Tutorial 11

Weekly summary and definitions and results for this tutorial

a) If $G = (V, E)$ is a connected graph embedded in $\mathbb{D}^2$ then $|V| - |E| + |F| = 2$, where F is the set of disconnected regions, or faces, in $\mathbb{D}^2 \setminus G$.
b) The complete graphs K_n, for $n \geqslant 5$, are not planar.
c) **Face-degree equation**: Let S be a polygonal surface without boundary, with e edges, vertex set V and F the set of faces. Then $\sum_{x\in V} \deg x = 2e = \sum_{y\in F} \deg y$.
d) A **platonic solid** is a solid made by gluing together regular n-gons of the same size with p edges meeting at every vertex. If the regular solid has vertex set V, edge set E and face set F then

$$\frac{1}{p} + \frac{1}{n} = \frac{1}{2} + \frac{1}{|E|} > \frac{1}{2}.$$

As a consequence, we saw that there are exactly five platonic solids:

Solid	n	p	$v = \frac{2e}{p}$	e	$f = \frac{2e}{p}$
Tetrahedron	3	3	4	6	4
Octahedron	3	4	6	12	8
Icosahedron	3	5	12	30	20
Cube	4	3	8	12	6
Dodecahedron	5	3	20	30	12

e) A **map** on a closed polygonal surface S is polygonal decomposition such that all vertices have degree at least 3, no region (or face), borders itself, no region contains a hole or another region and no internal region has only two borders.
f) A *colouring* of a map on a surface S is a colouring of the faces of the map so that polygons sharing a common edge (a.k.a countries that share a border) have different colours.
g) The **chromatic number** $C_M(S)$ of the *map* M is the minimum number of colours needed to colour M. The **chromatic number** of the *surface* S is

$$C(S) = \max\{C_M(S) \mid M \text{ a map on } S\}.$$

h) Heawood's estimate says that

$$C(S) \leqslant \begin{cases} 6, & \text{if } S = S^2 \text{ or } S = \mathbb{P}^2, \\ \frac{7+\sqrt{49-24\chi(S)}}{2}, & \text{otherwise} \end{cases}$$

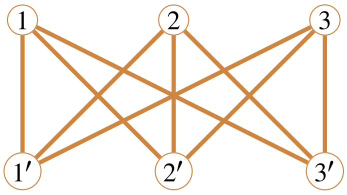
The key to proving this when $\chi(S) \leqslant 0$ is that $\partial_F \leqslant 5$, where $\partial_F = \frac{2|E|}{|F|}$ is the average degree of a face.
Heawood's estimate is *sharp* (i.e. exactly right), except when $S = S^2$ or $S = \mathbb{K}$. We proved that every map on S^2 or, equivalently (by stereographic projection), a map on $\mathbb{D}^2$, requires at most 5 colours. In fact, every map on S^2 is 4-colourable.
i) A **knot** is a closed path in $\mathbb{R}^3$ with no self-intersections.
j) A **knot projection** is a drawing of a knot in $\mathbb{R}^2$ with over and under crossings being used to indicate the relative positions of the strings and with no more than two strands meeting at any crossing.
k) A **polygonal decomposition** of a knot is a sequence of line segments with consecutive endpoints identified. Any knot is equivalent to a polygonal knot. Two polygonal knots are **equivalent** if there exists a polygonal knot that is a subdivision of both knots.
l) Two knot projections correspond to equivalent knots if and only if one can be transformed into the other using the three Reidemeister moves: twisting, looping and sliding.
m) The **segments** of a knot projection are the connected components of the knot projection.

Questions to complete *during* the tutorial

1. Recall that the complete graph K_5 on 5 vertices is not planar. That is, K_5 cannot be drawn on the plane or on the sphere without edge crossings.

a) Is it possible to draw K_5 without edge crossings on the Möbius band $\mathbb{M}$?
b) Is it possible to draw K_5 without edge crossings on the annulus $\mathbb{A}$?
[*Hint:* Argue by contradiction thinking about the relationship between $\mathbb{A}$ and S^2.]

2. Show that the complete bipartite graph $K_{3,3}$



is not planar.
[*Hint:* Argue by contradiction and first show that each face has four or six edges.]

3. A ball is constructed from squares and regular hexagons sewn along edges such that at each vertex 3 edges meet. Each square is surrounded by hexagons, and each hexagon by 3 squares and 3 hexagons. How many squares and hexagons are used in the construction?

4. a) Show that there is no regular polygonal decomposition of the torus by pentagons.
b) For which n is there a regular polygonal decomposition of the torus into n-gons?

5. The Degenerate Regular Decompositions of the Sphere

a) Show that for each integer $p \geq 2$ there is regular decomposition of the sphere into p two sided polygons.
b) Dually, show that for each integer $n \geq 2$ there is a regular decomposition of the sphere into 2 polygons with n sides.