MODULAR REPRESENTATION THEORY, ALCOVE COMBINATORICS AND A CATEGORY OF SHEAVES ZÜRICH 2018

MARTINA LANINI

Abstract. These are notes for a minicourse held during the Workshop "Interactions of low-dimensional topology and "higher" representation theory" (Zürich, September 17–21, 2018), organised by Daniel Tubbenhauer.

In this minicourse we will explain how the alcove combinatorics is related to the representation theory of algebraic groups over an algebraically closed field if the characteristc of the field is $\gg 0$, by stating the periodic version of Lusztig conjecture (proven by Andersen, Jantzen and Soergel under the assumption of big enough characteristic). In the third lecture we will discuss recent joint work with Peter Fiebig, in which a certain category of *sheaves on the alcoves* is introduced in order to get some understanding of the relevant representation category whenever the base field has characteristic bigger than the Coxeter number.

1. Representations of algebraic groups

Main reference for this first section is [J].

An algebraic group G over a field $\mathbb{K}$ is an algebraic variety which has also a structure of a group (meaning that the multiplication map $m : G \times G \to G$, $(g, h) \mapsto gh$, and the inversion map $\iota : G \to G$, $g \mapsto g^{-1}$, are continuous maps). An algebraic group is said to be reductive if it does not have any non-trivial smooth connected nomal unipotent subgroup.

Let G be a connected reductive (linear) algebraic group over the algebraically closed field $\mathbb{K}$. For instance, $GL_n(\mathbb{K})$ $(n \geq 1)$, $Sp_{2n}(\mathbb{K})$, $SL_n(\mathbb{K})$ $(n \geq 2)$ are all connective reductive algebraic groups.

A (rational) rapresentation of G is a $\mathbb{K}$-vector space V together with a homomorphism of algebraic groups

$$\varphi : G \to GL(V).$$

We denote by $\mathbb{G}_m := GL_1 \simeq \mathbb{K}^{\times}$.

An algebraic group is said to be an algebraic torus if it is a direct product of some copies of $\mathbb{G}_m$. We will often omit the word algebraic and just spaek of a torus.

If G is an algebraic torus, then all its irreducible rational representations are one-dimensional (as for the case of finite abelian groups), since by definition, it consists of commuting operators.

Example 1.1. *If $G = \mathbb{G}_m$, and k is a 1-dimensional $\mathbb{K}$- vector space, then any morphism of algebraic groups is of the form*

$$G = \mathbb{G}_m \to GL(V), \quad g \mapsto (v \mapsto g^j v)$$

for some fixed $j \in \mathbb{Z}$. Therefore irreducible representations of a 1-dimensional algebraic torus are parametrized by integers.

Let T be a torus. Then $X(T) := \mathrm{Hom}(T, \mathbb{G}_m)$ is its character lattice (where we are taking homomorphisms in the category of algebraic groups).

Example 1.2. *By the previous example, we see that if T is one dimensional, then $X(T) \simeq \mathbb{Z}$.*

Let G be a reductive connected algebraic group, we fix once and for all a maximal (algebraic) torus T and we denote by X its character lattice $X(T)$.

Example 1.3. *If $G = SL_2(\mathbb{K})$, then we can take*

$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{K}^\times \right\} \simeq \mathbb{G}_m.$$

Let M be a T-module. Since T is abelian, it consists of commuting operators and therefore we can look at the weight space decomposition of M: $M = \bigoplus_{\lambda \in X} M_\lambda$, where, for $\lambda \in X$,

$$M_\lambda := \{ m \in M \mid t \cdot m = \lambda(t) m, \ \forall t \in T \}$$

and it is called the λ-weight space of M.

Denote now

$$\mathfrak{h} := \mathrm{Lie}(T), \ \ \mathfrak{g} := \mathrm{Lie}(G).$$

Then $\mathfrak{g}$ is a T-module under the adjoint action and the root system $R \subset X$ is the set of non-zero weights such that $\mathfrak{g}_\alpha \neq 0$.

Example 1.4. *Let $G = SL_2$, so that*

$$\mathfrak{sl}_2 := \mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, c, b \in \mathbb{K} \right\}.$$

Under the T-action, we have the following root space decomposition:

$$\mathfrak{sl}_2 = \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha},$$

with

$$\mathfrak{g}_0 = \mathbb{K} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathfrak{g}_{-\alpha} = \mathbb{K} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathfrak{g}_\alpha = \mathbb{K} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

where

$$\alpha : \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mapsto a^2.$$

We conclude that in this case $R = \{\pm\alpha\}$.

We denote by $Y := \mathrm{Hom}(\mathbb{G}_m, T)$ the coweight lattice. There is a natural pairing

$$\langle \ , \ \rangle : X \times Y \to \mathbb{Z}, \quad (\varphi, \psi) \mapsto \varphi \circ \psi$$

given by the composition (where we are identifying as before $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)$ with $\mathbb{Z}$). For any $\alpha \in R$, there is a unique element $\alpha^\vee \in Y$ such that $\langle \alpha, \alpha^\vee \rangle = 2$.

Fix now once and for all a Borel subgroup B of G, that is a maximal closed connected and solvable algebraic subgroup of G. We denote by $\mathfrak{b} := \mathrm{Lie}(B)$.

The set of positive roots R^+ is the set of non-zero weights appearing in the weight decomposition of $\mathfrak{b}$, that is $\mathfrak{b} = \mathfrak{g}_0 \oplus_{\gamma \in R^+} \mathfrak{g}_\gamma$. It turns out that $R = R^+ \sqcup R^-$, where $R^- := -R^+$. Moreover, there exists a unique subset $\Delta \subseteq R^+$ such that all roots in R^+ are a $\mathbb{Z}_{\geq 0}$-combination of elements in Δ. Δ is called the set of simple roots.

Example 1.5. *If $G = SL_2$, we can take*

$$B := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{K}^\times \ b \in \mathbb{K} \right\}$$

so that

$$\mathfrak{b} = \left\{ \begin{pmatrix} c & d \\ 0 & -c \end{pmatrix} \mid c, d \in \mathbb{K} \right\}$$

and it is immediately seen that $R^+(= \Delta) = \{\alpha\}$.

The choice of a Borel, and hence of the set of positive roots, allows us to define a partial order on X:

$$\mu \leq \lambda \quad \Leftrightarrow \quad \lambda - \mu \in \mathbb{Z}_{\geq 0} R^+.$$

The partial order just defined is called the dominance order.

1.1. **Irreducible representations.** We want to construct irreducible representations for G.

We start by considering the 1-dimensional T representation $\mathbb{K}_\lambda$: this is isomorphic to $\mathbb{K}$ as a vector space, and the action of T is given by $t \cdot r = \lambda(t) r$ for any $t \in T$ and $r \in \mathbb{K}_\lambda$.

We have that $B = U \rtimes T$, where U is a unipotent subgroup.

Example 1.6. *If $G = SL_2$, T the subgroup of diagonal matrices and B the subgroup of upper triangular matrice as before, then*

$$U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}.$$

It is immediate to verify that U is normal in B and that $B = U \rtimes T$.

We can inflate $\mathbb{K}_\lambda$ to B via letting U act trivially:

$$u \cdot r = r \qquad \forall u \in U, \ r \in \mathbb{K}_\lambda.$$

The Weyl module $W(\lambda)$ is hence defined as

$$W(\lambda) : = \mathrm{Ind}_B^G(\mathbb{K}_\lambda) \quad = \{f : G \to \mathbb{K}_\lambda \text{ regular } \mid f(gb) = \lambda^{-1} f(g), \forall b \in B, g \in G\}.$$

Example 1.7. *If $G = SL_2$, we can consider the action of SL_2 on $\mathbb{K}[x,y]$, given by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot p(x,y) = p(ax + cy, bx + dy).$$

This comes from the right action of SL_2 on $\mathbb{K}^2$ if we see x, resp. y as the basis coordinate $(1,0)$, resp. $(0,1)$. Let $r \in \mathbb{Z}_{\geq 0}$, then $\mathbb{K}_r \simeq \mathbb{K}x^r$, as

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot x^r = (ax)^r = a^r x^r, \quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot x^r = (1 \cdot x)^r = x^r.$$

Moreover, $Ind_B^G \mathbb{K}_r = \mathbb{K}[x,y]_r$, where $\mathbb{K}[x,y]_r$ is the vector space of homogeneous polynomials of degree r.

Recall that the socle $\mathrm{soc}(M)$ of a module M is the submodule generated by its simple submodules.

Theorem 1.1. (1) *If $W(\lambda) \neq (0)$, then $soc(W(\lambda))$ is simple.*
(2) *$W(\lambda) \neq (0)$ if and only if $\lambda \in X_+$, where X_+ is the set of dominant weights, that is*

$$X_+ = \{\lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \ \forall \alpha \in R^+\}.$$

We denote by $L(\lambda) := \mathrm{soc}(W(\lambda))$ for any $\lambda \in X^+$.

Example 1.8. *Let $G = SL_2$. If $char(\mathbb{K}) = 0$, then it is easy to see that $W(r) = \mathbb{K}[x,y]_r$ is simple for every $r \in \mathbb{Z}_{\geq 0}$. But as soon as the characteristic is positive, the Weyl module does not have to be irreducible. Let $char(\mathbb{K}) = p > 0$, then $\mathbb{K}\{x^p, y^p\}$ is the unique simple (proper) submodule of $\mathbb{K}[x,y]_p$, so that it holds*

$$soc(\mathbb{K}[x,y]_p) = \mathbb{K}\{x^p, y^p\}.$$

Theorem 1.2. (1) *$dim L(\lambda)_\lambda = 1$.*
(2) *$\{L(\lambda) \mid \lambda \in X_+\}$ is a full set of representatives of isomorphism classes of simple (rational) G-modules.*

1.2. **Character theory.** Any G-module has an induced structure of T modules and hence a decomposition in weight spaces. Let $M = \bigoplus_{\lambda \in X} M_\lambda$. Its character is defined as

$$\mathrm{ch}(M) := \sum_{\lambda \in X} \dim(M_\lambda) e^\lambda \in \mathbb{Z}[X].$$

Example 1.9. *Let $G = SL_2$ and suppose that $char(\mathbb{K}) = 2$. Then, under the identification $X \leftrightarrow \mathbb{Z}$, we have*

$$char(W(2)) = e^2 + e^0 + e^{-2}, char(L(2)) = e^2 + e^{-2}.$$

The character is an additive function, that is, if $M' \subset M$ is a submodule then $\mathrm{ch}(M) = \mathrm{ch}(M') + \mathrm{ch}(M/M')$.

Example 1.10. *Let $G = SL_2$ and $char(\mathbb{K}) = 2$. Then according with the additive property just stated, we have that $ch(W(2)/L(2)) = e^0$.*

From the way $L(\lambda)$ is constructed and the fact that the highest weight space has dimension 1 we know that

$$\mathrm{ch}(L(\lambda)) = e^\lambda + \sum_{\mu < \lambda} a_\mu L(\mu).$$

Therefore $\{L(\lambda) \mid \lambda \in X_+\} \subset \mathbb{Z}[X]$ is a linearly independent set. We deduce that the knowledge of all simple characters allows us to determine Jordan-Hölder multiplicities for a modules from its character and viceversa.

The central problem is hence to compute $\mathrm{ch}(L(\lambda))$ for any $\lambda \in X_+$.

If $\mathrm{char}(\mathbb{K}) = 0$, then all $W(\lambda)$ are irreducible and their characters are known to be given by the Weyl character formula.

Thus from now on, we assume $\mathrm{char}(\mathbb{K}) > 0$. In the full generality we have stated it, this problem is still open. We will discuss how certain polynomials govern the simple characters for big enough characteristic.

Before moving on, let us notice that the SL_3 case was completely solved by Braden in 1976; in the Seventies, Jantzen was able to determine all characters in the cases SL_4, Sp_4, G_2 (via his sum formula). All further cases are not yet completely solved (even for SL_5 there are still some cases missing!).

1.3. **Steinberg tensor product theorem.** From here on, to avoid technicalities, we assume moreover, that our reductive connected algebraic group is simple and simply connected. This is the case, for example, of SL_n. Let A_0 be an $\mathbb{F}_p$-algebra such that $\mathbb{K}[G] = A_0 \otimes_{\mathbb{F}_p} \mathbb{K}$. Then there is an algebra morphism

$$\mathbb{K}[G] = A_0 \otimes_{\mathbb{F}_p} k \to \mathbb{K}[G] = A_0 \otimes_{\mathbb{F}_p} k, \quad a \otimes x = a^p \otimes x$$

which induces a (surjective) morphism of algebraic groups

$$\mathrm{Fr}: \ G \to G.$$

Example 1.11. *If $G = SL_n(\mathbb{K})$, then $Fr(a_{ij}) = (a_{ij}^p)$.*

Let M be a G-module, then the pullback of the G structure along the Frobenius morphism induces a new G-module structure on it, that we denote by $M^{(1)}$. Explicitely, $M^{(1)} = M$ as a vector space, but $g \cdot_{(1)} m = \mathrm{Fr}(g) \cdot m$.

Lemma 1.1. *Let $\lambda \in X_+$, then $L(\lambda)^{(1)} \simeq L(p\lambda)$.*

Example 1.12. *For $G = SL_2$ and char$(\mathbb{K}) = p$ it is immediate that $L(1)^{(1)} = (\mathbb{K}\{x, y\})^{(1)} \simeq \mathbb{K}\{x^p, y^p\} = L(p)$.*

Let $j \in \mathbb{Z}_{>0}$, we denote by $M^{(j)}$ the module obtained by Frobenius twisting M j times

Let $X_1 := \{\lambda \in X \mid 0 \leq \langle \lambda, \alpha^\vee \rangle < p \ \forall \alpha \text{ simple }\}$ be the set of restricted weights.

Theorem 1.3 (Steinberg's Tensor product Theorem)**.** *Let $\lambda \in X_+$ and write $\lambda = \lambda_0 + p\lambda_1 + p^2\lambda_2 + \ldots + p^n\lambda_n$, where $\lambda_i \in X_1$ for any $i = 1, \ldots, n$. Then there is an isomorphism*

$$L(\lambda) \simeq L(\lambda_0) \otimes L(\lambda_1)^{(1)} \otimes \ldots \otimes L(\lambda_n)^{(n)}.$$

Exercise 1.1. *Prove the previous theorem in the SL_2 case.*

We deduce that it is enough to know $\mathrm{ch}(L(\lambda))$ for $\lambda \in X_+$ to determine all simple characters.

1.4. **Modules over restricted Lie algebras.** We want now to rephrase the problem in terms of modules over a quotient of the enveloping algebra of the Lie algebra of G.

As before, we denote by $\mathfrak{g} = \mathrm{Lie}(G)$. Denote by $[,]$ the Lie bracket on $\mathfrak{g}$. For any $x \in \mathfrak{g}$, recall the endomorphism $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$, $y \mapsto [x, y]$.

Since G is defined over a field of characteristic p, then $\mathfrak{g}$ is a p-Lie algebra, that is it admits a p-operation ${}^{[p]} : \mathfrak{g} \to \mathfrak{g}$ satisfying the following properties:

- $(ax)^{[p]} = a^p x^{[p]}$ for any $a \in \mathbb{K}$ and $x \in \mathfrak{g}$,
- $\mathrm{ad}(x^{[p]}) = (\mathrm{ad}(x))^p$ for any $a \in \mathbb{K}$ and $x \in \mathfrak{g}$,
- $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} [\mathrm{ad}^{p-1}(tx+y)(x)]_{i-1}/i$, where $[\mathrm{ad}^{p-1}(tx+y)(x)]_{i-1}$ is the coefficient of t^{i-1} in the expansion of $\mathrm{ad}^{p-1}(tx+y)(x)$.

If we think of $\mathfrak{g}$ as a subalgebra of the algebra of the derivations $\mathrm{Der}(\mathbb{K}[G], \mathbb{K}[G])$ (precisely, the elements of $\mathfrak{g}$ are the derivations which commute with the morphisms between affine algebras induced by $\lambda_x : G \to G$, $g \mapsto gx$ for any $x \in G$), then the p-operation is just the composition of a derivation with itself p-times.

Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Any G-module admits also a structure of $\mathfrak{g}$-module by differentiating the action. Now the problem is that modules that were simple for the G-action may now become reducible as $U(\mathfrak{g})$-modules. Moreover, we loose information on the characters: $\mathfrak{h}$ acts in the same way (as 0) on $\mathbb{K}_0$ and $\mathbb{K}_{p\lambda}$ for any $\lambda \in X_+$.

The solution is to look at restricted $U(\mathfrak{g})$-modules: $U(\mathfrak{g})$-modules such that $x^{[p]} \cdot m = x^p \cdot m$, where x^p denotes the p-power of x in the $U(\mathfrak{g})$.

If we define the restricted enveloping algebra as the quotient

$$U^{\mathrm{res}} := U(\mathfrak{g})/(x^p - x^{[p]},\ x \in \mathfrak{g}),$$

then the category of restricted $U(\mathfrak{g})$-modules is equivalent to the category of U^{res}-modules. We now focus on the category $\mathcal{C}$, whose objects are U^{res}-modules M equipped with an X-grading, i.e. a direct sum decomposition $M = \bigoplus_{\lambda \in X} M_\lambda$, such that $\mathfrak{h}$ acts on M_λ via $\overline{\lambda}$ (=character of $\mathfrak{h}$ obtained by differentiating λ).

We hence obtain a functor

$$\{\text{rational } G\text{-representations}\} \to \mathcal{C}$$

via differentiating and remembering the T-action.

Denote by $U(\mathfrak{b})^{\mathrm{res}}$ the image of $U(\mathfrak{b}) \subseteq U(\mathfrak{g})$ in U^{res}. For any $\lambda \in X$ we define

$$Z(\lambda) = U^{\mathrm{res}} \otimes_{U^{\mathrm{res}}(\mathfrak{b})} \mathbb{K}_\lambda$$

(where by abuse of notation we denote by $\mathbb{K}_\lambda$ the image of $\mathbb{K}_\lambda$ under the above functor). Analogously as for a Verma module, we have

$$Z(\lambda) = \bigoplus_{\substack{\{\alpha_1,\dots,\alpha_m\} = R^- \\ 0 \le r_i < p}} (x_{\alpha_1}^{r_1} \dots x_{\alpha_m}^{r_m}) 1 \otimes 1.$$

where $(x_{\alpha_1}^{r_1} \dots x_{\alpha_m}^{r_m}) 1 \otimes 1 \in Z(\lambda)_{\lambda + r_1\alpha_1 + \dots r_m\alpha_m}$, and x_α is a generator of the (one-dimensional) weight space $\mathfrak{g}_\alpha$.

The module just defined is called Baby Verma module. The terminology is due to the fact hat it looks similar to a Verma, but it is finite dimensional.

By definition, every proper submodule of $Z(\lambda)$ is contained in $\bigoplus_{\mu<\lambda} Z(\lambda)_\mu$, so that $Z(\lambda)$ admits a unique irreducible quotient, which we denote by $L'(\lambda)$. The set $\{L'(\lambda)\}$ is a full set of isomorphism class representatives for the irreducible modules in $\mathcal{C}$.

For us the fundamental result is the following:

Theorem 1.4. *If $\lambda \in X_1$, then $ch(L'(\lambda)) = ch(L(\lambda))$.*

We can hence rephrase the original problem in terms of certain simple objects in the category $\mathcal{C}$. By construction, the character of any Verma module is easy to compute, therefore we would be able to solve our problem if we could determine the multiplicities

$$[Z(\mu) : L(\lambda)]$$

of $L(\lambda)$ in a(ny) Jordan-Hölder series for $Z(\mu)$, for $\lambda \in X_1$ and $\mu \in X_+$.

We will see that in fact the set of pairs of weights whose multiplicities we need to compute can be further reduced.

2. Affine Weyl group and the set of alcoves

We keep the notation of the previous sections. In particular, remember that we have restricted to the case of G simple and simply connected.

Let $V = X \otimes_{\mathbb{Z}} \mathbb{R}$. Then the affine Weyl group $\widehat{W}$ is a subgroup of the group of affine transformations $\mathrm{Aff}(V)$ of V. More precisely, $\widehat{W}$ is the subgroup of $\mathrm{Aff}(V)$ generated by elements $s_{\alpha,n}$ for $\alpha \in R^+$ and $n \in \mathbb{Z}$ defined as

$$s_{\alpha,n}(\lambda) = \lambda - (\langle, \lambda, \alpha^\vee\rangle - pn)\alpha.$$

We denote by W_0 the subgroup of $\widehat{W}$ generated by the elements $s_{\alpha,0}$, $\alpha \in R^+$. This is isomorphic to the Weyl group of G.

By definition, $\widehat{W}$ naturally acts on V. Nevertheless, this is not the action we want to look at, but we will consider the one obtained by translating the origin by $-\rho$, where ρ is the Weyl vector: $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$. Such an action is referred to as the dot action. Explicitely:

$$w \cdot (\lambda) = w(\lambda+\rho) - \rho, \quad \forall w \in \widehat{W},\ \lambda \in V,$$

where $w(\lambda+\rho)$ is the image of $\lambda+\rho$ under the natural affine action of $\widehat{W}$ on V.

Let $\alpha \in R^+$ and $n \in \mathbb{Z}$. Consider the hyperplane fixed by $s_{\alpha,n}$ under the dot action:

$$H_{\alpha,n} := \{v \in V \mid \langle v+\rho, \alpha^\vee\rangle = pn\}.$$

The set of alcoves $\mathcal{A}$ is the set of connected component of $V \setminus \bigcup_{\alpha,n} H_{\alpha,n}$.

Let h be the Coxeter number of G, i.e. $h = 1 + \mathrm{ht}(\theta)$, where θ is the unique highest root of R, i.e. the unique root such that if $\theta = \sum_{\alpha \text{ simple}} a_\alpha \alpha$, the value of the sum $\mathrm{ht}(\theta) = \sum_{\alpha \text{ simple}} a_\alpha$ is maximal.

Example 2.1. *If $G = SL_n$, then $\theta = \sum_{\alpha \text{ simple}} \alpha$ and, since there are $n-1$ simple roots, $h = n$.*

If $p \geq h$, then the origin $0 \in V$ does not lie in any of the hyperplanes $H_{\alpha,n}$. There is hence a unique alcove containing it. Let's denote it by A_0 and call it the fundamental alcove. More explicitely,

$$A_0 := \{\lambda \in V \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle \ \forall \alpha \text{ simple, and } \langle \lambda + \rho, \theta^\vee \rangle < p\}$$

The affine Weyl group $\widehat{W}$ acts freely and transitively on $\mathcal{A}$, so that we have a bijection

$$\widehat{W} \to \mathcal{A}, \quad w \mapsto A_w := \text{ unique alcove containing } w \cdot 0.$$

With this notation, we have $A_e = A_0$.

The above bijection induces a right action of $\widehat{W}$ on $\mathcal{A}$ given by

$$A_w \cdot x := A_{wx} \qquad (w, x \in \widehat{W}.$$

2.1. **Back to representation theory.** Thanks to the linkage and translation principles (sorry, no time to discuss them!), it is enough to compute

$$[Z(w \cdot 0) : L'(x \cdot 0)]$$

for $x \cdot 0 \in X_1$ and $w \cdot 0 \in X_+$ to determine all relevant multiplicities.

It will be convenient for us to denote by $Z(A_w)$, resp. $L'(A_x)$, the module $Z(w \cdot 0)$, resp. and $L'(x \cdot 0)$.

2.2. **The Hecke algebra of a Coxeter system.** Our main reference for this section is [Soe].

Let (W, S) be a Coxeter system, that is a group given by a finite set of generators S and relations among all pairs of them:

$$(s_i s_j)^{m_{ij}}$$

where $m_{ij} \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ are such that $m_{ii} = 1$ and $m_{ji} = m_{ij} \geq 2$ if $i \neq j$. We will sometime refer to elements of W as to words, meaning that they can be thought of as words in the alphabet S. If $w = s_{i_1} \ldots s_{i_r}$ is an expression of w as a product of a minimal number of generators, then such a product is called a reduced expression for w.

The Coxeter group W is equipped with a partial order $\leq$, called Bruhat(-Chevalley) order. The Bruhat order has several characterisations. The definition we give here is in terms of subwords: $y \leq w$ if and only if y is obtained by removing some letters from a reduced expression for w.

We have now recalled everything we need to define the Hecke algebra $\mathcal{H} = \mathcal{H}(W, S)$.

The Hecke algebra $\mathcal{H}$ is the free $\mathbb{Z}[v^{\pm 1}]$-module with basis $\{H_x \mid x \in W\}$ and whose algebra structure is uniquely determined by

$$H_x H_s = \begin{cases} H_{xs} & \text{if } xs > x, \\ (v - v^{-1}) H_x + H_{xs} & \text{if } xs < x, \end{cases} \qquad (x \in W, s \in S)$$

It follows that if $x = s_{i_1} \ldots s_{i_r}$ is a reduced expression of x, then

$$H_x = H_{s_{i_1}} \cdot \ldots \cdot H_{s_{i_r}}.$$

Moreover, we notice that, being H_e the unity of $\mathcal{H}$, if we apply the multiplication rule with $x = s$ we obtain that

$$H_s^{-1} = H_s - (v - v^{-1})$$

and we deduce that $H_w^{-1} \in \mathcal{H}$ for any $w \in W$.

We can hence define the $\mathbb{Z}$-linear involution $\bar{\cdot} : \mathcal{H} \to \mathcal{H}$ given by

$$H_x \mapsto H_{x^{-1}}^{-1},\ v^{\pm 1} \mapsto v^{\mp 1}.$$

The following theorem was proven by Kazhdan and Lusztig in their seminal paper [KL]:

Theorem 2.1. *For any $y \in W$ there exists a unique element $\underline{H}_y \in \mathcal{H}$ such that*

(1) $\overline{\underline{H}_y} = \underline{H}_y$,
(2) $\underline{H}_y = H_y + \sum_{x \neq y} h_{x,y} H_x$, *where* $h_{x,y} \in v^{-1}\mathbb{Z}[v^{-1}]$.

The polynomials $h_{x,y}$ are called Kazhdan-Lusztig polynomials.

Example 2.2. (1) *It is immediate to see that $\underline{H}_e = H_e$.*
(2) *Since we know the inversion formula for H_s, we can easily verify that $\underline{H}_s = H_s + v^{-1}$:*

$$\overline{H_s + v^{-1}} = \overline{H_s} + \overline{v^{-1}} = (H_s - (v - v^{-1})) + v = H_s + v^{-1}.$$

Clearly, we could have also defined $\mathcal{H}$ as the $\mathbb{Z}[v^{\pm 1}]$ module admitting a right $\mathcal{H}$-module structure, given by right multiplication. Such a structure is uniquely determined by the following rule:

$$H_x \underline{H}_s = \begin{cases} H_{xs} + v^{-1} H_x & \text{if } xs > x, \\ X_{xs} + v H_x & \text{if } xs < x. \end{cases} \qquad (x \in W, s \in S)$$

On top of being a more symmetric rule than the one given by right multiplication by H_s, the above formula is the one we will use as an inspiration to define the periodic module.

2.3. **The periodic module.** From now on, $\mathcal{H}$ will be the Hecke algebra of the affine Weyl group, seen as a Coxeter system with set of simple reflections $S = \{s_{\alpha,0} \mid \alpha \in \Delta\} \cup \{s_{\theta,1}\}$.

Before giving the definition of periodic module we need to equip the set of alcoves with a partial order which will play the role of the Bruhat order in the alcove setting.

Let $\alpha \in R^+$, $n \in \mathbb{Z}$. Then the hyperplane $H_{\alpha,n}$ divides V into two halfspaces:

$$H_{\alpha,n}^+ = \{v \in V \mid \langle v + \rho, \alpha^\vee \rangle > pn\},$$

$$H_{\alpha,n}^- = \{v \in V \mid \langle v + \rho, \alpha^\vee \rangle < pn\}.$$

The generic (or semi-infinite) order $\preceq$ on $\mathcal{A}$ is the partial order relation generated by

$$A \prec s_{\alpha,n} A \quad \Leftrightarrow \quad A \subset H_{\alpha,n}^-.$$

Notice that the generic order is invariant by weight translation: $A \preceq B$ if and only if $A + p\lambda \preceq B + p\lambda$ for any $\lambda \in X$.

Now we have set everything we need to define the periodic module $\mathcal{P}$: it is the free $\mathbb{Z}[v^{\pm 1}]$-module with basis $\{A \mid A \in \mathcal{A}\}$ equipped with a right $\mathcal{H}$-module structure given by

$$A\underline{H}_s = \begin{cases} As + v^{-1}A & \text{if } As \succ A, \\ As + vA & \text{if } As \prec A, \end{cases} \qquad (a \in \mathcal{A},\ s \in S).$$

For a $\lambda \in X$, define

$$E_\lambda := \sum_{x \in W_0} v^{-\ell(x)}(A_x) + p\lambda.$$

Let $\mathcal{P}^0$ be the $\mathcal{H}$-submodule of $\mathcal{P}$ generated by the E_λ's.

The last notion we need before stating the periodic version of Kazhdan-Lusztig's theorem is the notion of skew-linear map: this is a $\mathbb{Z}$-linear map $f : \mathcal{P} \to \mathcal{P}$ such that $f(ph) = f(p)\overline{h}$ for any $h \in \mathcal{H}$, $p \in \mathcal{P}$.

Theorem 2.2 ([Lu])**.** (1) *There exists a unique skewlinear involutive map* $\bar{\cdot} : \mathcal{P} \to \mathcal{P}$ *such that* $\overline{E_\lambda} = E_\lambda$.
(2) *For any* $A \in \mathcal{A}$ *there exists a unique element* $\underline{P}_A = A + \sum_{B \neq A} p_{B,A}B$, *where* $p_{A,B} \in v^{-1}\mathbb{Z}[v^{-1}]$.

Example 2.3. *In the* SL_2 *case, any alcove is of the form* $A = (-1 + mp, -1 + (m+1)p)$ *for an* $m \in \mathbb{Z}$ *(where* (a, b)*, for* $a < b$ *denotes the corresponding interval on the real line). Then, under the identification of* X *with* $\mathbb{Z}$*, we have that* $\underline{P}_A = E_m$.

The periodic version of Lusztig's conjecture is

$$[Z(A) : L(B)] = p_{w_0A, w_0B}(1),$$

where A corresponds to a regular weight in X_+ and B to a weight in $X_+ \cap (\widehat{W} \cdot 0)$. This is actually a theorem due to Andersen, Jantzen and Soergel if $p \gg 0$ [AJS]. The above equality was expected to be valid for $p \geq h$, but thanks to work of Williamson [W] we know by now that this was a way to optimistic bound. There is an explicit bound (which is huge!) due to Peter Fiebig [F]. What really happens in between such a huge bound and h is still sort of mysterious. In the following section I will sketch a construction of a category recently introduced in [FL1] which should provide new insight into the problem.

3. Sheaves on the alcoves

We start by defining a topology on the set of alcoves $\mathcal{A}$. Recall that we have equipped $\mathcal{A}$ with a partial order $\preceq$. We now consider $\mathcal{A}$ as a topological space by declaring open the poset ideals, that is $\mathcal{J} \subseteq \mathcal{A}$ is open if and only if whenever $A \in \mathcal{J}$ and $B \preceq A$, then also $B \in \mathcal{J}$.

Remember that the partial order $\preceq$ was invariant under translation by any p-multiple of a weight, and hence also by any p-multiple of an element of the root lattice $\mathbb{Z}R^+$. We deduce that for any $\lambda \in \mathbb{Z}R^+$ we obtain a homeomorphism of topological spaces:

$$\mathcal{A} \to \mathcal{A}, \quad A \mapsto A + p\lambda.$$

We consider now the quotient of $\mathcal{A}$ by the translation action of $p\mathbb{Z}R$ and denote it by $\mathcal{V}$. There is a left action of the finite Weyl group W_0 on $\mathcal{V}$ since $\widehat{W} =$

$W_0 \ltimes p\mathbb{Z}R^+$. Moreover, $\mathcal{V}$ is a homogeneous space for W_0, being $\mathcal{A}$a homogenous space for the action of $\widehat{W}$.

Denote by $S = \mathrm{Sym}(Y \otimes_{\mathbb{Z}} k)$ the symmetric algebra of the vector space dual to V. Then we can define the structure algebra as

$$\mathcal{Z} := \left\{ (z_x) \in \bigoplus_{x \in \mathcal{V}} S \mid z_x - z_{s_{\alpha,0}x} \in \alpha^\vee S \right\}.$$

The structure algebra admits a diagonal action of C, and it is indeed a C algebra, where addition and multiplication are defined componentwise.

Remark 3.1. *The structure algebra computes equivariant cohomology of the Langlands dual flag variety: $\mathcal{Z} \simeq H_{T^\vee}(G^\vee/B^\vee, k)$. If char$(k)$ is a good prime for G (cf.* [AJS, Appendix D]*), then $\mathcal{Z} \simeq C \otimes_{C^{W_0}} C$. Finally, if $k = \mathbb{C}$, $\mathcal{Z}$ is isomorphic to the centre (of a deformed version) of a principal block in the BGG category $\mathcal{O}$ for the Lie algebra $\mathfrak{g} = Lie(G)$, or, equivalently (cf. Michael Ehrig's 2nd lecture), to the endomorphism ring of the "big" projective (i.e. the one obtained by taking the direct sum of all indecomposible projectives in the principal block).*

We are interested in studying categories of (pre)-sheaves of $\mathcal{Z}$-modules on $\mathcal{A}$. Of course, it will be too much to hope that any presheaf of $\mathcal{Z}$-modules on $\mathcal{A}$ could help us in solving our representation theoretical problem. We need hence to require some technical conditions to hold. The first one is reasonably easy to state, while we will only see an approximation of the others, and then refer the interested reader to the preprints [FL1], [FL2].

Let $x \in \mathcal{V}$ and denote by ι_x the inclusion $\iota_x : x \hookrightarrow \mathcal{A}$. We denote by $\mathcal{Z}^x$ the image of the projection map $\mathcal{Z} \to S$, given by $(z_x) \mapsto z_x$ and, for a $\mathcal{Z}$-module M we define $M^x := M \otimes_Z \mathcal{Z}^x$. For a presheaf of $\mathcal{Z}$-modules we denote by $\mathscr{F}^x$ the sheaf of $\mathcal{Z}^x$-modules obtained by applying $\cdot^x$ to its modules of sections.

Thus we say that a sheaf $\mathscr{F}$ satisfies the support condition if the map

$$\mathscr{F}^x \to \iota_{x,*}\iota_x^*\mathscr{F}^x.$$

is an isomorphism.

Consider now the set of alcoves with a finer topology defined as follows: first of all for a positive root $\alpha \in R^+$ consider $S^\alpha := S[\beta^{-1} \mid \beta \in R^+ \setminus \{\alpha\}]$. Then define a new partial order $\preceq_\alpha$ on $\mathcal{A}$ given by

$$A \preceq_\alpha B \text{ if either } B \in A + p\mathbb{Z}R^+ \text{ or } (B = s_{\alpha,n}A \text{ and } A \subset H^-_{\alpha,n}).$$

We define the open sets as before, by replacing $\preceq$ by $\preceq_\alpha$, and denote by $\mathcal{A}^\alpha$ the set of alcoves equipped with this new topology. The map

$$\gamma : \mathcal{A}_\alpha \to \mathcal{A}, \qquad A \mapsto A$$

is continuous (**Exercise!**) and we can define

$$\mathscr{F} \boxtimes_S S^\alpha := (\gamma^*(\mathscr{F} \otimes_S S^\alpha))^+$$

where $\mathscr{G}^+$ denotes the maximal quotient of $\mathscr{G}^+$ which satisfies the support condition. Now, $\mathscr{F} \boxtimes_S S^\alpha$ is a presheaf of $\mathcal{Z} \otimes_S S^\alpha$-modules on $\mathcal{A}^\alpha$ and it is not a sheaf in general.

The category we are interested in is the following

$$\mathcal{S} = \left\{ \begin{array}{c} \text{full subcategory of the category of sheaves of } \mathcal{Z}\text{-modules on } \mathcal{A} \\ \text{whose objects are flabby reflexive sheaves } \mathscr{F} \text{ satisfying} \\ \text{the support condition and such that } \mathscr{F} \boxtimes_S S^\alpha \text{ is a sheaf } \forall \alpha \in R^+ \end{array} \right\}$$

Even if $\mathcal{S}$ is not an abelian category, it inherits the exact structure from the category of sheaves on $\mathcal{A}$, so that we can talk about projective objects therein. Denote by $\Delta(A)$ the skyscraper sheaf on A.

Theorem 3.1. (1) *For any $B \in \mathcal{A}$, there is a unique indecomposable projective object $\mathscr{P}(B) \in \mathcal{S}$ such that there is a unique epimorphism $\mathscr{P}(B) \to \Delta(B)$.*

(2) *The sheaf $\mathscr{P}(B)$ admits a finite filtration whose subquotients are isomorphic to skyscraper sheaves $\Delta(A)$'s.*

(3) *The multiplicities of skyscraper sheaves in these particular projective sheaves compute the relevant representation theoretical multiplicities:*

$$(\mathscr{P}(B) : \Delta(A)) = [Z(A) : L'(B)].$$

Remark 3.2. *Skyscraper sheaves play in this theory the same role that Verma modules play in the more classical setting of category $\mathcal{O}$ for a simple complex finite dimensional Lie algebra, and the filtration in point (2) of the above theorem is the analogue of a Verma flag of an indecomposable projective object.*

3.1. **Remarks on the proof.** To prove the first two statements of Theorem 3.1, we first directly construct $\mathscr{P}(B)$ in the case $B = A_0 + p\lambda$ ($\lambda \in X$), and then we proceed by induction via translation functor techniques. It is not hard to prove the existence of translation funtors satisfying certain functoriality conditions in the category of flabby, reflexive $\mathcal{Z}$-modules, but it is hard work to prove that they preserve our category $\mathcal{S}$.

As for the third statement, we have to rely on Andersen, Jantzen and Soergel's work [AJS]. More precisely, in [AJS], the authors define a combinatorial category (usually referred to as the AJS-category) and show the existence of *special* projective objects (again by first constructing some base object and then applying translation functors to it), indexed by $\mathcal{A}$, which in some way encode the multiplicities $[Z(A) : L'(B)]$. We hence define a functor from our category to the AJS-category (which is far from being an equivalence) and show that it sends any $\mathscr{P}(B)$ to the corresponding special projective.

References

[AJS] H.H. Andersen, J.C. Jantzen, W. Soergel, Representations of quantum groups at a p-th root of unity and of semisimple groups in characteristic p: independence of p, Astérisque **220** (1994), 338 pages.

[F] P. Fiebig, An upper bound on the exceptional characteristics for Lusztig's character formula, J. Reine Angew. Math. **673** (2012), 1–31.

[FL1] P. Fiebig, M.Lanini, Sheaves on the alcoves and modular representations I, preprint 2018, arXiv:1801.03959

[FL2] P. Fiebig, M.Lanini, Sheaves on the alcoves and modular representations II, preprint 2018, arXiv:1801.03958

[J] J.C. Jantzen, Representations of Algebraic Groups: Second Edition, American Mathematical Soc., 2003, 576 pages.

[KL] D. Kazhdan, G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math. **53** Vol. 2 (1979), 165–184.

[Lu] G. Lusztig, Hecke algebras and Jantzen's generic decomposition patterns, Advances in Mathematics **37** Vol. 2 (1980), 121–164.

[Soe] W. Soergel,Kazhdan-Lusztig polynomials and a combinatoric[s] for tilting modules, Representation Theory **1** (1997), 83–114.

[W] G. Williamson, Schubert calculus and torsion explosion, J. Amer. Math. Soc. **30** (2017), 1023–1046.