EXERCISES 7: LECTURE REPRESENTATION THEORY

Exercise 1. Let χ be a character of G. Show that $\chi(g)$ is an algebraic integer for all $g \in G$.

Here is the character table of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ for comparison:

```
-----------------------------------
Class |  1  2   3    4    5     6
Size  |  1  1   1    1    1     1
Order |  1  2   3    3    6     6
-----------------------------------
p  =  2  1  1   4    3    3     4
p  =  3  1  2   1    1    2     2
-----------------------------------
X.1  +   1  1   1    1    1     1
X.2  +   1 -1   1    1   -1    -1
X.3  0   1  1   J -1-J -1-J     J
X.4  0   1 -1 -1-J    J   -J   1+J
X.5  0   1 -1   J -1-J  1+J    -J
X.6  0   1  1 -1-J    J    J  -1-J
```

Here J is a primitive third root of unity.

Exercise 2. Let p be a prime and let G be a group of order p^2. Show that G is abelian.

Hint: Use that the dimension of simple representations divide the order of G.

It gets a bit more complicated from p^3 onward:

Small groups of prime power order p^n are given as follows:

- Order p: The only group is cyclic.
- Order p^2: There are just two groups, both abelian.
- Order p^3: There are three abelian groups, and two non-abelian groups. One of the non-abelian groups is the semidirect product of a normal cyclic subgroup of order p^2 by a cyclic group of order p. The other is the quaternion group for $p = 2$ and a group of exponent p for $p > 2$.
- Order p^4: The classification is complicated, and gets much harder as the exponent of p increases.

Exercise 3. Let p, q be primes with $p < q$ and $q \not\equiv 1 \bmod p$. Show that any group G of order pq is abelian.

Hint: Use that the dimension of simple representations divide the order of G.

Exercise 4. Let G be a non-abelian group of order 39. Determine the dimensions of the simple representations of G and how many simple representations G has of each dimensions (up to equivalence).

Hint: Use that the dimension of simple representations divide the order of G, and the number of 1-dimensional simple representations also divide the order of G.

Addendum: There is only one such group and Magma can find it:

```
F<a, b> := FreeGroup(2);
G<x, y>, phi := quo< F | a^13 = 1, b^3 = 1, b*a = a^3*b >;
Order(G)
```

▶ The exercises are optimal and not mandatory. Still, they are highly recommend.

▶ There will be 12 exercise sheets, all of which have four exercises.

▶ The sheets can be found on the homepage www.dtubbenhauer.com/lecture-rt-2022.html.

▶ Slogan: "Everything that could be finite is finite, unless stated otherwise.". For example, groups are finite and representations are on finite dimensional vector spaces.

▶ There might be typos on the exercise sheets, my bad, so be prepared.