Potential master or Ph.D. project of I WANT YOU in 2023

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Key information

Candidate. I WANT YOU.

Email. YOUR EMAIL.

Research areas. Algebra, representation theory, and topology, more specifically Soergel bimodules and their 2-representations.

Title. *"The two-color Soergel calculus in action".*

First read. [Lib17].

Kazhdan and Lusztig introduced a particular basis of Hecke algebras, nowadays known as the Kazhdan–Lusztig (KL) basis. This basis was conjectured to have certain positivity properties. One way to prove this positivity would be to construct an additive monoidal category whose Grothendieck ring is isomorphic to H, and whose indecomposable objects descend to the KL basis.

In the early 1990s, Soergel constructed an algebraic categorification of the Hecke algebra using certain bimodules, now called Soergel bimodules Soergel conjectured that the indecomposable Soergel bimodules should descend to the KL basis, when defined over a field of characteristic zero, though in the absence of geometric tools there is no a priori reason this should be true. This conjecture was – after more than 20 years – proven by the Elias and Williamson using the diagrammatic methods they developed in [EW16].

Soergel bimodules were (and are) in the heart of an explosion of new discoveries in representation theory, algebraic combinatorics, algebraic geometry and knot theory. And, in particular, their 2-representations ("categorified representations") turned out be of fundamental interest in modern parts of mathematics.

The main questions concerning Soergel bimodules nowadays are to understand their intricate structure and their 2-representations using methods from graph theory, combinatorics and algebra alike.

Minimal goal. Summarize the paper of Elias in your own words.
Average goal. Add a bit about their 2-representations, *i.e.* [MT19].
Optimal goal (for Ph.D.). Address some of the open question mentioned below.
Key. Be concise with the basics. Be precise with the categorical actions. Find a self-containing way to summarize various results scattered over the literature.

The thesis in details – minimal goal

The minimal master project should be structured as follows.

- Write an introduction about why Soergel bimodules are interesting, see *e.g.* [Lib17].
- Summarize basics about dihedral groups and their Kazhdan–Lusztig combinatorics.
- Explain the main results in [Eli16].

The thesis in details – average goal

As above, but add:

- Summarize basics about 2-representations as *e.g.*in [Maz17].
- Summarize some basics about zigzag algebras attached to arbitrary simply-laced type.
- Explain the main results in [MT19].

The thesis in details – optimal goal

Here are some open questions which (if time suffices) deserve further study.

- Classification problems in higher ranks, see *e.g.* [MMMT20, CP 5.34]?
- Is there any relation to knot polynomials or 3-manifold invariants via the connection of [MT19] to [KO02]?
- Is there any hope to generalize the underlying quantum Satake [Eli17] to higher ranks?
- Is there any precise relationship to the physics literature where the same classification problems show up, *e.g.*in [Zub98]?

References

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[EW16] B. Elias and G. Williamson. Soergel calculus. *Represent. Theory*, 20:295–374, 2016. URL: https://arxiv.org/abs/1309.0865, doi:10.1090/ert/481.

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