Topology – week 11

Math3061

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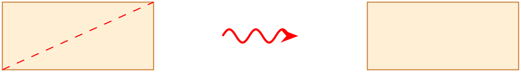
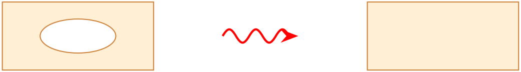
Map coloring assumptions

A map on a surface S is a polygonal subdivision such that:

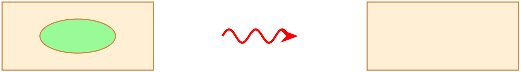
- All vertices have degree at least 3
- No region (i.e. face or polygon) has a border with itself



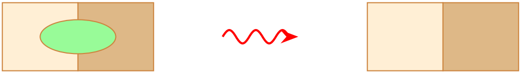
- No region contains a hole



- No region is completely surrounded by another



- No internal region has only two borders (i.e. edges)



The last three assumptions are purely for convenience because, in each case, we can color these maps using the same number of colors

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Moreover,

- $\partial_V \geq 3$ since vertices have degree at least 3
- $\partial_F \leq |F| - 1$ because no region borders itself
- If M is a map on a closed surface S, then we proved that

$$\partial_F \leq 6\Big(1 - \frac{\chi(S)}{|F|}\Big)$$

Lemma

Let M be a map on a closed surface S with $\chi(S) \leq 0$. Then

$$\partial_F \leq \frac{1}{2}\left(5 + \sqrt{49 - 24\chi(S)}\right)$$

Maps on surfaces with $\chi(S) \leq 0$

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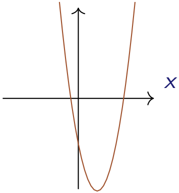
$y = x^2 - 5x + 6(\chi - 1)$



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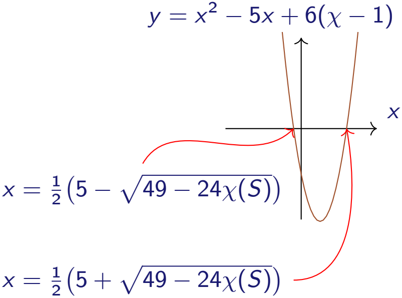

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$$\begin{aligned}
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\Longleftrightarrow \quad & \partial_F^2 - 5\partial_F + 6\big(\chi(S) - 1\big) \leq 0 \\
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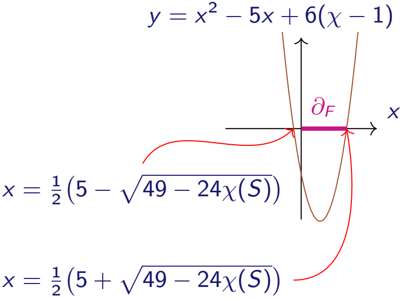

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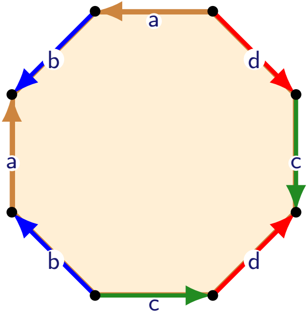

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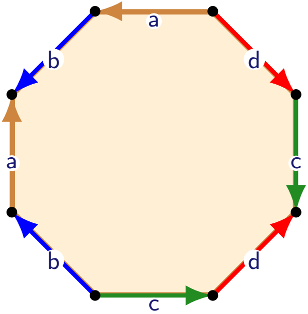

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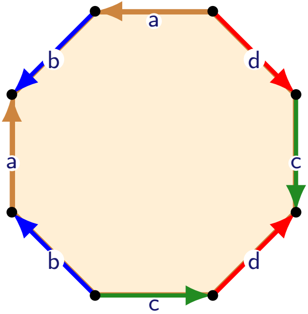

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This is not a contradiction because we are assuming that no region has a border with itself, which is never true for a polygonal decomposition that has only one face

Heawood's theorem

Theorem

Suppose that S is a closed surface. Then

$$C(S) \leq \begin{cases} 6, & \text{if } S = S^2 \text{ or } S = \mathbb{P}^2, \\ \frac{7+\sqrt{49-24\chi(S)}}{2}, & \text{otherwise} \end{cases}$$

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Since $\partial_F < c$ there is at least one face f with $\deg(f) < c$

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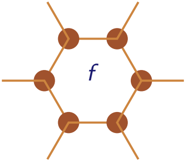

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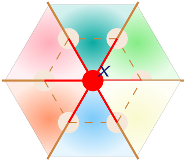

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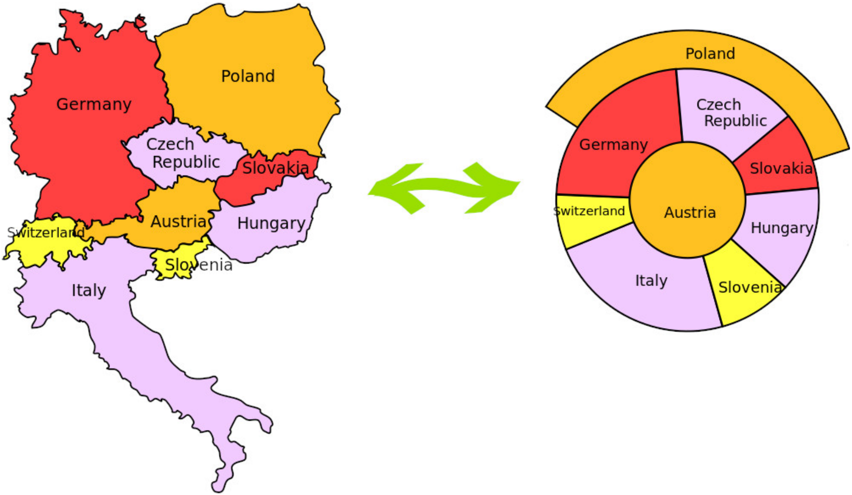
3. If $S = S^2$ then $\chi(S^2) = 2$ so $\frac{7+\sqrt{49-24\chi(S)}}{2} = 4$!?

Why is $C(S^2) \geq 4$ easy to see? Well:



Coloring the torus

Heawood's estimate for the torus is $C(\mathbb{T}) \leq \frac{7+\sqrt{49-24\chi(\mathbb{T})}}{2} \leq 7$

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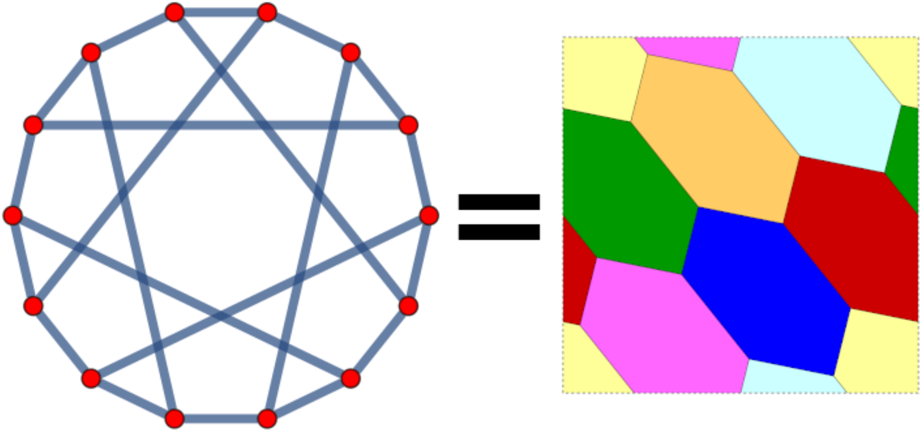
Hence, $C(\mathbb{T}) = 7$ (see the tutorials)

Hexagons on the torus



Coloring the projective plane

Heawood's estimate for the projective plane $\mathbb{P}^2$ is

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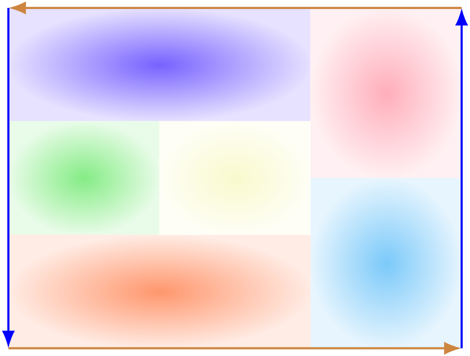
Here is a map on $\mathbb{P}^2$ that requires 6 colors:



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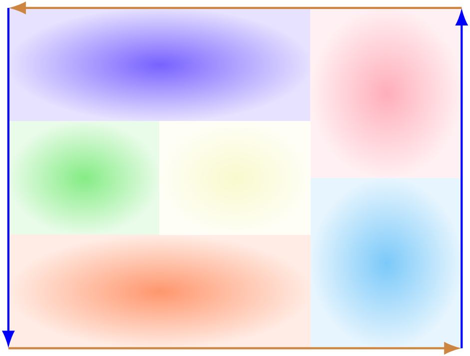

Hence, $C(\mathbb{P}^2) = 6$

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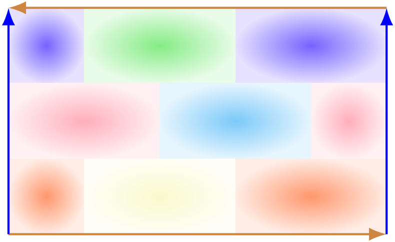
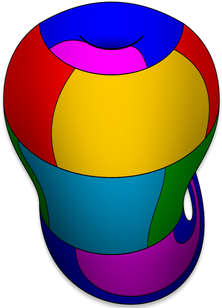
In fact, Franklin (1930) proved that $C(\mathbb{K}) = 6$





Using these maps you can show that $C(\mathbb{K}) \geq 6$

The four color theorem

Theorem

Every map on $\mathbb{D}^2$ *can be colored using four colors.*
That is, $C(\mathbb{D}^2) = C(\mathbb{R}^2) = C(S^2) = 4$

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By stereographic projection, it is enough to show that $C(S^2) \leq 5$

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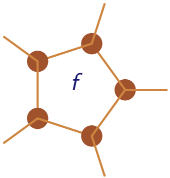

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As in the proof of Heawood's theorem, the idea is now to modify the 5-coloring on N to give a 5-coloring on M. This time the proof is more complicated and there are several cases to consider

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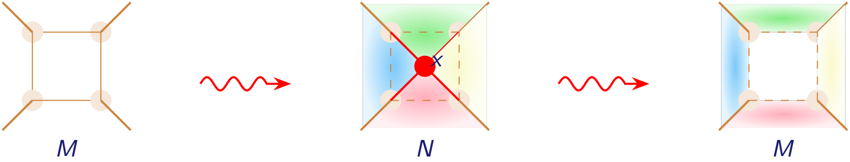

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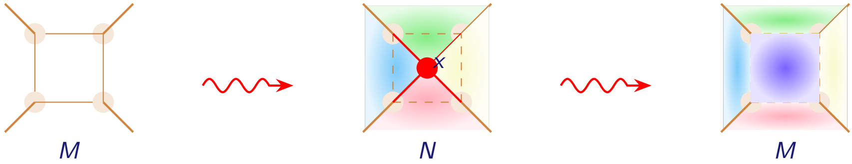

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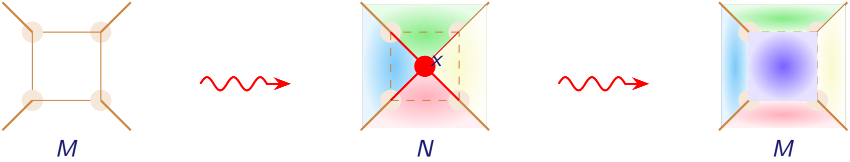

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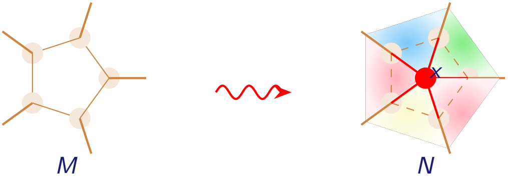

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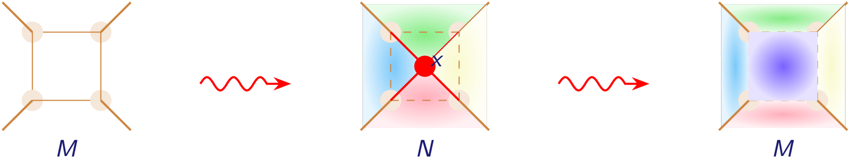

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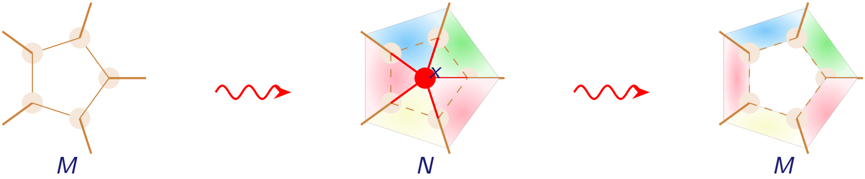

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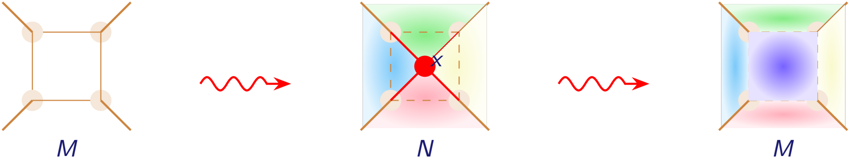

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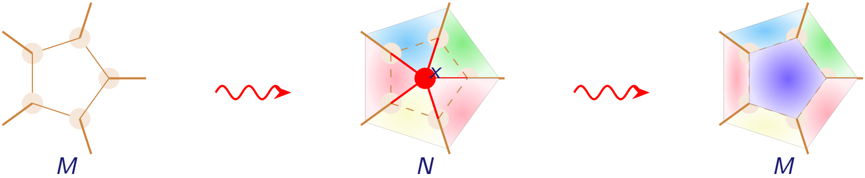
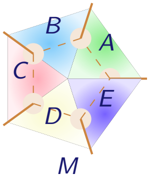


As we have used at most 4 colors in N around x, it follows that M is 5-colorable

Case 3: $\deg(f) = 5$ and all of the colors in N around x are different



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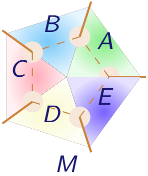

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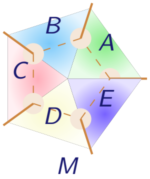

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Consider the polygonal decomposition P contained in N that has these five faces together with all of the regions in N that have the same colors as the faces A and C

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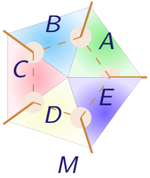

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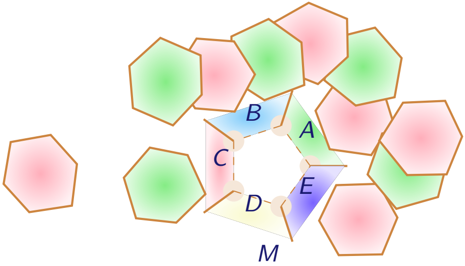

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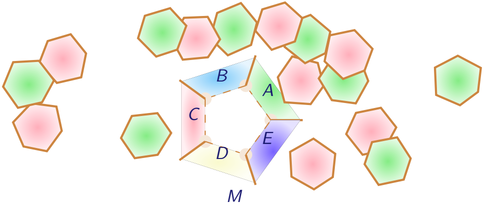

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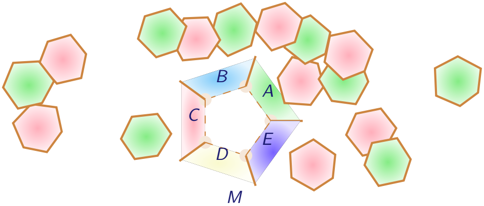
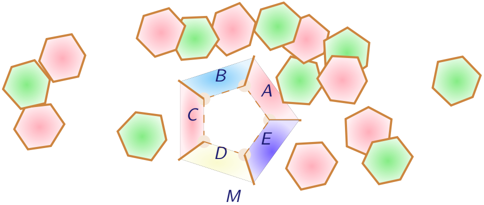


$\Longrightarrow$ Swapping the colors A and C in the connected component of P that contains A gives a new map N' with a valid coloring



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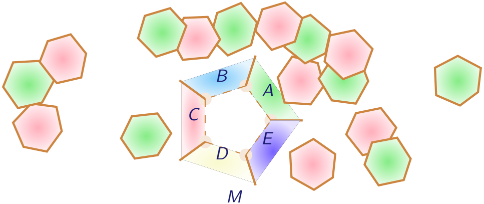

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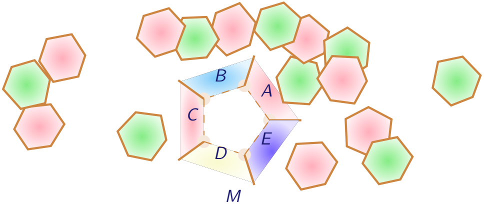

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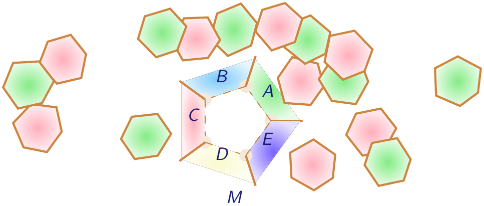

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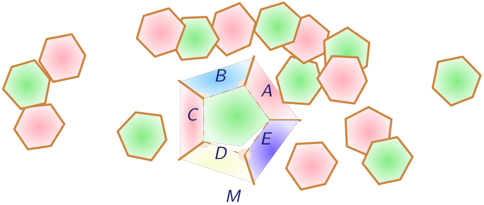

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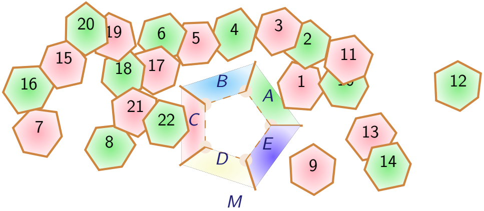

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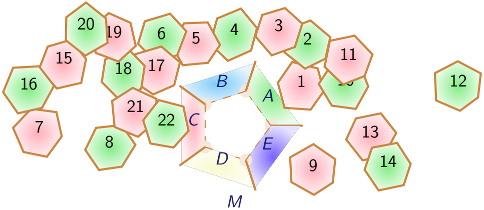

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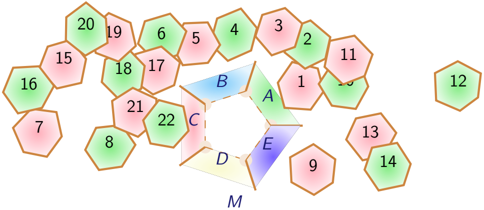

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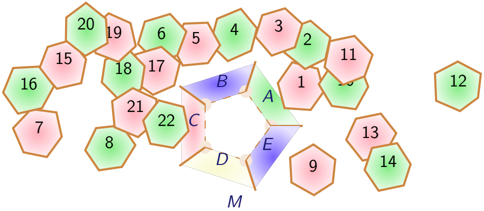

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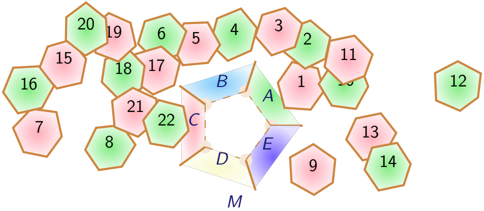

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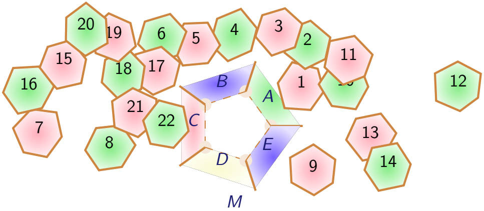

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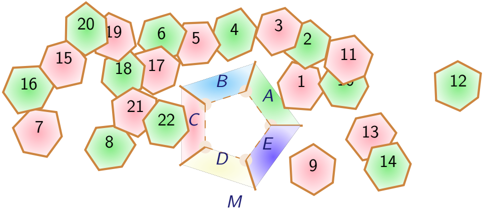

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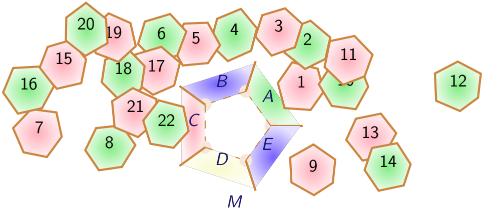

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This completes the proof of the Five color Theorem

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A knot is the image of an injective continuous map from S^1 into $\mathbb{R}^3$, where $S^1 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$ is the unit circle in $\mathbb{R}^2$

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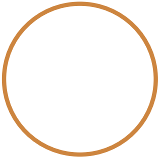
Equivalently, a knot is a closed path in $\mathbb{R}^3$ that has no self-intersections

Examples



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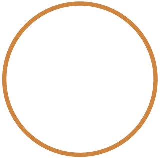

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Examples

Unknot   Trefoil   Reverse trefoil

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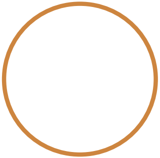

Unknot

Trefoil

Reverse trefoil

Heart knot

Knots

Intuitive definition A knot is a piece of string with the ends tied together

Definition

A knot is the image of an injective continuous map from S^1 into $\mathbb{R}^3$, where $S^1 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$ is the unit circle in $\mathbb{R}^2$

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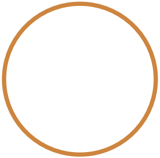
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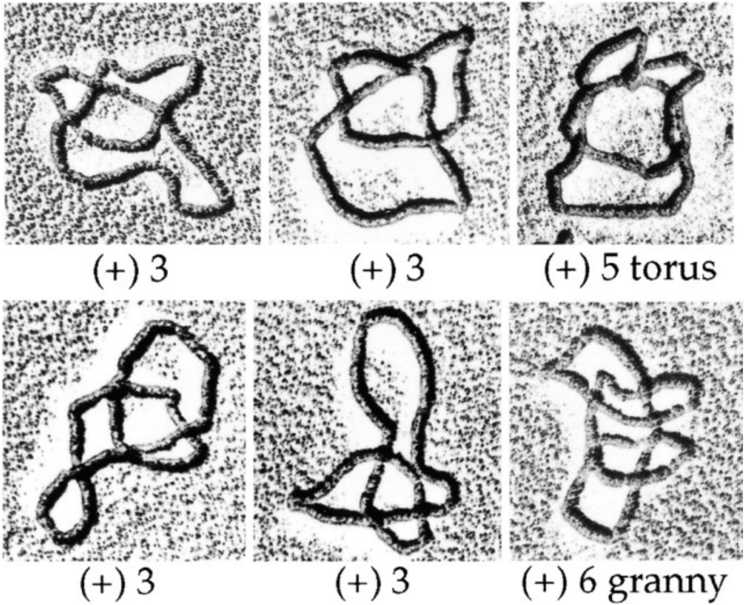
Heart knot

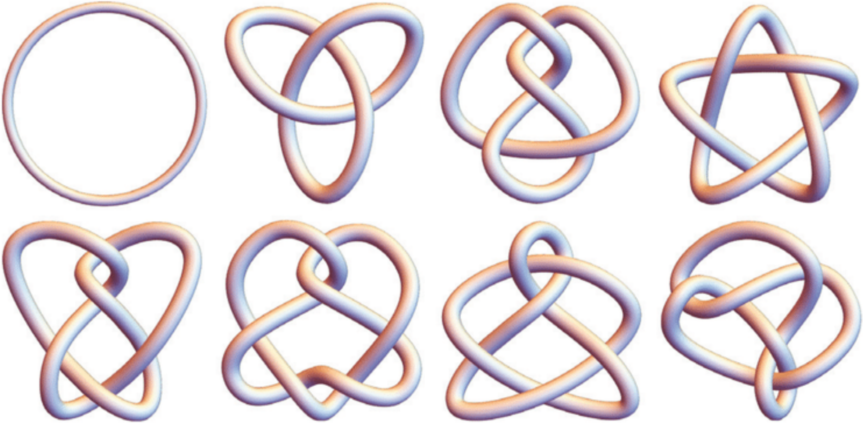
Knot theory is a beautiful mathematical subject with applications in mathematics, computer science, computer chip design, biology, ...

A picture of life



Another picture of life

More knots



Basic question in knot theory

Question

When is a knot the unknot?

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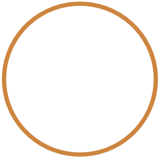

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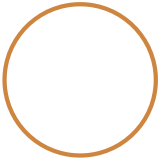

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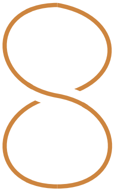

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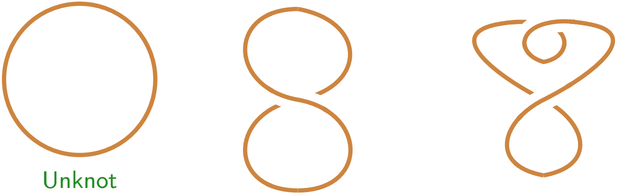

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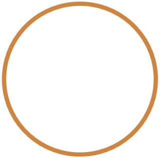

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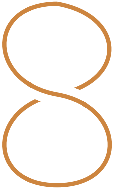

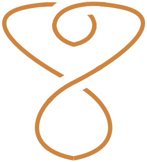

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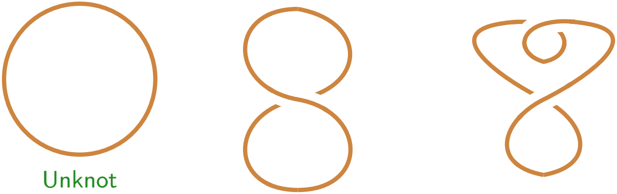

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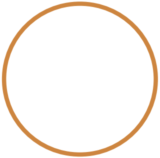

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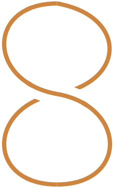
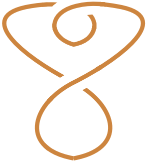
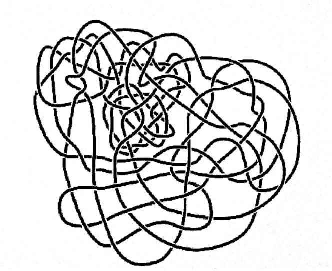




Another unknot

It is difficult to tell if a knot is the unknot



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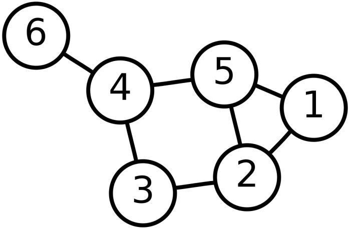
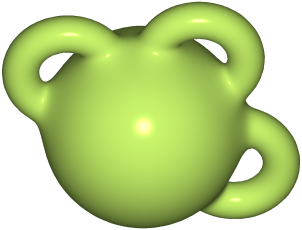
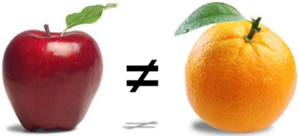
A knot K is trivial if it is equivalent to the unknot otherwise it is non-trivial

Objects	Graphs	Surfaces	Knots
Equivalence	Isomorphism of graphs	Homeomorphism	Equivalence of knots

In other words, graphs, surfaces and knots should never be directly compared – they are different beasts







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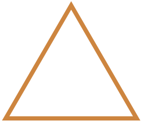

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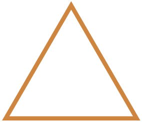

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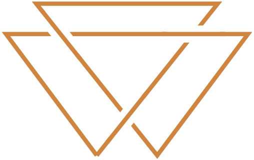

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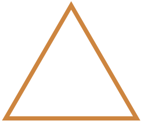

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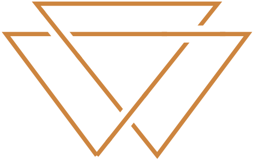

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Figure eight

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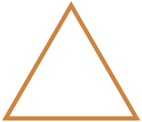

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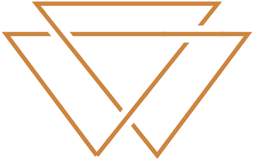

Trefoil

Figure eight

Remark Two polygonal knots K and L are equivalent if they have a common subdivision

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From now on all knots are polygonal knots and we drop the adjective polygonal

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This is not a huge restriction: anything you can draw is polygonal. Any "finite thing" is a polygonal knot, but "limits" are not so we ignore them

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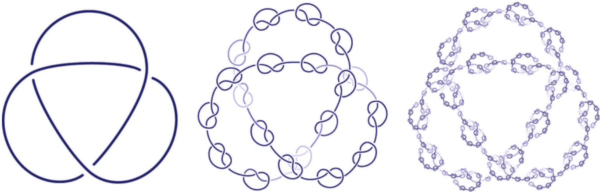
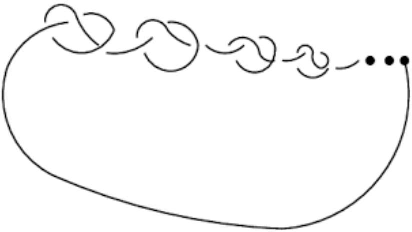
Good (but the limit is not):

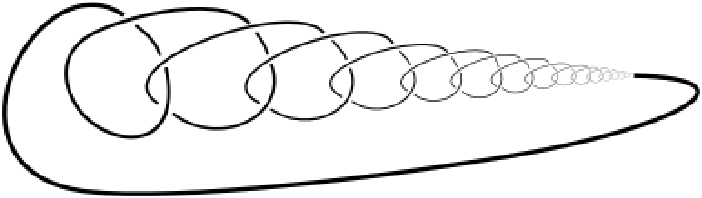


Not good:



Polygonal knots avoid pathologies

These are not polygonal knots:



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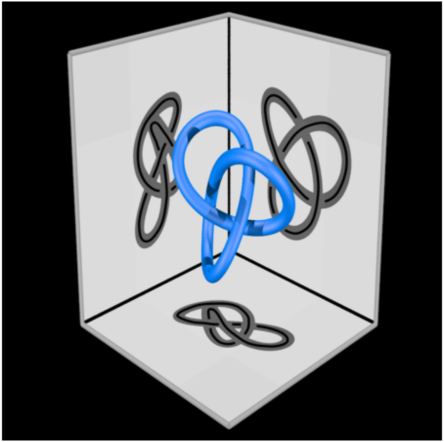
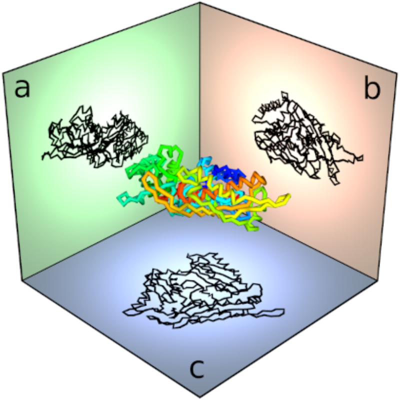
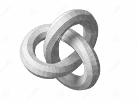
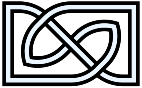
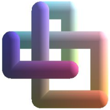
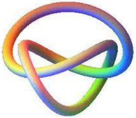
$\Longrightarrow$ Knot projections can be misleading so we have to check that our constructions are independent of the choice of knot projection

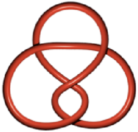
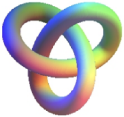
Projections = shadows





The trefoil knot times nine

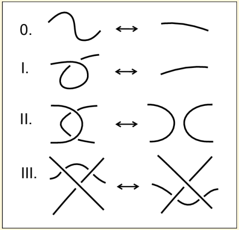






Reidemeister's theorem

Theorem

Two knot diagrams represent the same knot if and only if they are related by a (finite) sequence of moves of the following three types



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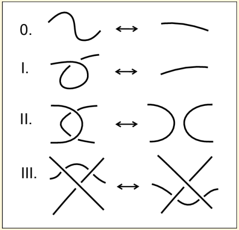

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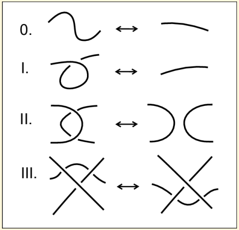

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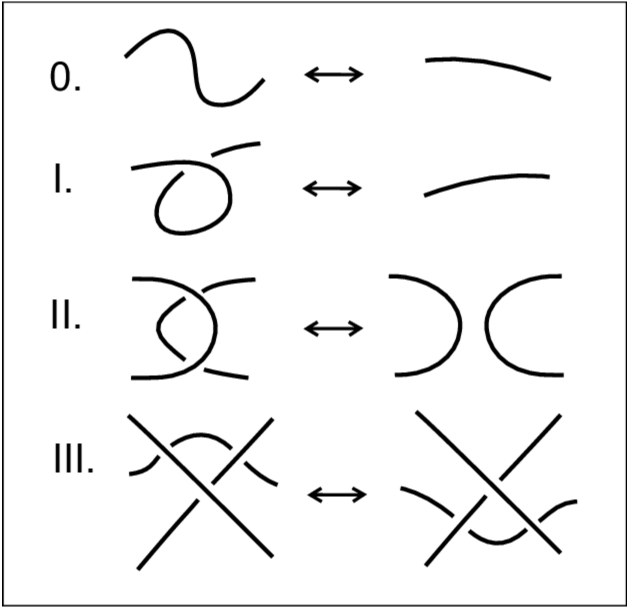
The point: Reidemeister's theorem reduces topology to combinatorics of diagrams

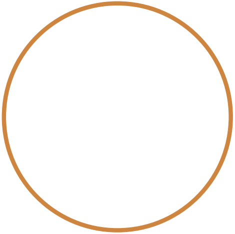
The Reidemeister moves on one slide

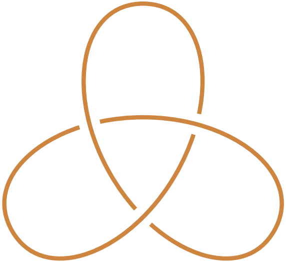


The knotty trefoil

Question

Is the trefoil knot equivalent to the unknot?





The knotty trefoil

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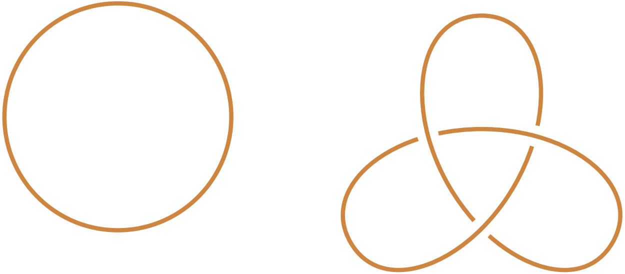

It seems clear that these two knots are different but, so far, we have not seen an easy way to distinguish between them

Knot colorings

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A coloring of a knot (projection) is the assignment of colors to the different segments,or connected components, so that at each crossing all segments have either the same color or they all have different colors and at least two colors are used

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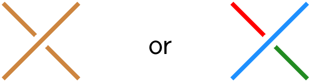

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Remark

- A knot can always be colored using a single color, so $C_3(K) \geq 3$ for all knots K
- As soon as more than one color is used we must use all three colors, so K is 3 colorable if and only if $C_3(K) > 3$

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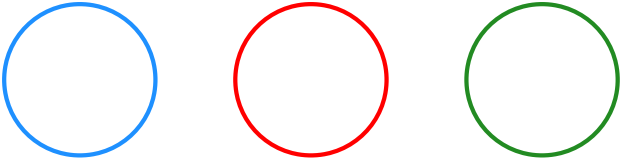

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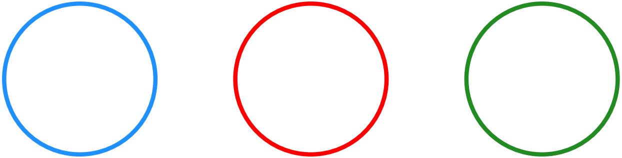

Which of the following are knots are 3-colorable?

coloring the trefoil knot

Question What is $C_3(T)$ if T is the trefoil knot?

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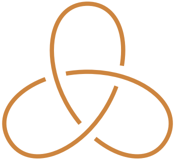

Claim $C_3(T) = 9$ since the components of T can be colored independently

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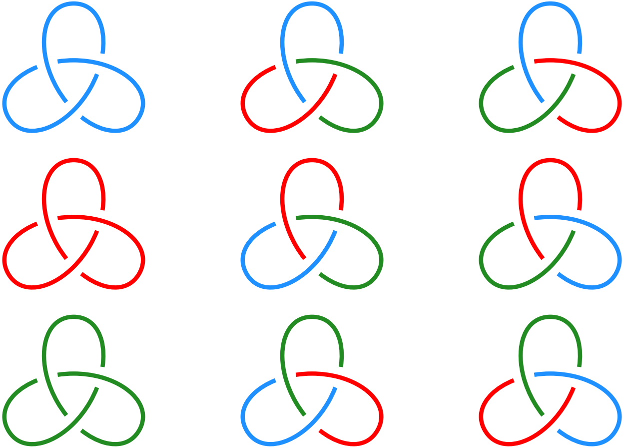

Three colorability

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The integer $C_3(K)$ is a knot invariant
That is, $C_3(K)$ depends only on K, up to ambient isotopy, and it is independent of the choice of knot projection

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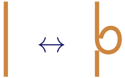
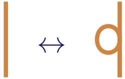
- Twisting  and 

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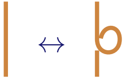
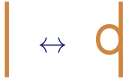
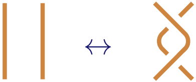
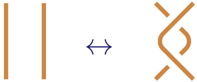
- Twisting  and 
- Looping  and 

Three colorability 2

- Braiding

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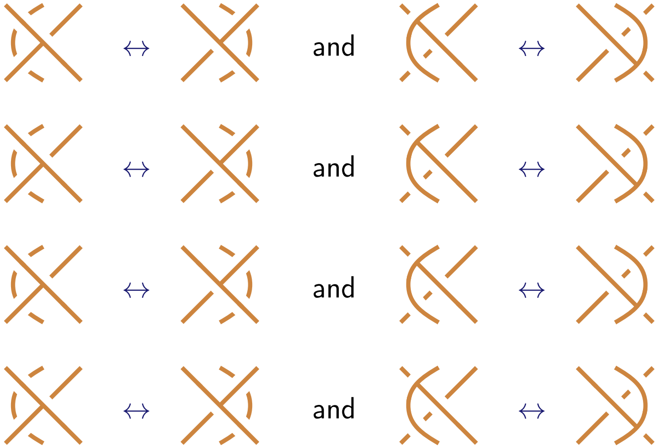

Key point For each Reidemeister move there is a unique way to complete any coloring given the existing colors of the segments going in and out