Tutorial 7

Here is a quick summary and main results from the lectures in Week 7. Please watch the lecture recordings to gain a better understanding of this material.

Weekly summary and definitions and results for this tutorial

a) A **graph** $\mathcal{G}$ consists of a set $V_{\mathcal{G}}$ of vertices and a *multiset* $E_{\mathcal{G}}$ of edges, which are pairs of vertices.

b) Two graphs $\mathcal{G}$ and $\mathcal{H}$ are **isomorphic** if there is a bijection $f : V_{\mathcal{G}} \longrightarrow V_{\mathcal{H}}$ such that $\{u, v\}$ is an edge of $\mathcal{G}$ if and only if $\{f(u), f(v)\}$ is an edge of $\mathcal{H}$.

c) The **degree** $\deg v$ of a vertex $v \in V_{\mathcal{G}}$ is the number of edges in $\mathcal{G}$ that start or end at v.

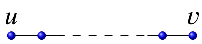
d) Proposition (Degree-Sum Formula) If G is a graph then $\sum_{v\in V_{\mathcal{G}}} \deg v = 2\#E_{\mathcal{G}}$.

e) A graph $\mathcal{H}$ is a **subgraph** of $\mathcal{G}$ if $V_{\mathcal{H}} \subseteq V_{\mathcal{G}}$ and $E_{\mathcal{H}} \subseteq E_{\mathcal{G}}$.

f) A graph is **planar** if it can be embedded (drawn) in $\mathbb{R}^2$ without edge crossings.

g) Every graph can be embedded in $\mathbb{R}^3$ without edge crossings.

h) A **subdivision** of a graph $\mathcal{G}$ is any graph that it obtained by replacing any number of edges •—• with •—•—• .

i) A **path** between two vertices $u, v \in V_{\mathcal{G}}$ is a sequence of consecutive edges joining u to v:

The path is a **circuit** if it starts and ends at the same vertex (so $u = v$).

j) A circuit is **contractible** if it is possible to reduce it to a path of length 0 by repeatedly removing pairs of *consecutive* repeated edges.

k) A graph is **connected** G if any two vertices $u, v \in V_{\mathcal{G}}$ can be joined by a path.

l) A **tree** is a connected graph in which every circuit is contractible.

m) If $\mathcal{G}$ is a graph then a **spanning tree** $\mathcal{T}$ for $\mathcal{G}$ is a subgraph of $\mathcal{G}$ such that $\mathcal{T}$ is a tree and $V_{\mathcal{T}} = V_{\mathcal{G}}$.

n) Theorem Every connected graph has a spanning tree.

o) The **Euler characteristic** of a graph $\mathcal{G}$ is $\chi(\mathcal{G}) = \#V_{\mathcal{G}} - \#E_{\mathcal{G}}$.

p) Theorem If $\mathcal{G}$ is a connected graph then $\chi(G) \leqslant 1$ with equality if and only if G is a tree.

q) Definition Let $\mathcal{G}$ be a connected graph. The number of **independent cycles** in $\mathcal{G}$ is $1 - \chi(\mathcal{G})$.

r) A **Eulerian circuit**, or **Eulerian cycle**, is a path through in a graph $\mathcal{G}$ that goes through every edge exactly once, and every vertex at least once.

Questions to complete *before* the tutorial

1. a) Draw a graph with 4 vertices and 5 edges.
b) Check that the Degree-sum Formula holds for the graph that you drew for part (a).

2. a) Draw a graph with a non-trivial circuit. That is, give an example of a graph that is not a tree.
b) Compute the Euler characteristic of your graph in part (a).
c) Draw a graph that is a tree.
d) Compute the Euler characteristic of your tree from part (c).

3. Determine the Euler characteristic of the cycle graphs C_n, for $n = 1, 2, 3 \ldots$ etc.

Questions to complete *during* the tutorial

4. a) The cube graph is the graph of vertices and edges of a cube. Make sketches to show that the cube graph is planar.
b) Make sketches to show that the octahedral graph, the graph formed by the vertices and edges of the regular octahedron, is planar.

5. Show that in any graph the number of vertices of odd degree must be even.

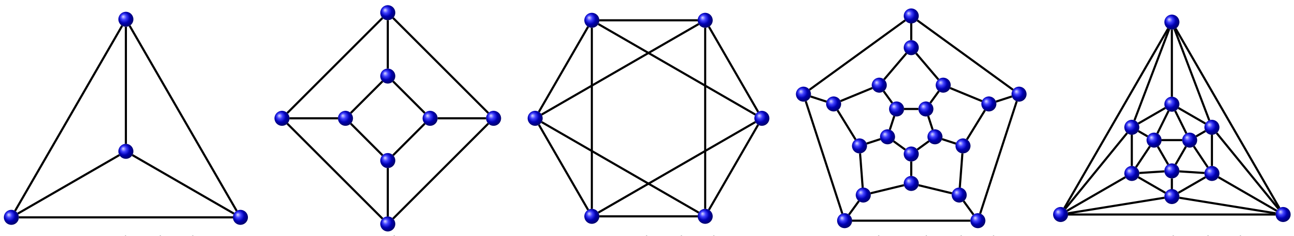
[*Hint:* Use the Degree-sum Formula.]

6. a) Show that if the connected components of a graph $\mathcal{G}$ are $\mathcal{G}_1, \ldots, \mathcal{G}_n$ then

$$\chi(\mathcal{G}) = \chi(\mathcal{G}_1) + \cdots + \chi(\mathcal{G}_n).$$

b) A *forest* is a graph such that all of its connected components are trees. If $\mathcal{F}$ is a forest show that $\chi(\mathcal{F})$ is equal to the number of trees in the forest.
c) Give an example of a graph $\mathcal{G}$ that is *not* a tree and $\chi(\mathcal{G}) = 1$.

7. The following graphs are projections of the five Platonic solids onto the plane:



Tetrahedral *Cubic* *Octahedral* *Dodecahedral* *Icosahedral*

a) Determine the Euler characteristic of each of these graphs.
b) A *Eulerian circuit* in a graph is a circuit that goes through every edge exactly once, and every vertex at least once. Find a Eulerian cycle for the octahedral graph.

8. The complete bipartite graph $\mathcal{K}_{m,n}$, for $m, n \geq 1$, has vertex set $V = M \sqcup N$ (disjoint union), where $m = |M|$, $n = |N|$, and with edge set $\{(x, y) \mid x \in M \text{ and } y \in N\}$. That is, $\mathcal{K}_{m,n}$ has mn edges that connect every element of M with every element of N. Show that all the bipartite graphs $\mathcal{K}_{2,n}$ are planar

Questions to complete *after* the tutorial

9. A **spanning tree** in a graph G is any subgraph of T that is a tree and has the same vertex set as G. Let t_n be the number of distinct spanning trees in the complete graph K_n.

a) Find t_2, t_3 and t_4.
b) (Harder.) What is t_5?