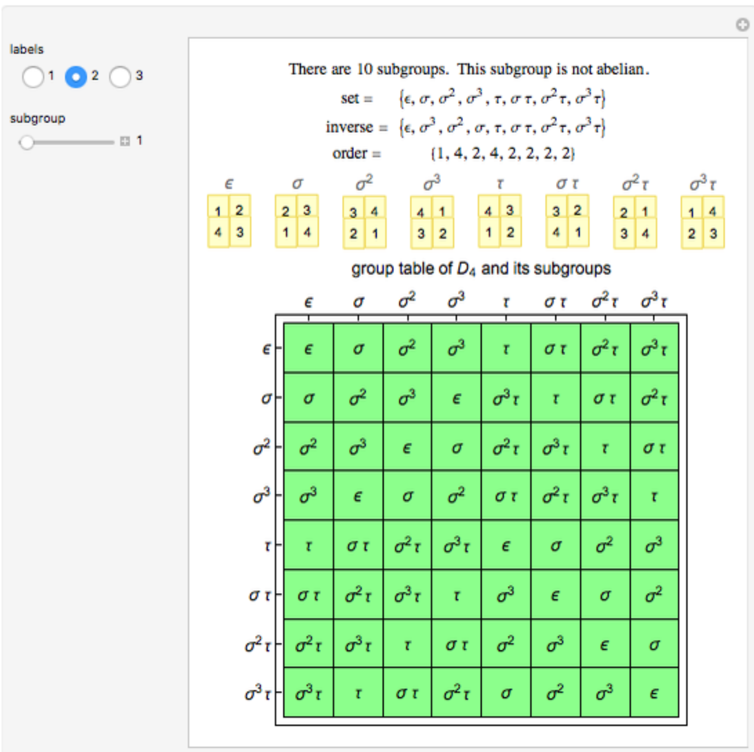
EXERCISES 5: LECTURE REPRESENTATION THEORY

Exercise 1. Here is the character table of S_3:

```
-----------------
Class |   1  2  3
Size  |   1  3  2
Order |   1  2  3
-----------------
p  =  2   1  1  3
p  =  3   1  2  1
-----------------
X.1   +   1  1  1
X.2   +   1 -1  1
X.3   +   2  0 -1
```

Using Schur's orthogonality relations, show that the existence and properties of the first tow characters imply existence and properties of the bottom one.

Exercise 2. Recall the following exercise from the third sheet: Let D_4 be the dihedral group with eight elements. Abstractly D_4 is generated by σ and τ and the multiplication table



	ϵ	σ	σ^2	σ^3	τ	$\sigma\tau$	$\sigma^2\tau$	$\sigma^3\tau$
ϵ	ϵ	σ	σ^2	σ^3	τ	$\sigma\tau$	$\sigma^2\tau$	$\sigma^3\tau$
σ	σ	σ^2	σ^3	ϵ	$\sigma^3\tau$	τ	$\sigma\tau$	$\sigma^2\tau$
σ^2	σ^2	σ^3	ϵ	σ	$\sigma^2\tau$	$\sigma^3\tau$	τ	$\sigma\tau$
σ^3	σ^3	ϵ	σ	σ^2	$\sigma\tau$	$\sigma^2\tau$	$\sigma^3\tau$	τ
τ	τ	$\sigma\tau$	$\sigma^2\tau$	$\sigma^3\tau$	ϵ	σ	σ^2	σ^3
$\sigma\tau$	$\sigma\tau$	$\sigma^2\tau$	$\sigma^3\tau$	τ	σ^3	ϵ	σ	σ^2
$\sigma^2\tau$	$\sigma^2\tau$	$\sigma^3\tau$	τ	$\sigma\tau$	σ^2	σ^3	ϵ	σ
$\sigma^3\tau$	$\sigma^3\tau$	τ	$\sigma\tau$	$\sigma^2\tau$	σ	σ^2	σ^3	ϵ

Show that D_4 has precisely four nonequivalent actions on $\mathbb{C}$ given by $\sigma \mapsto \pm 1$, $\tau \mapsto \pm 1$, all of which give simple representations.

Here is the character table of D_4:

```
----------------------
Class |   1  2  3  4  5
Size  |   1  1  2  2  2
Order |   1  2  2  2  4
----------------------
p  =  2   1  1  1  1  2
----------------------
X.1   +   1  1  1  1  1
X.2   +   1  1 -1  1 -1
X.3   +   1  1  1 -1 -1
X.4   +   1  1 -1 -1  1
X.5   +   2 -2  0  0  0
```

Show that this is indeed the character table of D_4 using the above exercise and Schur's orthogonality relations.

Exercise 3. Find the character table for the quaternion group Q_8. This group has eight elements and the following multiplication table:

×	**e**	$\bar{\mathbf{e}}$	**i**	$\bar{\mathbf{i}}$	**j**	$\bar{\mathbf{j}}$	**k**	$\bar{\mathbf{k}}$
e	e	$\bar{e}$	i	$\bar{i}$	j	$\bar{j}$	k	$\bar{k}$
$\bar{\mathbf{e}}$	$\bar{e}$	e	$\bar{i}$	i	$\bar{j}$	j	$\bar{k}$	k
i	i	$\bar{i}$	$\bar{e}$	e	k	$\bar{k}$	$\bar{j}$	j
$\bar{\mathbf{i}}$	$\bar{i}$	i	e	$\bar{e}$	$\bar{k}$	k	j	$\bar{j}$
j	j	$\bar{j}$	$\bar{k}$	k	$\bar{e}$	e	i	$\bar{i}$
$\bar{\mathbf{j}}$	$\bar{j}$	j	k	$\bar{k}$	e	$\bar{e}$	$\bar{i}$	i
k	k	$\bar{k}$	j	$\bar{j}$	$\bar{i}$	i	$\bar{e}$	e
$\bar{\mathbf{k}}$	$\bar{k}$	k	$\bar{j}$	j	i	$\bar{i}$	e	$\bar{e}$

Exercise 4. Show that $D_4 \not\cong Q_8$ as groups, but $\mathbb{C}[D_4] \cong \mathbb{C}[Q_8]$.
Hint: Use the character tables of D_4 and Q_8.

- The exercises are optimal and not mandatory. Still, they are highly recommend.
- There will be 12 exercise sheets, all of which have four exercises.
- The sheets can be found on the homepage www.dtubbenhauer.com/lecture-rt-2022.html.
- Slogan: "Everything that could be finite is finite, unless stated otherwise.". For example, groups are finite and representations are on finite dimensional vector spaces.
- There might be typos on the exercise sheets, my bad, so be prepared.