17] is to answer both questions affirmatively, i.e. to prove the functoriality of Khovanov–Rozansky link homologies under link cobordisms.

We prove the general functoriality statement in a framework that is different to and more general than Khovanov–Rozansky's construction in [KR08]. For example, the so-called Murakami–Ohtsuki–Yamada state-sum model determines the Reshetikhin–Turaev invariants of links whose components are colored by fundamental representations. One categorical level up, there is an analogous extension of Khovanov–Rozansky's original uncolored (i.e. colored only with the vector representation) construction to the colored case. In [ETW17], we work in this generality.

Note also that we have decided to present the results of the paper [ETW17] using the ground ring $\mathbb{C}$. This is for notational convenience: with minimal adjustments our proof of functoriality also works over $\mathbb{Z}$.

Our main tool is the graphical calculus of foams, i.e. certain singular cobordisms locally modeled on



These foam models provide an accessible way to define and study the link homologies.

2C. **Relative cellular algebras.**

Paper [ET17]

M. Ehrig and D. Tubbenhauer, *Relative cellular algebras*, https://arxiv.org/abs/1710.02851.

Abstract. In this paper we generalize cellular algebras by allowing different partial orderings relative to fixed idempotents. For these relative cellular algebras the construction and classification of simples still works similarly as for cellular algebras, but they are e.g. homologically quite different.

We give several examples of algebras which are relative cellular, but not cellular. Most prominently, the restricted enveloping algebra and the small quantum group for $\mathfrak{sl}_2$, and an annular version of arc algebras.

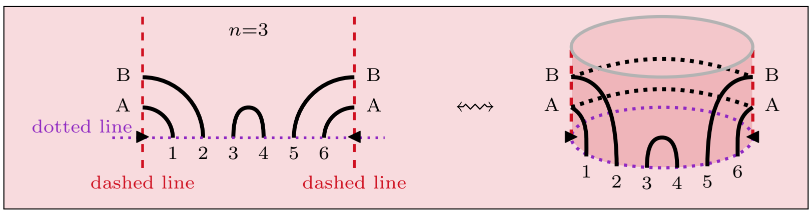
2C.1. *Cellular algebras.* In pioneering work Graham–Lehrer [GL96] introduced the notion of a cellular algebra, i.e. an algebra equipped with a so-called cell datum. Examples of such algebras are semisimple algebras, commutative algebras, Hecke algebras of the symmetric group, various diagram algebras as e.g. Temperley–Lieb or Brauer algebras, arc algebras and many more.

Next, of key importance for this paper, the cell datum comes with a set X and a partial order $<$ on it. The cell datum provides a method to systematically reduce hard questions about the representation theory of such algebras to problems in linear algebra. Hereby the partial order $<$ on the set X plays an important role since it yields an "upper triangular way" to construct certain modules, called cell modules, which have a crucial role in the theory. In well-behaved cases the linear algebra problems can be solved giving e.g. a parametrization of the isomorphism classes of simple modules via a subset of X, and a construction of a representative for each class.

2C.2. *Relative cellular algebras.* In the paper [ET17] we (strictly) generalize the notion of a cellular algebra to what we call a relative cellular algebra, i.e. an algebra equipped with a relative cell datum. For example, the relative cell datum comes with a set X, but now with several partial orders $<_{\varepsilon}$ on it, one for each idempotent from a preselected set of idempotents. Taking only one idempotent $\varepsilon = 1$, namely the unit, and only one partial order $<_1 = <$, we recover the setting of Graham–Lehrer. Surprisingly, most of the theory of cellular algebras still works in this relative setup. However, with fairly different proofs, carefully incorporating the various partial orders.

Examples which are only cellular in the relative sense are small quantum groups for $\mathfrak{sl}_2$ as well as annular versions of arc algebras. All of our examples semm to come from characteristic p representation theory, but we do not know whether this is a coincidence.

2C.3. *A key examples.* One of our main examples of an algebra which is relative cellular, but not cellular in the usual sense is an annular version of Khovanov-type arc algebras [Kho02] with the following picture to be kept in mind:



For this algebra there are various partial orderings $<_{\varepsilon_i}$ (on the same set) attached to certain idempotents ε_i. Here an example of how these partial orderings can vary.

First, on one idempotent ε_2 we have

$$\begin{tikzcd}
 & & \boxed{\vee\vee\wedge\wedge} \arrow[dr, "<_{\varepsilon_2}"] & & \\
\boxed{\vee\wedge\wedge\vee} \arrow[r, "<_{\varepsilon_2}"] & \boxed{\wedge\vee\wedge\vee} \arrow[ur, "<_{\varepsilon_2}"] \arrow[dr, "<_{\varepsilon_2}"] & & \boxed{\vee\wedge\vee\wedge} \arrow[r, "<_{\varepsilon_2}"] & \boxed{\wedge\vee\vee\wedge} \\
 & & \boxed{\wedge\wedge\vee\vee} \arrow[ur, "<_{\varepsilon_2}"] & &
\end{tikzcd}$$

but on a different idempotent ε_5 we have

$$\begin{tikzcd}
 & & \boxed{\wedge\vee\vee\wedge} \arrow[dr, "<_{\varepsilon_5}"] & & \\
\boxed{\vee\vee\wedge\wedge} \arrow[r, "<_{\varepsilon_5}"] & \boxed{\vee\wedge\vee\wedge} \arrow[ur, "<_{\varepsilon_5}"] \arrow[dr, "<_{\varepsilon_5}"] & & \boxed{\wedge\vee\wedge\vee} \arrow[r, "<_{\varepsilon_5}"] & \boxed{\wedge\wedge\vee\vee} \\
 & & \boxed{\vee\wedge\wedge\vee} \arrow[ur, "<_{\varepsilon_5}"] & &
\end{tikzcd}$$

2D. **Categorical representations of trihedral Hecke algebras.**

Paper [MMMT18]

M. Mackaay, V. Mazorchuk, V. Miemietz and D. Tubbenhauer, *Trihedral Soergel bimodules*, https://arxiv.org/abs/1804.08920.

Abstract. The quantum Satake correspondence relates dihedral Soergel bimodules to the semisimple quotient of the quantum $\mathfrak{sl}_2$ representation category. It also establishes a precise relation between the simple transitive 2-representations of both monoidal categories, which are indexed by bicolored ADE Dynkin diagrams.

Using the quantum Satake correspondence between affine A_2 Soergel bimodules and the semisimple quotient of the quantum $\mathfrak{sl}_3$ representation category, we introduce trihedral Hecke algebras and Soergel bimodules, generalizing dihedral Hecke algebras and Soergel bimodules. These have their own Kazhdan–Lusztig combinatorics and simple transitive 2-representations corresponding to tricolored generalized ADE Dynkin diagrams.

2D.1. *Motivation: higher representation theory.* In pioneering work [KL79], Kazhdan–Lusztig defined their celebrated bases of Hecke algebras for Coxeter groups. Crucially, on these bases the structure constants of the algebras belong to $\mathbb{Z}_{\geq 0} = \mathbb{Z}_{\geq 0}$. This started a program to study $\mathbb{Z}_{\geq 0}$-algebras, which have a fixed basis with non-negative integral structure constants, see e.g. [Lus87], [EK95], where these algebras are called $\mathbb{Z}_{+}$-rings.

As proposed by the work of Kazhdan–Lusztig, for $\mathbb{Z}_{\geq 0}$-algebras it makes sense to study and classify $\mathbb{Z}_{\geq 0}$-representations, i.e. representations with a fixed basis on which the fixed bases elements of the algebra act by non-negative integral matrices, see e.g. [EK95]. The first examples are the so-called cell representations, which were originally defined for Hecke algebras [KL79], but can be defined for all $\mathbb{Z}_{\geq 0}$-algebras. As it turns out, $\mathbb{Z}_{\geq 0}$-representations are interesting from various points of view, with applications and connections to e.g. graph theory, conformal field theory, modular tensor categories and subfactor theory.

Categorical analogs of $\mathbb{Z}_{\geq 0}$-algebras are finitary 2-categories. These decategorify to $\mathbb{Z}_{\geq 0}$-algebras, because the isomorphism classes of the indecomposable 1-morphisms form naturally a $\mathbb{Z}_{\geq 0}$-basis. For example, Hecke algebras of Coxeter groups are categorified by Soergel bimodules, see Section 1A or also Section 1B.

The categorical incarnation of $\mathbb{Z}_{\geq 0}$-representation theory is 2-representation theory. Any 2-representation decategorifies naturally to a $\mathbb{Z}_{\geq 0}$-representation, with the $\mathbb{Z}_{\geq 0}$-basis given by the isomorphism classes of the indecomposable 1-morphisms. However, not all $\mathbb{Z}_{\geq 0}$-representations can be obtained in this way.

In 2-representation theory, the simple transitive 2-representations play the role of the simple representations, see in Section 1B above. Although their decategorifications need not be simple as complex representations, they are the "simplest" 2-representations, as attested e.g. by the categorical Jordan–Hölder theorem [MM16b]. This naturally motivates the problem of classification of simple transitive 2-representations of 2-categories

2D.2. *The dihedral story.* For finite Coxeter types, the classification of the simple transitive 2-representations of Soergel bimodules is only partially known, see e.g. [KMMZ16], [MMMZ18], [Zim17]. There are two exceptions: For Coxeter type A, the cell 2-representations exhaust the simple transitive 2-representations of Soergel bimodules [MM16b], so the classification problem has been solved. For Coxeter type of dihedral type there also exists a complete classification of simple transitive 2-representations [KMMZ16], [MT16] (under the additional assumption of gradeability), which is completely different from the one for type A. In this case, the simple transitive 2-representations of rank greater than one are classified by bicolored ADE Dynkin diagram.

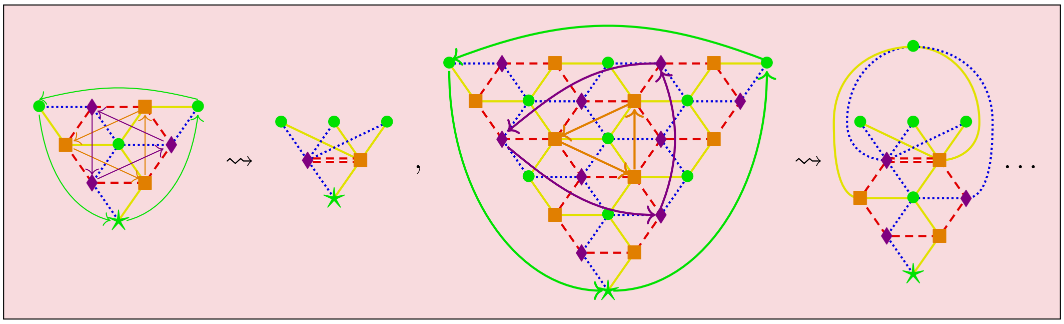
Let us explain this in a bit more detail. Equivalence classes of simple transitive 2-representations of finitary 2-categories (or graded versions of them) correspond bijectively to Morita equivalence classes of simple algebra 1-morphisms in the abelianizations of these 2-categories. This was initially proved for semisimple tensor categories [Ost03] and later generalized to certain finitary 2-categories with duality [MMMT18]. Kirillov–Ostrik [KJO02] classified the simple algebra 1-morphisms in the semisimple quotient of quantum $\mathfrak{sl}_2$ up to Morita equivalence, under some natural assumptions, in terms of ADE Dynkin diagrams. From their results, via the quantum Elias Satake correspondence [Eli16], [Eli17], we can get all indecomposable algebra 1-morphisms, in our Soergel bimodule 2-category.

Note here that the semisimple quotient of quantum $\mathfrak{sl}_2$ is semisimple, and the quiver underlying the 2-representation of ADE Dynkin diagrams are trivial. However, the quiver underlying the corresponding simple transitive 2-representation in the Soergel bimodule case is the so-called doubled quiver of the corresponding type, which has two oppositely oriented edges between each pair of adjacent vertices. Its quiver algebra, the zig-zag algebra, was for example studied by Huerfano–Khovanov [HK01]. It has very nice properties and shows up in various mathematical contexts nowadays.

2D.3. *The trihedral story.* Elias also defined a quantum Satake correspondence for $\mathfrak{sl}_3$ [Eli17]. In this paper, we study certain subquotients of Soergel bimodules, and their 2-representation theory. Our construction uses the quantum Satake correspondence, and to the best of our knowledge, these subquotients are new and have not been studied before.

In fact, even their decategorifications, seem to be new. We call these trihedral Hecke algebras. These algebras have their own Kazhdan–Lusztig combinatorics and interesting $\mathbb{Z}_{\geq 0}$-representations. We see the trihedral Hecke algebras as rank three analogues of (the small quotients of the) dihedral Hecke algebras. There are many similarities, but also some differences. For example, as far as we can tell, the trihedral Hecke algebras are not deformations of any group algebra. But they are semisimple algebras and the classification of their irreducible representations runs in parallel to the analogous classification for dihedral Hecke algebras, and their $\mathbb{Z}_{\geq 0}$-representation theory has also a very similar behavior. Similarly, we define their categorifications called trihedral Soergel bimodules and we see them as rank three version of dihedral Soergel bimodules.

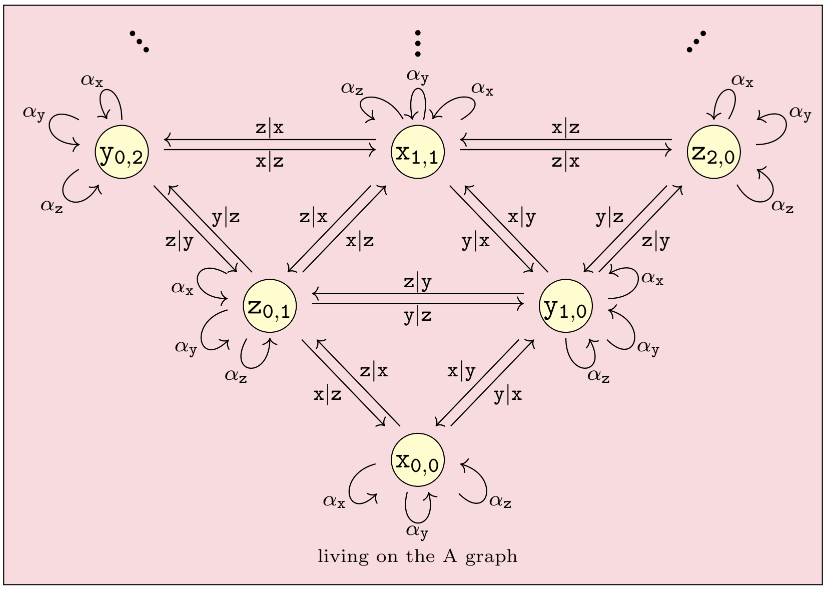
Coming back to representation theory, people have studied the $\mathbb{Z}_{\geq 0}$-representations of the Grothendieck group of the semisimple quotient of quantum $\mathfrak{sl}_3$, as they arise in conformal field theory and the study of fusion categories/modular tensor categories, see e.g. [Gan94], [EP10], [Sch17] and related works. This time, four families of graphs play an important role and, by analogy with the $\mathfrak{sl}_2$ case, their types are called A, conjugate A, D and E, although they are not Dynkin diagrams, e.g. the type A graphs can be seen as a cut-off of the positive Weyl chamber of $\mathfrak{sl}_3$, just as the usual type A Dynkin diagrams can be seen as cut-offs for $\mathfrak{sl}_2$. Finally, the type D graphs for $\mathfrak{sl}_3$ come from a $\mathbb{Z}/3\mathbb{Z}$-symmetry of these cut-offs



just as the type D Dynkin diagrams come from a $\mathbb{Z}/2\mathbb{Z}$-symmetry

Simple algebra 1-morphisms in the semisimple quotient of quantum $\mathfrak{sl}_2$ and the corresponding simple transitive 2-representations have also been studied e.g. in [Sch17] and are closely related to these ADE type graphs. Via the quantum Satake correspondence, we therefore get indecomposable algebra 1-morphisms in the trihedral 2-category and the corresponding simple transitive 2-representations of the trihedral Soergel bimodules.

Computing the quiver algebras proved to be much harder this time, but here is a picture:



living on the A graph

These algebras are the trihedral analogs of the zig-zag algebras of type A, e.g. the endomorphisms algebras of their vertices are the cohomology rings of the full flag variety of flags in $\mathbb{C}^3$, instead of the flags in $\mathbb{C}^2$ as in the dihedral case. For this reason, we call them trihedral zig-zag algebras.

3. Ongoing projects

3A. **Webs calculi in representation theory – potential directions.** As sketched in Section 2A, Howe duality in its various flavors has played an important role in the more recent developments of diagrammatic representations theory. Similarly, it also played and plays an important role on the level of categorification.

Hence, keeping all potential directions in mind, I would like to continue my research in this area, and here are two explicit questions which remain open.

The quantization of Howe duality in [ST17] features a so-called coideal subalgebra together with a quantum group, and is asymmetric. This is a hint that there is also an alternative way to do it where the coideal subalgebra and the quantum group swap roles. This should lead to new way to study link polynomials outside of type A. Moreover, a categorification of this story is completely mysterious at present, and its a worthwhile goal to be studied.

3B. **Link homologies – potential directions.** In his pioneering work, Khovanov introduced the so-called arc algebra [Kho02]. One of his main purposes was to extend his celebrated categorification of the Jones polynomial [Kho00] from links to tangles. His idea was to interpret the link homology as certain bimodules of the arc algebra.

One of Khovanov's main ideas (as developed further by Bar-Natan [BN05]) was that the arc algebras are obtained via 1+1-dimensional topological quantum fields theories (TQFTs) as originate in work of physicists and axiomatized by Atiyah–Segal in the end of the 1980ties. Note hereby that the formulation by Khovanov–Bar-Natan can be seen as the $\mathfrak{sl}_2$ case of a broader story, where the generalization of the usage of 1+1-dimensional TQFTs is replaced by what is called a singular 1+1-dimensional TQFTs.

Left aside its knot theoretical origin, the arc algebra has interesting representation theoretical, algebraic geometrical and combinatorial properties, see e.g. [BS11a], [BS11b]. For instance, there are relations to (cyclotomic) KL–R algebras, knot homologies, to the Alexander polynomial and knot Floer homology, to the representation theory of Brauer's centralizer algebras and to Lie superalgebras – just to name a few.

In [ETW16], we constructed the type D version of Khovanov's arc algebra using a singular TQFT approach à la Khovanov–Bar-Natan, revealing its potential application in low-dimensional topology. Note hereby that (again) only Lie type A is well-understood and any steps outsides of type A are novel and interesting. There is one particular striking question which remains open:

While the representation theoretical origin, connections and properties of the type D version of Khovanov's arc algebra are fairly well-understood, its connection to low-dimensional topology remains mysterious.

This is a question which I am currently investigating together with Stroppel–Wilbert, and we have some ideas about potential link homologies for links in certain orbifolds, generalizing classical links.

3C. **Relative cellular algebras – potential directions.** In [ET17] we generalize the notion of cellular algebras to what we call relative cellular algebras and develop a theory of such algebras. We also give several examples which are relative cellular, but not cellular. In fact, most of our example are related to characteristic p representation theory.

So the question whether this is just a coincidence or whether there is some deeper connection between the theories comes up naturally, but remains open.

3D. **Categorical representations – potential directions.** The subject of finitary 2-representation theory is rapidity growing with several new results published every year, but it is still widely open at the moment, and a lot of natural and interesting questions remain open. I want continue my research on 2-representations of Soergel bimodules and related 2-categories.

More specifically, the following are questions on which I work jointly with (a subset of) Mackaay, Mazorchuk and Miemietz.

The construction in [MT16] is very similarly to and motivated by [AT17]. Since the ideas from [AT17] heavily draw from a construction of Khovanov–Seidel [KS02] (which is related to categorical actions of braid groups), one might expect connections from [MT16] to braid groups. Another path we are exploring is that we hope that the construction presented in [MT16] generalizes to other e.g. (infinite) Coxeter types.

Moroever, in [MMMT18] we defined a new algebras and its categorification and even some basic questions of this are still open.

Finally, in [MMMT16] we made some connections of finitary 2-representation theory to the theory of tensor categories, but in an example. We are trying to develop the abstract 2-representation theory further and hope to make some connections to this broad field on a more general level.

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D.T.: Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, Campus Irchel, Office Y27J32, CH-8057 Zürich, Switzerland, www.dtubbenhauer.com

Email address: daniel.tubbenhauer@math.uzh.ch