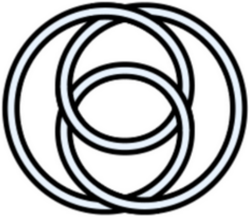
Topology – week 12

Math3061

Daniel Tubbenhauer, University of Sydney

Reidemeister moves are powerful but might be tricky

This is the unknot: $K =$ 

These two knots are equivalent: $K =$  , $K' =$ 

How to show that? Use Reidemeister moves (this is a strongly recommended exercise). But that might be tricky in general, so invariants is what we want.

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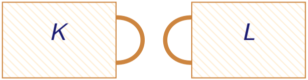

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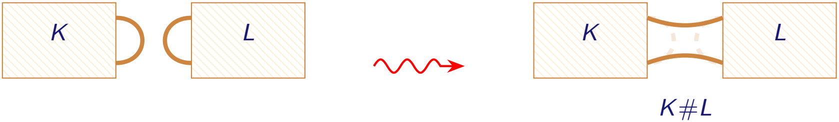

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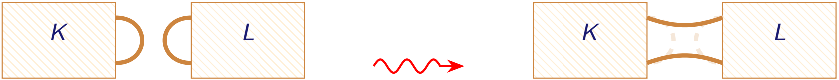

$K \# L$

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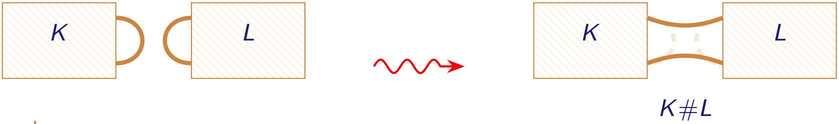

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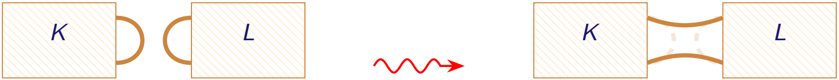

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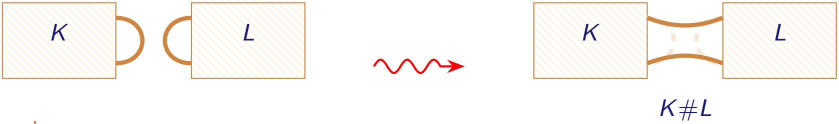

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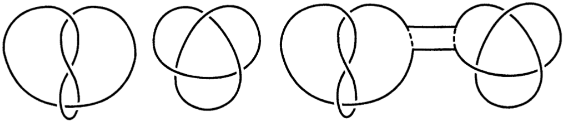
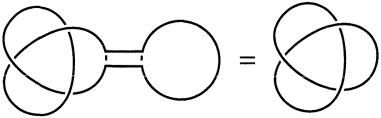
  - $(K\#L)\#M \cong K\#(L\#M)$

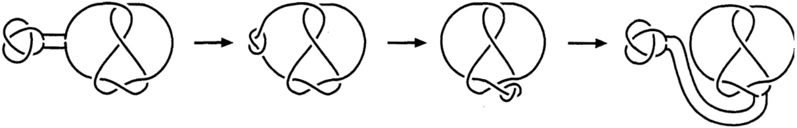
Examples of



J K $J\#K$



K unknot K



Three colorability and connected sums

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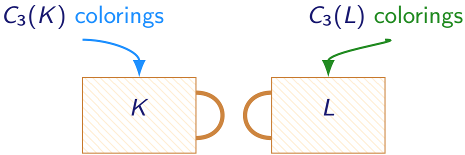

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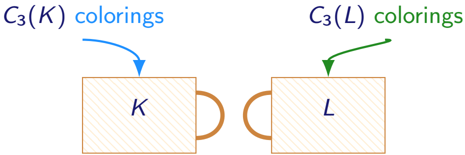

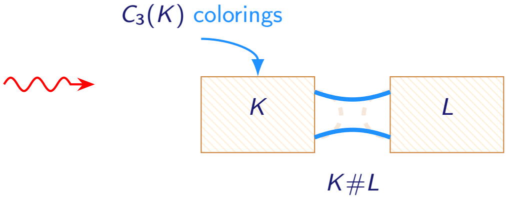

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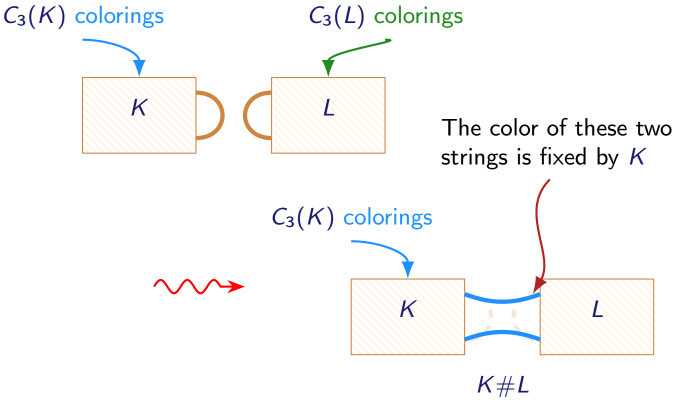

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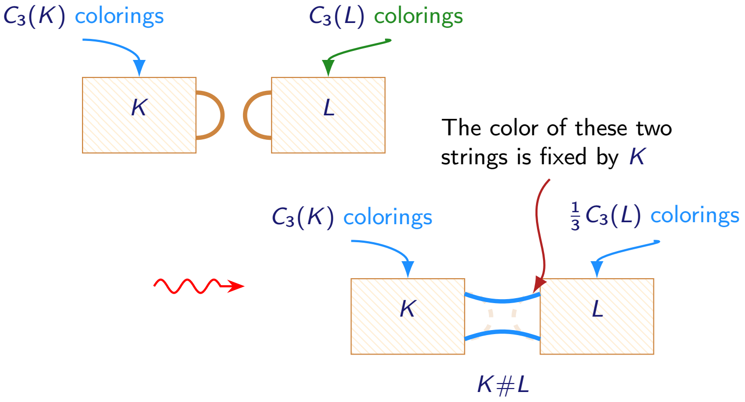

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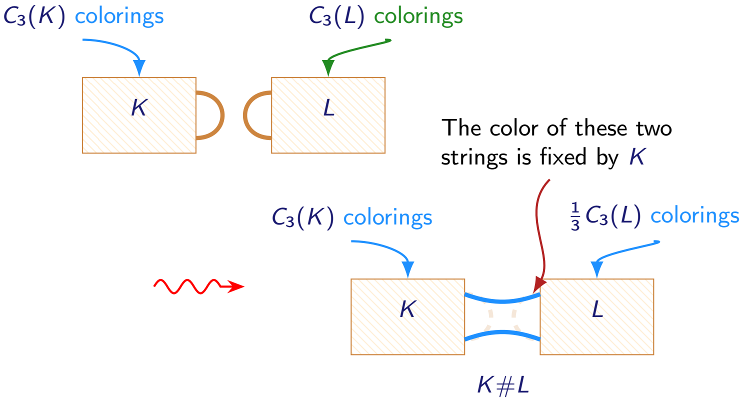

Since the colors of the connecting strands are fixed, there are only $\frac{1}{3}C_3(L)$ ways to 3-color the strands of L inside $K\#L$

How many knots are there?

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More generally, the same argument shows that if K is 3-colorable then the knots K, $\#^2 K$, $\#^3 K$,... are all inequivalent

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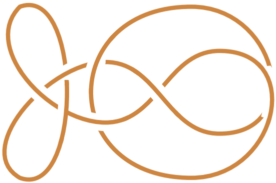

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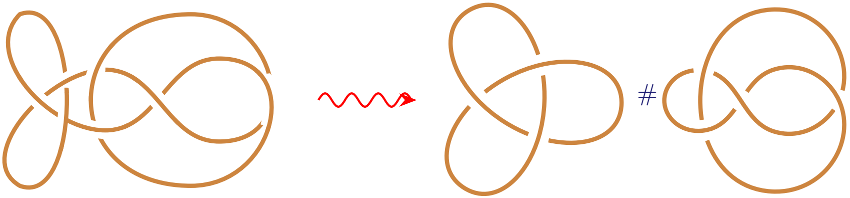

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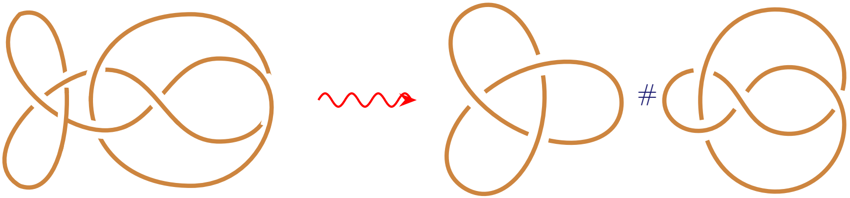

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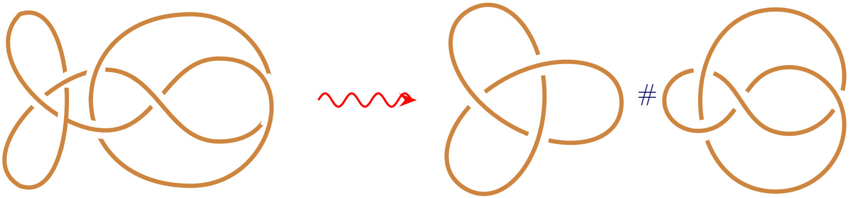

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In fact, we don't yet know that the figure eight knot is not the unknot!!

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Remark It is a big open question if $\mathrm{cross}(K\#L) = \mathrm{cross}(K) + \mathrm{cross}(L)$

This is only known to be true for certain types of knots such as alternating knots, which we will meet soon

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Conversely, we can ask how many prime knots there are

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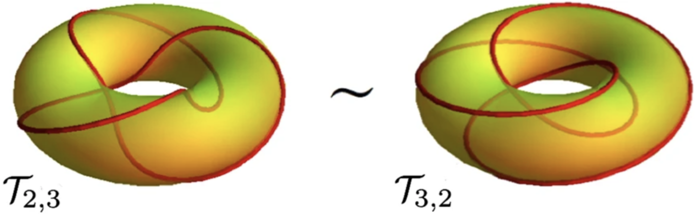

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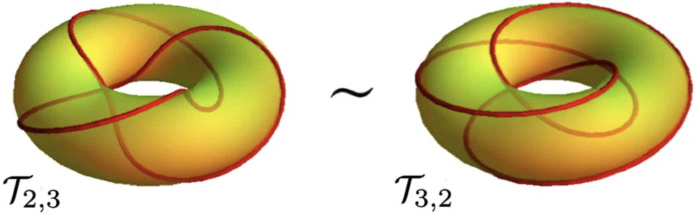

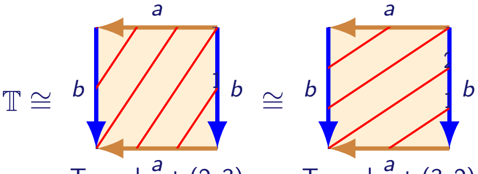

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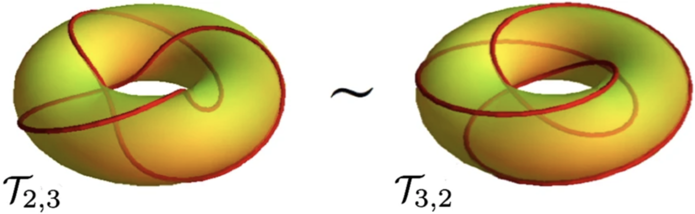

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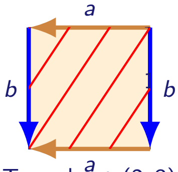

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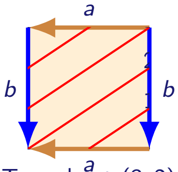

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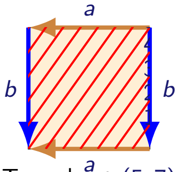

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Proof If $p < q$ then $\mathrm{cross}(\mathcal{T}_{p,q}) = (p-1)q$ — true but won't prove

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Proof If $p < q$ then $\mathrm{cross}(\mathcal{T}_{p,q}) = (p-1)q$ — true but won't prove

$\Longrightarrow$ the torus knots $\mathcal{T}_{2,q}$ with $q > 2$ odd are all inequivalent

Torus knots are prime knots

Theorem

Suppose that $\gcd(p,q)=1$. *Then the* (p,q)*-torus knot is prime*

This is intuitively clear because whenever we try to write a torus knot as the connected sum of two smaller knots, each of the smaller knots is the unknot; we sketch the proof momentarily

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The number of prime knots with n-crossings

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	0	1	1	2	3	7	21	49	165	552	2176	9988	46972

As is common, knots and their mirror images are only counted once

Torus knots are prime - proof sketch

Proof

For $p, g \geq 2$ let the (p, q)-torus knot K lie on an unknotted torus $T \subset S^3$ and let the 2-sphere S define a decomposition of K. We assume that S and T are in general position, that is, $S \cap T$ consists of finitely many disjoint simple closed curves.

Such a curve either meets K, is parallel to it or it bounds a disk D on T with $D \cap K = \emptyset$. Choose γ with $D \cap S = \partial D = \gamma$. Then γ divides S into two disks D', D'' such that $D \cup D'$ and $D \cup D''$ are spheres, $(\cup D') \cap (\cup D'') = D$; hence, D' or D'' can be deformed into D by an isotopy of S^3 which leaves K fixed. By a further small deformation we get rid of one intersection of S with T.

Torus knots are prime - proof sketch

Proof Continued

Consider the curves of $S \cap T$ which intersect K. There are one or two curves of this kind since K intersects S in two points only. If there is one curve it has intersection numbers $+1$ and -1 with K and this implies that it is either isotopic to K or nullhomotopic on T. In the first case K would be the trivial knot. In the second case it bounds a disk D_0 on T and $D_0 \cap T$, plus an arc on S, represents one of the factor knots of K; this factor would be trivial, contradicting the hypothesis.

Proof Continued

The case remains where $S \cap T$ consists of two simple closed curves intersecting K exactly once. These curves are parallel and bound disks in one of the solid tori bounded by T. But this contradicts $p, q \geq 2$

Prime factorisation of knots

Theorem

Suppose that K *is not the unknot. Then* $K = P_1 \# P_2 \# \ldots \# P_n$*, for prime knots* $P_1, \ldots, P_n$*. Moreover, the multiset of prime knots is a knot invariant*

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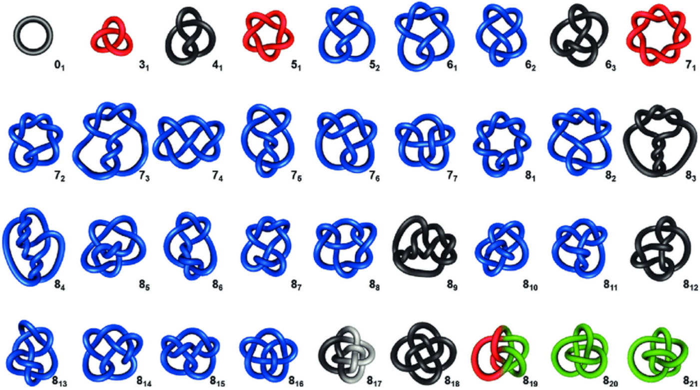
Here is a table of the unknot and the first 36 prime knots:



Colorable knots

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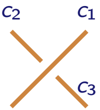
Question

What can we say about $c_1 + c_2 + c_3$ for a 3-coloring?



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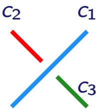
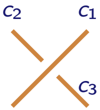


Possible colorings and the values of $c_1 + c_2 + c_3$

Allowed colorings



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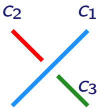
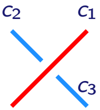


Disallowed colorings



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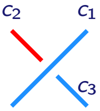

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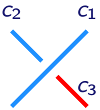
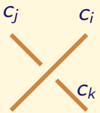


Knot colorings with p-colors

Definition

Let $p \in \mathbb{N}$. A p-coloring of a knot K is a coloring of the segments of K that using colors from $\{0, 1, \ldots, p-1\}$ such that



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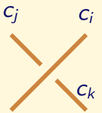

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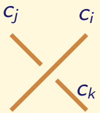

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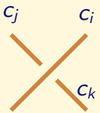

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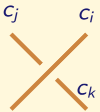

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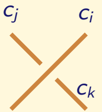

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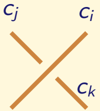

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$$\Longrightarrow \quad C_p(K) \geq p$$
$$\Longrightarrow \quad K \text{ is } p\text{-colorable if and only if } C_p(K) > p$$

Colorability is a knot invariant

Theorem

Suppose that $p \geq 3$. Then $C_p(K)$ and p-colorability are both knot invariants

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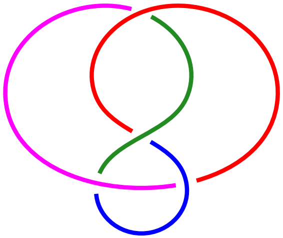
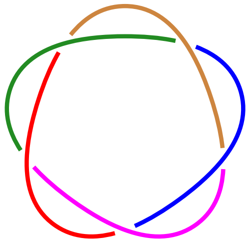
Question

Is there an easy way to tell if a knot is p-colorable?

Examples of p-colorings

Are the following knots 4-colorable, 5-colorable, ... ?





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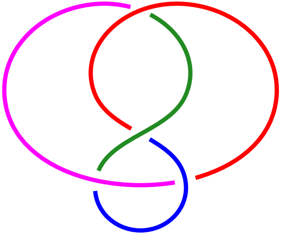

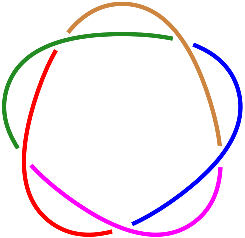

We need a better way to determine if a knot is p-colorable!

$\rightsquigarrow$ Use linear algebra!

The trefoil knot is knotted!

Corollary

The trefoil knot is not the unknot

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The trefoil knot is not equivalent to the figure eight knot

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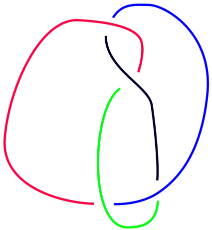
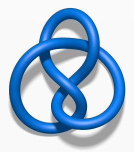


The trefoil knot in comparison

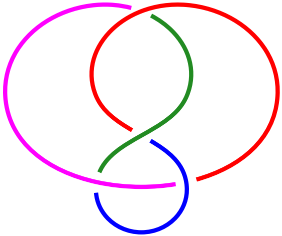
$\neq$

or



Colorful linear algebra

Consider the figure eight knot.



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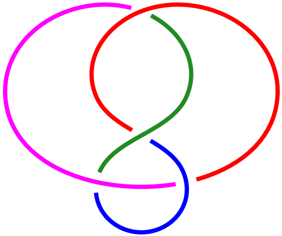
Label the segments c_1, c_2, c_3, c_4 in traveling order around the knot



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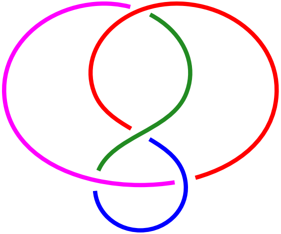

$\Longrightarrow$ We require:

$$\begin{array}{rrrrl} 2c_1 & & -c_3 & -c_4 & \equiv 0 \\ -c_2 & 2c_2 & & -c_4 & \equiv 0 \\ -c_1 & -c_2 & 2c_3 & & \equiv 0 \\ & -c_2 & -c_3 & 2c_4 & \equiv 0 \end{array}$$

In matrix form this becomes $M_K \underline{C} \equiv \underline{0} \pmod p$, where

$$M_K = \begin{bmatrix} 2 & 0 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & -1 & 2 & 0 \\ 0 & -1 & -1 & 2 \end{bmatrix} \quad \text{and } \underline{C} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

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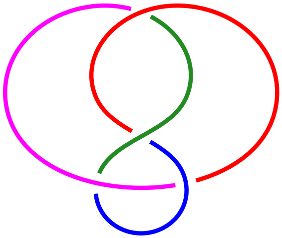

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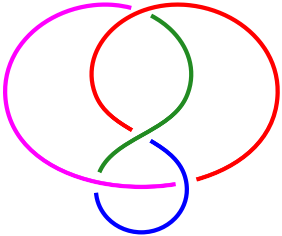

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We have reduced finding $c_1, \ldots, c_4$ to linear algebra!

The knot matrix

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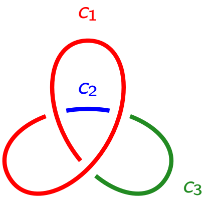
An atypical example



$$M_K = \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} 2-1 & -1 & 0 \\ 2 & -1 & -1 \\ 2-1 & 0 & -1 \end{bmatrix} \end{array}$$

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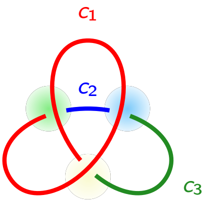
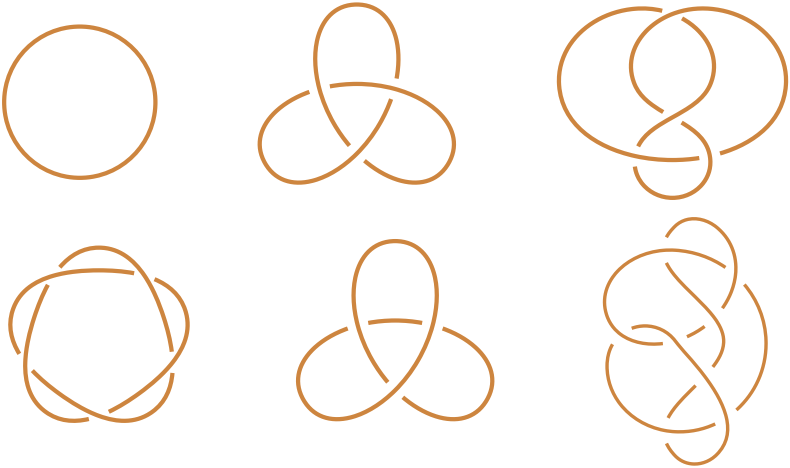


$$M_K = \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} 2-1 & -1 & 0 \\ 2 & -1 & -1 \\ 2-1 & 0 & -1 \end{bmatrix} \end{array}$$

We mainly consider colorings of alternating knots



Alternating knots

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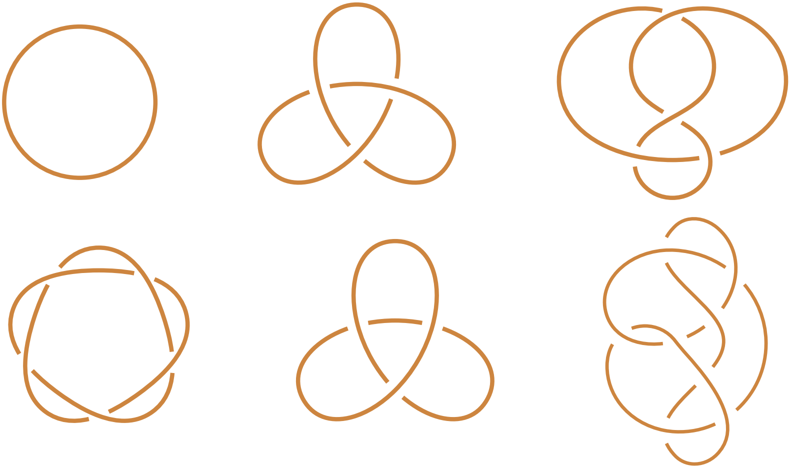
A knot projection is alternating if the crossings alternate between over and under crossings as you travel around the knot in an anti-clockwise direction



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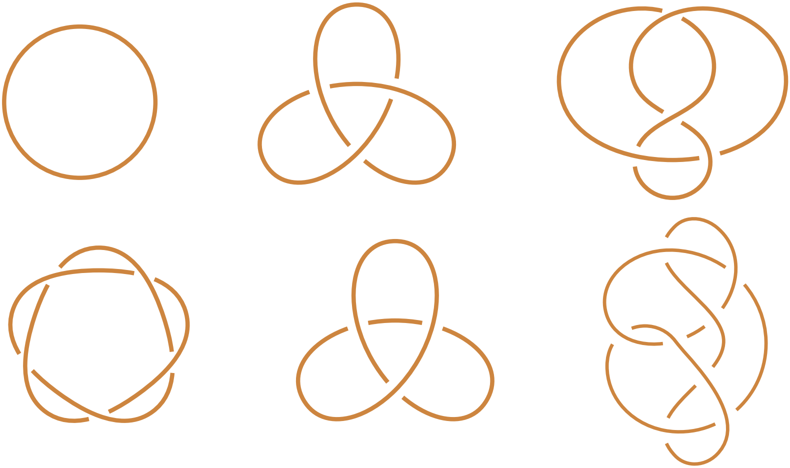

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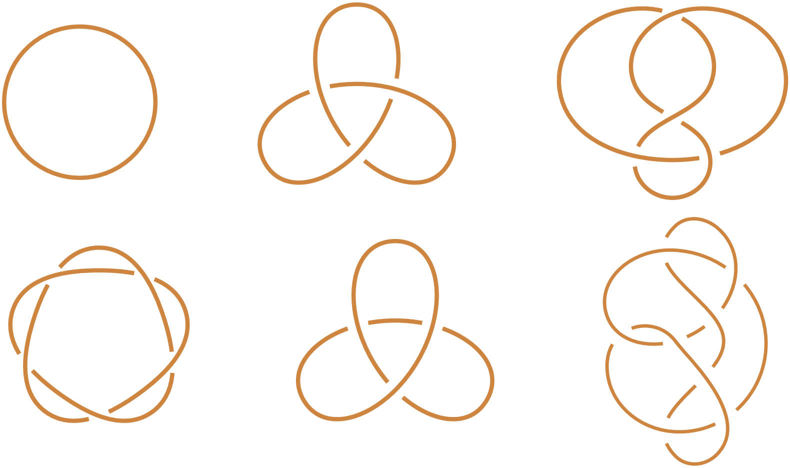
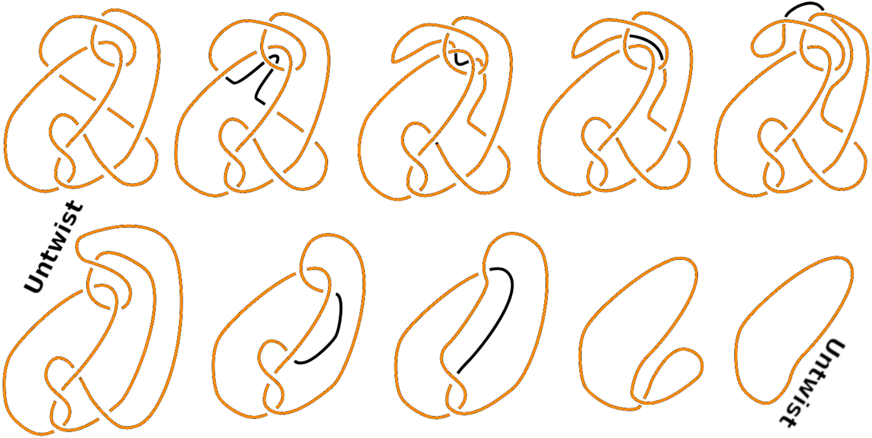


$\Longrightarrow$ Being alternating is not a knot invariant

The unknot is alternating, but it can have non-alternating projections:



Similarly, for other knots

If K is an alternating knot then:

Knot matrices for alternating knots

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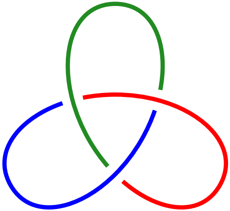
We will mainly consider colorings of alternating knots

Knot matrix examples

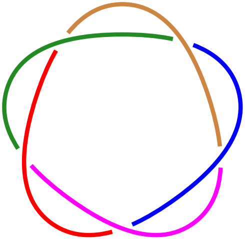
$$M_K = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$K =$



$$M_L = \begin{pmatrix} 2 & 0 & 0 & -1 & -1 \\ -1 & 2 & 0 & 0 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$L =$



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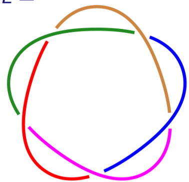
Proof

(1) Since the knot is alternating every colored strand contributes 2 once and -1 twice (see below) and dually from crossings

$$M_L = \begin{pmatrix} 2 & 0 & 0 & -1 & -1 \\ -1 & 2 & 0 & 0 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$L =$ 

Properties of the knot matrix

Proof Continued

(2) By (1), the respective vector is an eigenvector with eigenvalue zero

(3) By (2) there is an zero eigenvector, so the kernel is nontrivial

Minors of a matrix

The (r, c)-minor of an $n \times n$ matrix M is the $(n-1) \times (n-1)$-matrix M_{rc} obtained by deleting row r and column c from M)

$$M = \begin{bmatrix} a_{11} & \cdots\cdots & a_{1c} & \cdots\cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{r1} & \cdots\cdots & a_{rc} & \cdots\cdots & a_{rn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots\cdots & a_{nc} & \cdots\cdots & a_{nn} \end{bmatrix}$$

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Then $$\det(M + \mathbb{I}) = \det \begin{bmatrix} m_{11}+1 & m_{12}+1 & \cdots & m_{1n}+1 \\ m_{21}+1 & m_{22}+1 & \cdots & m_{1n}+1 \\ \vdots & \ddots & \ddots & \vdots \\ m_{n1}+1 & m_{n2}+1 & \cdots & m_{nn}+1 \end{bmatrix}$$

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By the same argument, if $1 \le r, c \le n$ then

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$$\iff \det(K) \neq 0 \pmod p$$

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3. If K is not alternating then the row sums of M_K are still 0. Therefore, the argument used to prove the theorem shows that K is p-colorable if and only if p divides $(M_K)_{rc}$, for some r, c.

Colorability of the figure eight knot

Summary of how to determine p-colorability

1. Label the segments in traveling order

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Colorability of the figure eight knot

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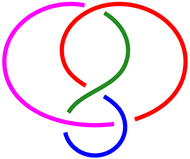
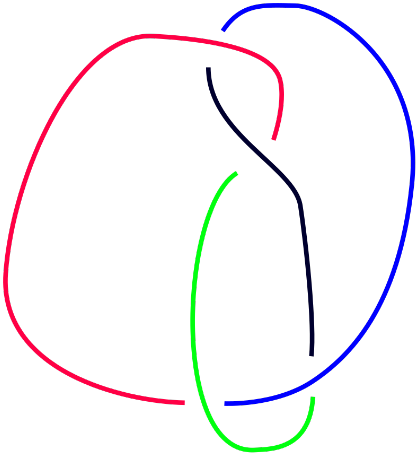
1. Label the segments in traveling order
2. Compute the entries of the knot matrix M_K
3. Compute the knot determinant $\det(K) = |\det(M_K)_{11}|$
4. Check if p divides $\det(K)$

$$M_K = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & 0 & 2 \\ 0 & 2 & -1 & -1 \end{pmatrix}$$

$K =$ 

The determinant is five, so the figure eight knot is five-colorable (and only five colorable)

Colorability of the figure eight knot – part 2



Thus, the figure eight knot is not trivial (it has strictly more than five 5-colorings) and also not the trefoil knot

Seifert surfaces

Definition

A Seifert surface for a knot K is an orientable surface that has K as its boundary

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We will prove this result by giving an algorithm for constructing a Seifert surface for any knot

Proof Real world version

Take a knot, build out of wire, and put it into soap

The minimal surface you get is a Seifert surface

Proof Math version

Step 1 Pick an orientation of the knot
That is, fix a direction to travel around the knot

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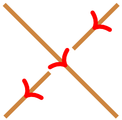
Step 2 At each crossing cut the over-string and join the incoming and outgoing strings; the knot is then a disjoint union of Seifert circles



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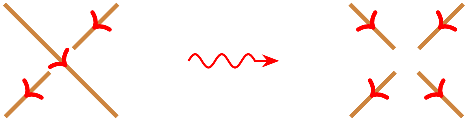

Constructing Seifert surfaces

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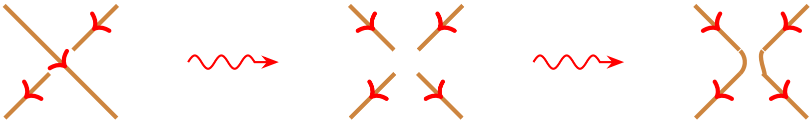

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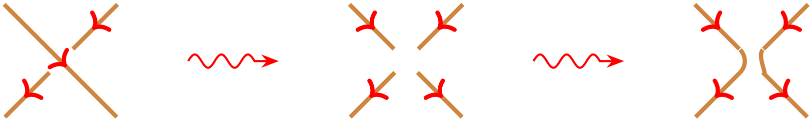

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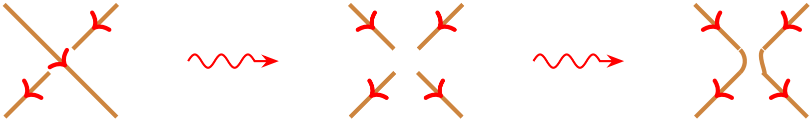

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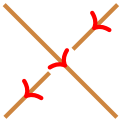
Step 4 Now each crossing in K, glue on a twisted strip that has the crossing as a boundary



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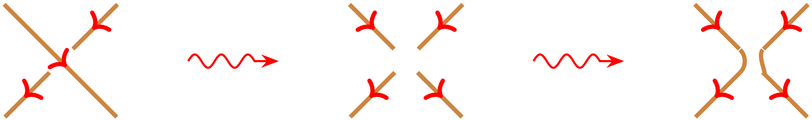

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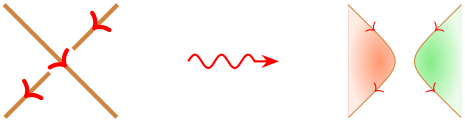

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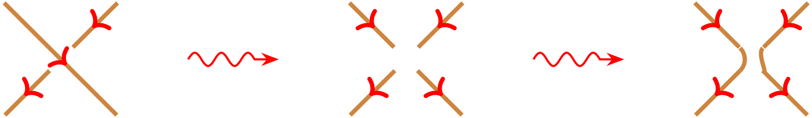

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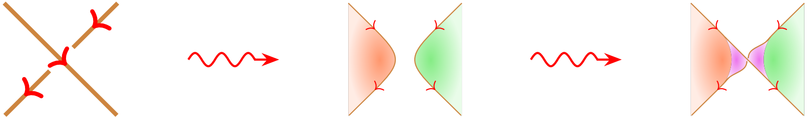
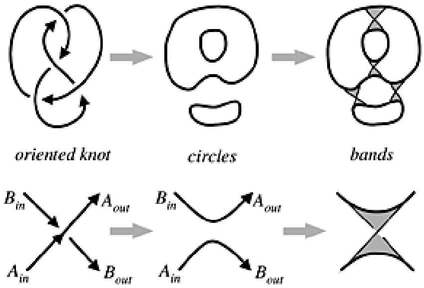
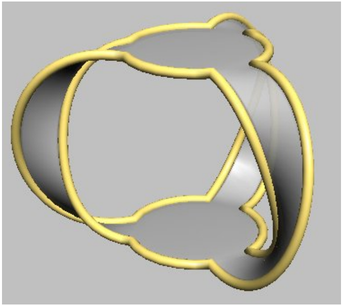
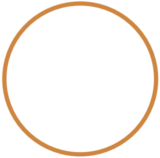


The platform construction





Examples of Seifert surfaces

- Unknot: 

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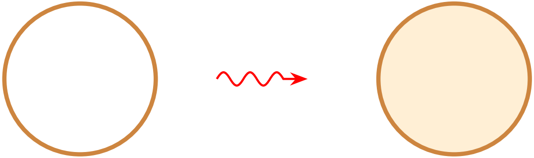
- Unknot: 

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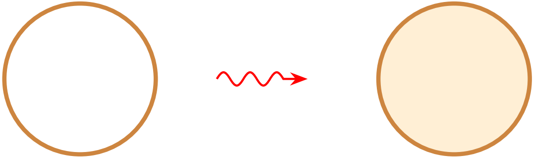
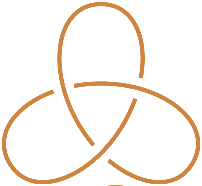
- Unknot: 
- Trefoil 
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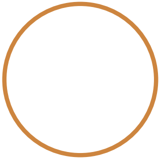
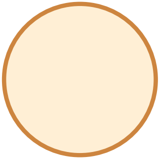
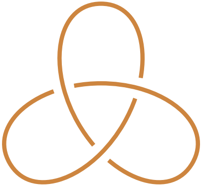
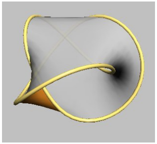




- Trefoil





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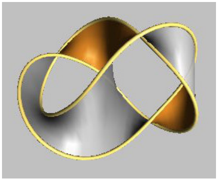

More examples of Seifert surfaces

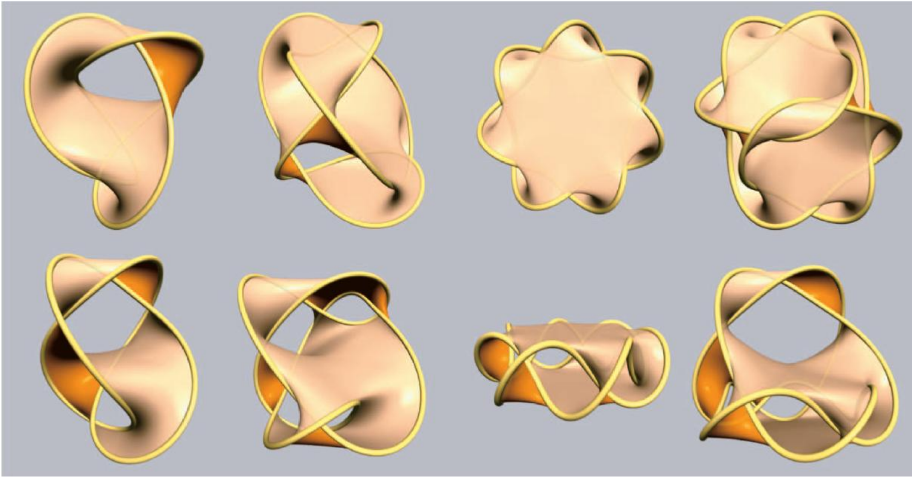

Figure 8$= 4_1$ $\quad 6_1 \quad 7_1 \quad 8_5$

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Definition

The genus of K is $g(K) = \min\left\{ \frac{1-\chi(S)}{2} \,\middle|\, S \text{ a Seifert surface of } K \right\}$

Remark Used to prove uniqueness of factorization of prime knots

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Fact $g(K) = 0 \iff K = \bigcirc$

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Fact $g(K) = 0 \iff K =$ 

Problem K is the trefoil:  . . . not very clear how to calculate $g(K)$!

Calculating the knot genus

Proposition

Let S be the Seifert surface with s Seifert circles that is constructed from a knot projection for a knot K with c crossings.
Then $\chi(S) = s - c$ and $g(K) \leq \frac{1+c-s}{2}$

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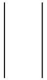
Proof Recall from tutorials that $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$

Write $S = A \cup B$, where A the union of the Seifert circles and B the union of the twists in S

$\Longrightarrow$ $A \cap B$ is a union of c pairs 

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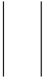

$$\Longrightarrow \quad \chi(S) = \chi(A) + \chi(B) - \chi(A \cap B) = s + c - 2c = s - c$$

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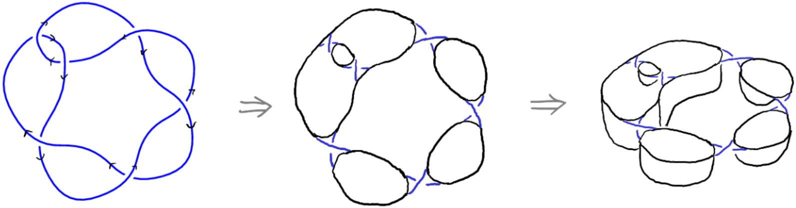
Hence, $g(K) \leq \frac{1-\chi(S)}{2} = \frac{1+c-s}{2}$

If K has c crossings and s Seifert circles then $g(K) \leq \frac{1+c-s}{2}$

So $g(K) \leq \frac{1+4-3}{2} = 1$



genus=1

Bad news: It can happen that $g(K) < \frac{1-\chi(S)}{2}$!!

Genus of alternating knots

Bad news: It can happen that $g(K) < \frac{1-\chi(S)}{2}$!!

The good news is that there is no bad news for alternating knots

Theorem

Let S be the Seifert surface constructed from an alternating knot projection of K. Then $g(K) = \frac{1-\chi(S)}{2}$

Proof Nontrivial and omitted!

Knot genus is additive

Theorem

Let K and L be knots. Then $g(K\#L) = g(K) + g(L)$

Start of proof It is not hard to see that $S_{K\#L} \cong S_K \#_{\text{strip}} S_L$ (connected sum along a strip connecting the surfaces and boundary cycles). This implies that $g(K\#L) \leq g(K) + g(L)$. The reverse implication is much harder!

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Corollary

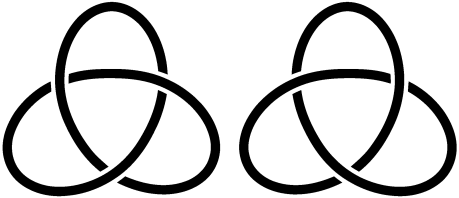
Let K and L be knots, which are not the unknot. Then $K \not\cong (K\#L)\#M$ *for any knot* M

Proof If such a knot M existed then

$$g(K) = g\big((K\#L)\#M\big) = g(K) + g(L) + g(M)$$
$$\implies \quad g(M) = -g(L) < 0 \qquad ↯↯↯$$

Left = right-handed trefoil? No idea...

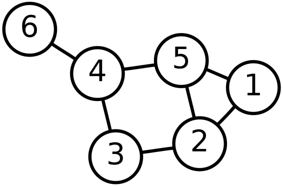
No method we have seen distinguishes these two fellows:

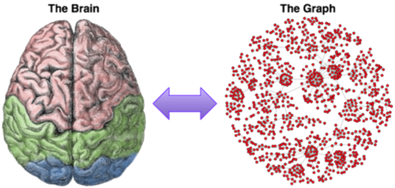


But that has to wait for another time...

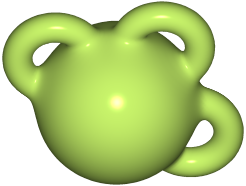
A few take away pictures

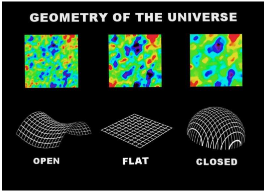
Topic 1: graphs!



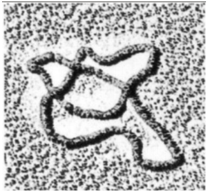


Topic 2: surfaces!





Topic 3: knots!



This was my last slide!