Tutorial 12 — Solutions

Weekly summary and definitions and results for this tutorial

a) A **p-colouring** of a knot is a colouring of the segments of the knots by colours $0, 1, 2, \dots, p-1$ in such a way that for each crossing $2c_i \equiv c_j + c_k \pmod{p}$. Let $C_p(K)$ be the number of p-colourings of the knot K.

b) A knot K is p**-colourable** if it has a p-colouring that uses at least two colours. Equivalently, K is p-colourable if and only if $C_p(K) > 0$.

c) Given a knot projection, read the segments $c_1, \dots, c_n$ in a direction around the knot. The **knot matrix** $M_K = (m_{ij})$ is the $n \times n$ matrix where m_{ij} is the sum of contributions of the jth segment to the ith crossing given by

$$m_{ij} = \begin{cases} 2, & \text{if } i = j, \\ -1, & \text{if } c_j \text{ goes under } c_i \text{ at the } i^{\text{th}} \text{ crossing}, \\ 0, & \text{otherwise.} \end{cases}$$

If K is alternating then the row and column sums in M_K are all zero.

d) If M is an $n \times n$ matrix and $1 \leqslant r, c \leqslant n$ then the (r, c)-minor of M is the $(n-1) \times (n-1)$ matrix obtained by removing all entries in row r and column c from M.

e) The **knot determinant** of an alternating knot is $\det(K) := |\det(M_K)_{rc}|$, where $(M_K$ is *any minor* of the knot matrix M_K. If p is a prime then the alternating knot K is p-colourable if and only if p divides $\det(K)$.

f) The **crossing number** $\mathrm{cross}(K)$ of a knot K is the minimum number of crossings in any knot projection of K. By definition, $\mathrm{cross}(K)$ is a knot invariant but it is difficult to calculate. We saw that $\mathrm{cross}(K) = 0$ if and only K is the unknot and $\mathrm{cross}(K \# L) \leqslant \mathrm{cross}(K) + \mathrm{cross}(L)$.

g) A knot K is a **composite knot** if $K = L \# M$ for non-trivial knots L and M. The knot K is **prime** if it is not composite.

h) Every knot can be written, uniquely, as a connected sum of prime knots.

i) A knot is **alternating** is the under and over crossings alternate as you you travel around the knot in a fixed direction.

j) Let K be a not. A **Seifert surface** for K is any surface that has K as its boundary. Seifert surfaces of K always exist but they are not unique. We gave an algorithm for constructing the Seifert surface of a knot given by putting an orientation on the knot, cutting the over-strings and then rejoining the using the orientation, gluing **Seifert circles** onto the result circles and then added **twists** with boundaries given by the previous crossings.

k) The **genus** of the knot S is the knot invariant

$$g(K) = \min\left\{ \tfrac{1}{2}\big(1 - \chi(S)\big) \,\middle|\, S \text{ is a Seifert surface of } K \right\}.$$

For knots K and L, $g(K \# L) = g(K) + g(L)$

l) If K has a knot projection with c crossings and the corresponding Seifert surface has s Seifert circles then $g(K) = \frac{1}{2}(1 + c - s)$.

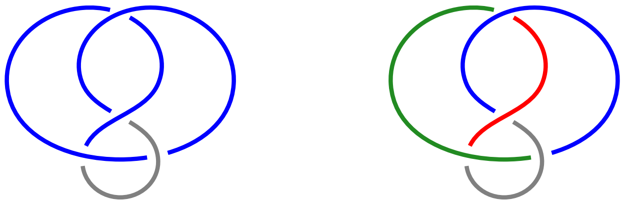
Questions to complete *before* the tutorial

1. Let $K = 4_1$ be the figure of eight knot:

$$4_1 \quad =$$

Show that 4_1 is not 3-colourable.

Solution Suppose by way of contradiction that the figure eight knot has a 3-colouring that uses all three colours. Then there are two possibilities: either all of segments meeting at the top crossing have the same colour, say blue, or they all have different colours:



Here we use gray strings for those strings for which the colour is still to be determined.

If all colours at the top crossing are the same then each of the remaining crossings force the remaining segment to be coloured blue, since two the segments in each of these crossings are already blue. In the second case, where the segments meeting the top crossing all have different colours, the three remaining crossings all force the remaining to coloured with the missing colour: the middle crosses forces the string to be coloured green, the bottom left crossing forces it to be coloured blue, and the bottom right crossing forces the segments to be coloured red. Hence, either way, we see that it is not possible to give a 3-colouring of the figure eight knot that uses all three colours.

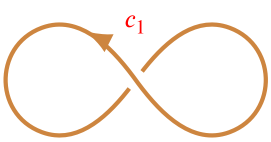
Alternative solution In lectures, we proved that a knot K is 3-colourable if and only if 3 divides its knot determinant. Below we show $\det(4_1) = 5$, which is not divisible by 3. Hence, 4_1 is not 3-colourable.

2. Compute the knot determinants of the knots:

Solution The first two knot are the unknot and the third knot is equivalent to the trefoil knot, so we expect that the knot determinants of these knots should be 1, 1 and 3, respectively.

The first knot is alternating but has only one segment and one crossing so we obtain:

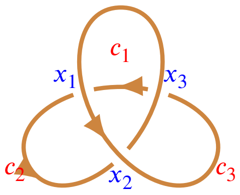
$K_1 =$



$$\rightsquigarrow \qquad M_{K_1} = \begin{pmatrix} 0 \end{pmatrix} \quad \Longrightarrow \det(K_1) = 1,$$

because the 1×1 minor of M_{K_1} is the empty matrix and, by convention, this matrix has determinant 1. Here, and below, the arrows show the orientation that we are using for the knot. The orientation is not essential when computing the knot matrix, but it is always better to put the segments into "travelling order".

The second knot is not alternating, so the segments do not naturally label the crossings. Labelling the crossings and segments as shown, we obtain:

$K_2 =$ 

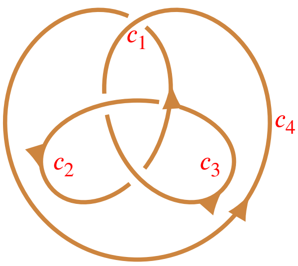
$$\begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{pmatrix}$$

As K_2 is not an alternating knot, to determine colourability we have to compute the determinants of *all* none of the 2×2 minors of the knot matrix M_{K_2}. We see that

$$M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1, \quad M_{12} = \begin{vmatrix} -1 & 2 \\ -1 & 1 \end{vmatrix} = 1, \quad M_{13} = \begin{vmatrix} -1 & -1 \\ -1 & 0 \end{vmatrix} = -1$$
$$M_{21} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1, \quad M_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 1 \end{vmatrix} = 1, \quad M_{23} = \begin{vmatrix} -1 & -1 \\ -1 & 0 \end{vmatrix} = -1$$
$$M_{31} = \begin{vmatrix} -1 & 2 \\ -1 & 2 \end{vmatrix} = 0, \quad M_{32} = \begin{vmatrix} -1 & 2 \\ -1 & 2 \end{vmatrix} = 0, \quad M_{33} = \begin{vmatrix} -1 & -1 \\ -1 & -1 \end{vmatrix} = 0.$$

In particular, this example shows that if a knot is not alternating then the determinants of the $(n-1) \times (n-1)$ minors of an $n \times n$ knot matrix are not necessarily equal.

Finally, the third knot is an alternating knot and

$K_3 =$ 

$$M_{K_3} = \begin{pmatrix} 2 & -1 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

To compute the knot determinant we can take any minor so it makes sense to take one that gives us more zeros in the minor. Taking the $(2,4)$-minor gives

$$\det(K_3) = \left| \det \begin{pmatrix} 2 & -1 & -1 \\ -1 & -1 & 2 \\ -1 & 0 & 0 \end{pmatrix} \right| = \left| \det \begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix} \right| = 3,$$

which is what we expect because K_3 is a knot projection of the trefoil knot.

Questions to complete *during* the tutorial

3. Let K be a knot with n crossings, $p > 2$ and suppose that $c_1, \ldots, c_n$ is a p-colouring of K.

a) Let $c'_k = c_k + 1 \pmod p$, for $1 \leqslant k \leqslant n$. Show that $c'_1, \ldots, c'_n$ is a p-colouring of K.

b) Using (a), or otherwise, for some p give five different p-colourings that use two or more colourings for the knots:

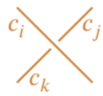
$4_1 =$  $5_2 =$ 

You will need to use different values of p for the two knots.

c) For knots K and L show that $C_p(K \# L) = \frac{1}{p} C_p(K) C_p(L)$.

Solution

a) Suppose that we have a crossing in a knot projection of K:



Then $2c_i' = 2(c_i + 1) = 2c_i + 2 \equiv c_j + c_k = (c_j + 1) + (c_k + 1) \equiv c_j' + c_k' \pmod{p}$, where the first equivalence follows because $c_1 \dots, c_n$ is a p-colouring of K. Hence, the colours $c_1', \dots, c_n'$ give a p-colouring of K.

Another way to see this is to recall from lectures that $c_1, \dots, c_n$ is a p-colouring of K if and only if

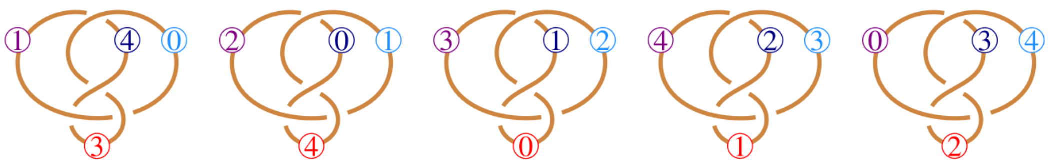
$$M_K \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \equiv \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \pmod{p},$$

where M_K is the knot matrix of K. Moreover, $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is a zero eigenvector of M_K, so

$$M_K \begin{pmatrix} c_1' \\ \vdots \\ c_n' \end{pmatrix} = M_K \begin{pmatrix} c_1 + 1 \\ \vdots \\ c_n + 1 \end{pmatrix} = M_K \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} + M_K \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = M_K \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \equiv \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \pmod{p},$$

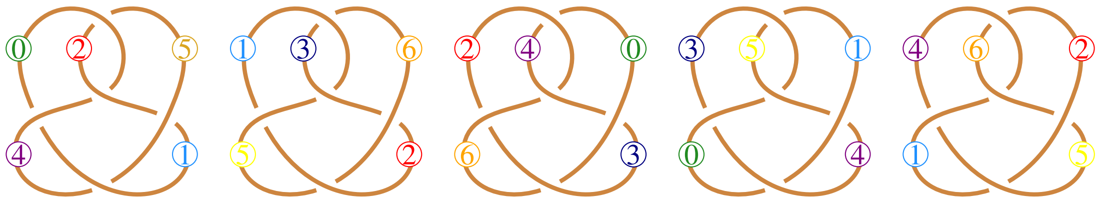
showing that $c_1', \dots, c_n'$ is a p-colouring of K

b) By part (a) it is enough to find one colouring and then we can generate the rest by adding k to the colours, modulo 5, for $0 \leqslant k < 5$. For the figure eight knot five possible 5-colourings are:

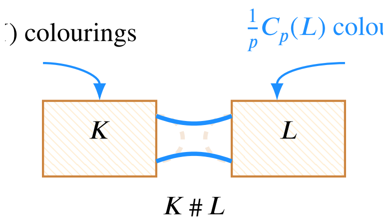


In each case, you can check that $2c_i \equiv c_j + c_k \pmod{5}$, where c_i is the colour of the over-crossing and c_j and c_k are the under-crossings. The latter colourings are obtained from the previous colouring by adding 1 to each colour, modulo 5.

Similarly, 7-colourings of the 5_1 knot are:

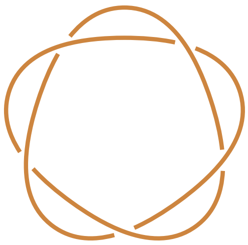


c) Given part (a), the proof is the same as that given in lectures for 3-colourings. The picture is that:



That is, we have $C_p(K)$ choices of colourings for the knot K. The choice of colouring for K fixes the colour of the string segment that is cut when forming the connected sum with L, so there are $\frac{1}{p}C_p(L)$ choices of colourings for L given the fixed colouring of the used to form the connected sum. Therefore, there are $\frac{1}{p}C_p(K)C_p(l)$ different p-colourings of $K \# L$.

4. Find the determinants of the knots 4_1, 5_1 and 5_2 and determine for which odd primes p they are p-colourable.

4_1 =  5_1 =  5_2 = 

Solution

a) Consider the knot 4_1. Using the same labelling as in Question 1 rows of the matrix M associated to 4_1 are

1st Row : $[2,-1,-1,0]$ because x_1 separates x_2 and x_3.
2nd Row : $[0,2,-1,-1]$ because x_2 separates x_3 and x_4.
3rd Row : $[-1,0,2,-1,]$ because x_3 separates x_4 and x_1
4th Row : $[-1,-1,0,2]$, because x_4 separates x_1 and x_2.

Note by construction all row sums are 0.

The matrix M associated in this way with 4_1 is

$$\begin{pmatrix} 2 & -1 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & 0 & 2 \end{pmatrix}$$

Observe the column sums of M are also all 0, as will always be the case for alternating knot diagrams. The knot determinant of 4_1 is thus the absolute value of the determinant of any 3×3

minor M_{ij} of M. *Recall the ijth minor of a matrix M is the matrix formed by ignoring the ith row and jth column of M.* Ignoring the fourth row and first column of M gives

$$M_{41} = \begin{pmatrix} -1 & -1 & 0 \\ 2 & -1 & -1 \\ 0 & 2 & -1 \end{pmatrix}$$

This minor has determinant -5:

$$\det M_{41} \overset{R_2=R_2+2R_1}{=} \begin{vmatrix} -1 & -1 & 0 \\ 0 & -3 & -1 \\ 0 & 2 & -1 \end{vmatrix} = -\begin{vmatrix} -3 & -1 \\ 2 & -1 \end{vmatrix} = -5.$$

In lectures we saw that if p is an odd prime then a knot K is p-colourable if and only if p divides its knot determinant $\mathrm{Det}(K)$. Thus the knot 4_1 is 5-colourable, but not p-colourable for any odd prime $p \neq 5$.

You can check

$$x_1 = 2, \quad x_2 = 3, \quad x_3 = 1, \quad x_4 = 0,$$

is a 5-colouring of 4_1.

b) Now consider the knot 5_1. No matter where you start labelling the segments of the knot diagram 5_1 long as you travel along the knot you obtain the knot matrix

$$M = \begin{pmatrix} 2 & 0 & -1 & -1 & 0 \\ 0 & 2 & 0 & -1 & -1 \\ -1 & 0 & 2 & 0 & -1 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{pmatrix}$$

Evaluating any 4×4 minor M_{ij} will show $\mathrm{Det}(5_1) = 5$. To see this it makes sense to pick a minor that removes some non-zero entries from the matrix, so let's consider M_{44}:

$$\det M_{44} = \begin{vmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{vmatrix} \overset{R_1=R_1+2R_4}{=} \begin{vmatrix} 0 & 0 & 3 & -2 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{vmatrix} = -\begin{vmatrix} 0 & 3 & -2 \\ 2 & 0 & -1 \\ -1 & -1 & 2 \end{vmatrix}$$

$$\overset{R_2=R_2+2R_3}{=} -\begin{vmatrix} 0 & 3 & -2 \\ 0 & -2 & 3 \\ -1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix} = 5.$$

Therefore, the knot 5_1 is 5-colourable, but not p-colourable for any odd prime $p \neq 5$.

c) Finally, consider 5_2. Labelling the segments in travelling order, starting at the inner crossing, gives the matrix

$$M = \begin{pmatrix} 2 & 0 & -1 & -1 & 0 \\ 0 & 2 & 0 & -1 & -1 \\ -1 & 0 & 2 & 0 & -1 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & -1 & -1 & 2 & 0 \end{pmatrix}$$

This time we compute M_{45}:

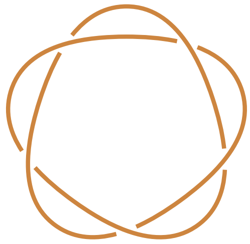
$$\det M_{45} = \begin{vmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & -1 & 2 \end{vmatrix} \overset{R_1=R_2+2R_3}{=} \begin{vmatrix} 0 & 0 & 3 & -1 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & -1 & 2 \end{vmatrix} = -\begin{vmatrix} 0 & 3 & -1 \\ 2 & 0 & -1 \\ -1 & -1 & 2 \end{vmatrix}$$

$$\overset{R_2=R_2+2R_1}{=} -\begin{vmatrix} 0 & 3 & -1 \\ 0 & -2 & 3 \\ -1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ -2 & 3 \end{vmatrix} = 7.$$

Hence, $\operatorname{Det}(5_2) = 7$. Thus the knot 5_2 is 7-colourable, but not p-colourable for any odd prime $p \neq 7$.

5. a) Calculate the genus of the three knots:

4_1 =

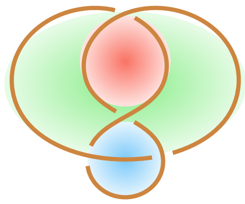
5_1 =

5_2 =


b) Using part (a), or otherwise, show that all of these knots are prime.

Solution

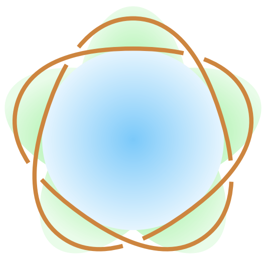
a) All of these knot projections alternating so, by a theorem stated in lectures, the genus of each of these knots is $\frac{1}{2}(1 + c - s)$, where c is the crossing number and s is the number of Seifert circles in the corresponding Seifert surface for the knot.

The figure eight knot 4_1 has three Seifert circles, which are the regions shown below:



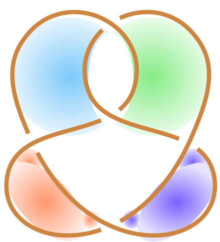
Therefore, $g(4_1) = \frac{1}{2}(1 + 4 - 3) = 1$.

The cinquefoil has two Seifert circles, which are the regions shown in:



Hence, $g(5_1) = \frac{1}{2}(1 + 5 - 2) = 2$.

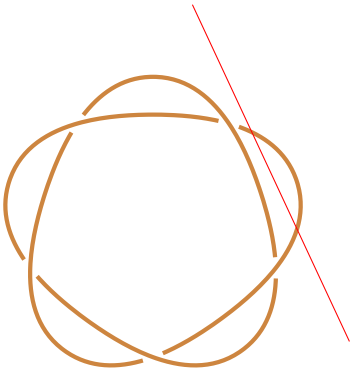
Finally, the knot 5_2 has four Seifert circles:



Therefore, the heart knot 5_2 has genus $\frac{1}{2}(1 + 5 - 4) = 1$.

b) By lectures, $g(K \# L) = g(K) + g(L)$, so the figure eight knot 4_1 and the heart knot 5_2 are both prime since they are of genus 1 by part (a).

The cinquefoil knot 5_1 has genus 2, so the genus alone is not sufficient to show that it is prime. The easiest way to see that this knot is prime is to notice that 5_1 is the $(5,2)$-torus knot, which means that it is prime. To see this explicitly, to write 5_1 and a connected sum $K \# L$ we need to cut the knot into two pieces by cutting exactly two strings in the knot in $\mathbb{R}^3$ with a plane. The only way to cut the cinquefoil knot this way is to cut off one of the "ears":



Hence, the 5_1 knot is prime.

6. a) What is the Euler characteristic of the double torus?
b) What is the minimum number of colours needed to be able to colour any map on the double torus so that no two adjacent regions have the same colour?

Solution

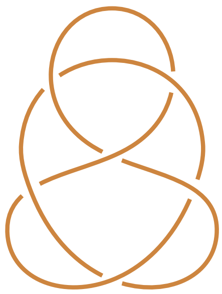
a) The double torus T has Euler characteristic -2.
b) By Heawood's theorem, we need at most

$$C_T < \tfrac{1}{2}\big(7 + \sqrt{49 - 24\chi(T)}\big) = \tfrac{1}{2}(7 + \sqrt{97}) \approx 8.4.$$

Hence, the maximum number of colours needed for a map on the double torus T is $C_T = 8$

Questions to complete *after* the tutorial

7. Find the determinants of the knots 6_1, 6_2 and 6_3 and determine for which odd primes p they are p-colourable.

6_1 =  6_2 =  6_3 = 

Solution All of these knots are alternating, so the knot determinant is the (absolute value of the), determinant of any minor of the knot matrix. As the knot determinant is the absolute value of the determinant of any minor of the knot matrix we ignore the signs of the determinants below when they do not affect the final answer.

First consider the knot 6_1. Labelling the segments in travelling order starting from the segment on the top left the knot matrix is

$$M_{6_1} = \begin{pmatrix} 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & -1 & -1 & 0 \\ -1 & 0 & 2 & 0 & 0 & -1 \\ 0 & -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix}$$

Note that all of the row and column sums are 0, as they must be. Taking the $(6,6)$-minor and using row and column operations to evaluate the determinant gives

$$\pm\det(6_1) = \begin{vmatrix} 2 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 & -1 \\ -1 & 0 & 2 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{vmatrix} \overset{C_1=C_1+2C_5}{=} \begin{vmatrix} 0 & 0 & 0 & 0 & -1 \\ -2 & 2 & 0 & -1 & -1 \\ -1 & 0 & 2 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 3 & -1 & 0 & 0 & 2 \end{vmatrix} = \begin{vmatrix} -2 & 2 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & -1 & 2 \\ 3 & -1 & 0 & 0 \end{vmatrix}$$

$$\overset{R_3=R_3+2R_1}{=} \begin{vmatrix} -2 & 2 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ -4 & 3 & -1 & 0 \\ 3 & -1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 2 \\ -4 & 3 & -1 \\ 3 & -1 & 0 \end{vmatrix} \overset{C_3=C_3+2C_1}{=} \begin{vmatrix} -1 & 0 & 0 \\ -4 & 3 & -9 \\ 3 & -1 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -9 \\ -1 & 6 \end{vmatrix} = 18 - 9 = 9.$$

For the third equality we expanded the determinant along the first row, for the fifth equality we expanded the determinant down the fourth column and for the seventh equality we expanded the determinant along the first row. Hence, $\det(6_1) = 9$, so 6_1 is 3-colourable.

Now consider the knot 6_2. Labelling the segments in travelling order start from the top of the knot, the knot matrix is:

$$M_{6_2} = \begin{pmatrix} 2 & 0 & -1 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 & -1 & -1 \\ -1 & 0 & 0 & 2 & 0 & -1 \\ 0 & -1 & -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Note that all of the row and column sums are 0, as they must be. Taking the $(6,6)$-minor and using row and column operations to evaluate the determinant gives

$$\pm\det(6_2) = \begin{vmatrix} 2 & 0 & -1 & -1 & 0 \\ 0 & 2 & 0 & -1 & -1 \\ 0 & 0 & 2 & 0 & -1 \\ -1 & 0 & 0 & 2 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{vmatrix} \overset{R_1=R_1+2R_4}{=} \begin{vmatrix} 0 & 0 & -1 & 3 & 0 \\ 0 & 2 & 0 & -1 & -1 \\ 0 & 0 & 2 & 0 & -1 \\ -1 & 0 & 0 & 2 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -1 & 3 & 0 \\ 2 & 0 & -1 & -1 \\ 0 & 2 & 0 & -1 \\ -1 & -1 & 0 & 2 \end{vmatrix} \overset{R_2=R_2+2R_4}{=} \begin{vmatrix} 0 & -1 & 3 & 0 \\ 0 & -2 & -1 & 3 \\ 0 & 2 & 0 & -1 \\ -1 & -1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 3 & 0 \\ -2 & -1 & 3 \\ 2 & 0 & -1 \end{vmatrix}$$

$$\overset{C_1=C_1+2C_3}{=} \begin{vmatrix} -1 & 3 & 0 \\ 4 & -1 & 3 \\ 0 & 0 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ 4 & -1 \end{vmatrix} = 1 - 12 = -11,$$

where we expanded the determinants down the first column twice and along the last row, respectively. Hence, $\det(6_2) = 11$, so 6_1 is 11-colourable.

Finally, consider 6_3. Labelling the segments in travelling order start from the top of the knot, the knot matrix is:

$$M_{6_2} = \begin{pmatrix} 2 & 0 & -1 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & -1 & -1 \\ -1 & -1 & 0 & 2 & 0 & 0 \\ 0 & -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

Again, it is good to check that the row and column sums are 0, as they must be. Taking the $(1,1)$-minor, is slightly and using row and column operations to evaluate the determinant gives

$$\pm\det(6_3) = \begin{vmatrix} 2 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 & -1 \\ -1 & 0 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & -1 & 2 \end{vmatrix} \overset{C_1=C_1+2C_5}{=} \begin{vmatrix} 0 & 0 & 0 & 0 & -1 \\ -2 & 2 & 0 & -1 & -1 \\ -1 & 0 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 4 & 0 & -1 & -1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 2 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ 4 & 0 & -1 & -1 \end{vmatrix} \overset{R_2=R_2+2R_4}{=} \begin{vmatrix} -2 & 2 & 0 & -1 \\ 7 & 0 & 0 & -2 \\ -1 & -1 & 0 & 2 \\ 4 & 0 & -1 & -1 \end{vmatrix} = \begin{vmatrix} -2 & 2 & -1 \\ 7 & 0 & -2 \\ -1 & -1 & 2 \end{vmatrix}$$

$$\overset{R_1=R_1+2R_3}{=} \begin{vmatrix} -4 & 0 & 3 \\ 7 & 0 & -2 \\ -1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} -4 & 3 \\ 7 & -2 \end{vmatrix} = 8 - 21 = -13,$$

where we have expanded the determinants along the first row, the third column, and the second column respectively. Hence, $\det(6_3) = 13$, so 6_3 is 13-colourable.

In particular, this shows that these three knots are pairwise inequivalent and that they are not equal to the unknot. As 6_2 is the first 11-colourable knot we have seen it is prime. Similarly, 6_3 is prime as it is the first 13-colourable knot. As 6_1 is 3-colourable, colourability does not prove that this knot is prime, however, arguing as in question 5 shows that 6_1 has genus 1, which proves that this knot is prime.