MAT562.1: SEMINAR ON 2D TQFTS – PLAN

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Disclaimer

Please do not hesitate to contact me in case of questions. (The contact data can be found below.) Never forget to give many examples during your talk!

What?

The philosophy is: Topology is interesting, but hard. So we want to translate topology into (way easier) algebra. Preferable into linear algebra. This is the main idea behind functors, which should be thought of as providing a way to turn questions in topology into questions in (linear) algebra. The point of this seminar is to understand how this works in the case of a 2D TQFT, where a beautiful classification in terms of algebra is possible.

The seminar follows the book [Kock04].

Who?

Master students and upwards interested in a mixture of topology, algebra and category theory.

Where and when?

- ▶ Time and date.
  - Every Monday from 13:15–15:00.
  - Room Y27H28, University Zurich, Institute of Mathematics.
  - First meeting: 24.Sep.2018.
- ▶ Preliminary meeting: 10.Sep.2018, 13:15–15:00, room Y27H28.
- ▶ Website http://www.dtubbenhauer.com/seminar-frob-2018.html

Schedule and some details.

- ▷ First talk "Prologue: Vocabulary from category theory".
  - **Speaker.** Mariya
  - **Date.** 24.Sep.2018, 13:15–15:00.
  - **Topic.** Generators and relations for monoidal categories.
  - **Plan.** The first step is to recall some notions about categories, functors, universal objects, monoidal categories and monoidal functors. The next step is explain what a monoidal category given by generators and relations is.
  - **Main goals.** After recalling some basic notions about categories [Kock04, Appendix A.1], functors [Kock04, Appendix A.2] and universal objects [Kock04,

Appendix A.3], briefly recall the notions of a strict monoidal category and their functors [EGNO15, Section 2.8] or [Kock04, Section 3.2], of a symmetric monoidal category, and mention that we use Mac Lane's coherence theorem [EGNO15, Theorem 2.8.5] to always assume that our monoidal categories are strict. Then explain carefully the notion of a monoidal category given by generators and relations [Kas95, Section XII.1].

- **Note.** Be brief in the first part (the audience has seen these notions before), otherwise you will run into time issues. Give many examples. Beware that [EGNO15, Theorem 2.8.5] is called *strictness theorem* therein and is actually equivalent to the coherence theorem.
- **Literature.** [Kock04, Appendices A.1 to A.3 & Section 3.2], [EGNO15, Section 2.8], [Kas95, Section XII.1].

▷ Second talk "Cobordisms topologically I"

- **Speaker.** Gilles
- **Date.** 01.Oct.2018, 13:15–15:00.
- **Topic.** The notion of cobordisms.
- **Plan.** After recalling some geometric preliminaries and basic facts from Morse theory, introduce the notion of a cobordism between manifolds, with the main focus on the case of dimension one or two. Finally, explain when two cobordisms are equivalent and what decomposition of cobordisms means.
- **Main goals.** Recall the basic geometric preliminaries [Kock04, Section 1.1], in particular, the main theorem of Morse theory [Kock04, Theorem 1.1.19]. Explain the definition of unoriented cobordisms and prove [Kock04, Lemmas 1.2.9 and 1.2.10]. Introduce oriented cobordisms and explain how they generalize the interval [Kock04, §1.2.13 and §1.2.14]. Introduce the categorical identities (aka cylinders) and tubes [Kock04, §1.2.16 and §1.2.18] and end with the notion of equivalence between cobordisms [Kock04, §1.2.17], and briefly explain the decomposition of cobordisms [Kock04, §1.2.20 to §1.2.22]. If time suffices, give [Kock04, Exercise 4 in Section 1.2] as an example.
- **Note.** Be brief in the first part, otherwise you will run into time issues. Skip [Kock04, §1.2.19], and do as many low-dimensional examples as possible. Moreover, the talk is supposed to be example-based with references to the literature for details.
- **Literature.** [Kock04, Section 1], starting at the beginning and ending with [Kock04, §1.2.22].

▷ Third talk "Cobordisms topologically II"

- **Speaker.** Nino
- **Date.** 08.Oct.2018, 13:15–15:00.
- **Topic.** The category of cobordisms.
- **Plan.** After introducing what a TQFT is topologically, prepare carefully the stage to give a categorical definition of a TQFT.

- **Main goals.** Explain why and how [Kock04, Section 1.2.23] and [Kock04, Section 1.3.32] are equivalent. To do so, explain the notion of cobordism pairings and state [Kock04, Corollary 1.2.28]. Introduce the category of cobordisms [Kock04, Section 1.3] by sketching how the gluing gives rise to composition [Kock04, Sections 1.3.1 to 1.3.14], and explain why this give an associative structure [Kock04, 1.3.15], what the units are [Kock04, Section 1.3.16] and various other properties. Explain why the cobordism category is symmetric monoidal [Kock04, §1.3.14 to §1.3.29], and finally end with [Kock04, §1.3.32].
- **Note.** You might run into time issues, so be brief at some points if necessary. For example, that gluing works is intuitively clear, but the rigorous proof is kind of a pain. Exercise 3 at the end of [Kock04, Section 1.3] is also recommended.
- **Literature.** [Kock04, Section 1], starting with [Kock04, §1.2.23] and ending with [Kock04, §1.3.32].

▷ 4$^{\text{th}}$ talk "Two-dimensional Cobordisms algebraically I"

- **Speaker.** Nino
- **Date.** 15.Oct.2018, 13:15–15:00.
- **Topic.** Generators for two-dimensional cobordisms.
- **Plan.** After recalling a classical example from combinatorics, and after recalling the idea of generators–relations, explain the main example, i.e. generators for two-dimensional cobordisms.
- **Main goals.** Start with the example of the symmetric group [Kock04, §1.4.1 to §1.4.4]. Please also present Exercise 1 at the end of [Kock04, Section 1.4] if time suffices. Then recall some basics about generators and relations, in particular why the skeleton is needed [Kock04, §1.4.5]. Then explain what the skeleton is in our case [Kock04, Proposition 1.4.9 and §1.4.10]. Finally, state and prove [Kock04, Proposition 1.4.13], which will take the remaining time of your talk.
- **Note.** There is, of course, some overlap with the first talk. Be brief in those parts and focus on the proof of [Kock04, Proposition 1.4.13].
- **Literature.** [Kock04, Section 1], starting at the beginning of [Kock04, Section 1.4] and ending with [Kock04, §1.4.23].

▷ 5$^{\text{th}}$ talk "Two-dimensional Cobordisms algebraically II"

- **Speaker.** Mariya
- **Date.** 22.Oct.2018, 13:15–15:00.
- **Topic.** Relations for two-dimensional cobordisms.
- **Plan.** State the generating relations of the category of two-dimensional cobordisms, prove that they hold in the topological setting and that they indeed generate all other relations.
- **Main goal.** Start by explaining all the relations without the twist carefully [Kock04, §1.4.24 to 1.4.28] and explain why they hold [Kock04, §1.4.30 to 1.4.34] using disks. Similarly for the relations involving the twist. Finally, carefully explain why these relations are sufficient [Kock04, §1.4.36 to 1.4.40], and prove

that they are in fact redundant [Kock04, Proposition 1.4.41]. Do not forget to do Exercise 5 at the end of [Kock04, Section 1.4].

- **Note.** You can also do some of the exercises, e.g. Exercise 5 is very nice and gives the audience a feeling how to work with generators an relations.
- **Literature.** [Kock04, Section 1], starting at with [Kock04, Section 1.4.24] and going all the way to the end.

▷ 6th talk "Frobenius algebras"

- **Speaker.** Fabio
- **Date.** 29.Oct.2018, 13:15–15:00.
- **Topic.** The definition of a Frobenius algebra.
- **Plan.** After some basic algebraic recollections, define the notion of a Frobenius algebra.
- **Main goals.** First, remind the audience of several notions related to vector spaces and their pairings [Kock04, §2.1.1 to 2.1.17]. Similarly in case of algebras and modules [Kock04, §2.1.18 to 2.1.36]. Finally, introduce the notion of a Frobenius algebra [Kock04, §2.2.1 and §2.2.9] and its basic properties. Then immediately go to the examples [Kock04, §2.1.12 to 2.1.23] and spent the remaining time carefully explaining as many of these as possible. In particular, the examples about the cohomology rings are very nice [Kock04, §2.1.23]. Exercises 6, 8 and 9 after [Kock04, §2.1.23] are also very useful examples, and should be included in the list of examples.
- **Note.** Be brief with the recollections about vector spaces and algebras. Focus more on Frobenius algebras (do not spent too much time showing that the various definitions given in [Kock04] are equivalent) and their various examples.
- **Literature.** [Kock04, Section 2], starting at the beginning and ending with [Kock04, §2.2.23].

▷ 7th talk "Frobenius algebras diagrammatically I"

- **Speaker.** Genta
- **Date.** 05.Nov.2018, 13:15–15:00.
- **Topic.** Diagrams for Frobenius algebras I.
- **Plan.** Start with some basic facts about Frobenius algebras with respect to the comultiplication and then carefully explain the diagrammatic calculus, and the category of Frobenius algebras.
- **Main goal.** After explaining some preliminaries about the comultiplication in Frobenius algebras [Kock04, §2.3.1 to §2.3.4], introduce the graphical calculus for Frobenius algebras [Kock04, §2.3.5 to §2.3.7]. Then explain all the gory details about the graphical calculus, the axioms and properties of Frobenius algebras [Kock04, §2.3.8 to §2.3.38]. Finish by explaining the graphical calculus in the view of the category of Frobenius algebras [Kock04, Section 2.4]. Exercise 1 after [Kock04, §2.4.12] is also very nice, and should be presented.

- **Note.** Joint with the 8$^{\text{th}}$ talk. So please be rather pedantic and do not worry about time issues to much.
- **Literature.** [Kock04, Section 2], starting at the beginning of [Kock04, Section 2.3] and go as far as you can. (The next talk will start where you ended.)

▷ 8$^{\text{th}}$ talk "Frobenius algebras III"

- **Speaker.** Genta
- **Date.** 12.Nov.2018, 13:15–15:00.
- **Topic.** Diagrams for Frobenius algebras II.
- **Plan.** Same as for the 7$^{\text{th}}$ talk.
- **Main goal.** Same as for the 7$^{\text{th}}$ talk.
- **Note.** Same as for the 7$^{\text{th}}$ talk.
- **Literature.** [Kock04, Section 2], starting where the previous talk stopped, and going all the way to the end.

Break since I am away.

▷ 9$^{\text{th}}$ talk "From Frobenius algebras to TQFTs"

- **Speaker.** Gilles
- **Date.** 10.Dec.2018, 13:15–15:00.
- **Topic.** The main result.
- **Plan.** Carefully explain the main result in the book [Kock04, Theorem 3.3.2].
- **Main goals.** Explain everything in [Kock04, Section 3.3] carefully. In the second part discuss the Exercises 2 and 3 after [Kock04, §3.3.5] carefully. Exercise 7 in the same section is also highly recommended and should be presented.
- **Note.** Be careful with the symmetry requirement [Kock04, §3.3.3] and carefully explain it.
- **Literature.** [Kock04, Section 3.3].

▷ 10$^{\text{th}}$ talk "Epilogue: Finite ordinals and cardinals diagrammatically"

- **Speaker.** Mariya
- **Date.** 17.Dec.2018, 13:15–15:00.
- **Topic.** The categories of finite ordinals and cardinals via generators–relations.
- **Plan.** Explain another instance of a diagrammatic generator–relation presentation, namely the simplex categories.
- **Main goals.** Introduce the category of finite ordinals [Kock04, §3.4.1 to §3.4.5]. Then introduce the graphical calculus for it [Kock04, §3.4.6 to §3.4.9], and its generator–relation presentation [Kock04, §3.4.10 to §3.4.14]. Similarly for the

case of finite cardinals, which is the all the rest of the corresponding section till [Kock04, §3.4.26]. Do not miss Exercise 1 after [Kock04, §3.4.25].

- **Note.** Carefully explain the case of ordinals, the case of cardinals is quite similar and you can be brief. Exercise 3 or 6 after [Kock04, §3.4.25] are also interesting.
- **Literature.** [Kock04, §3.4.1] to [Kock04, §3.4.26].

References

[EGNO15] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik. *Tensor categories*. Mathematical Surveys and Monographs, **205**. American Mathematical Society, Providence, RI, 2015.

[Kas95] C. Kassel. *Quantum groups*. Graduate Texts in Mathematics, **155**. Springer-Verlag, New York, 1995.

[Kock04] J. Kock. *Frobenius algebras and 2D topological quantum field theories*. London Mathematical Society Student Texts, **59**. Cambridge University Press, Cambridge, 2004.

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