Topology – week 10

Math3061

Daniel Tubbenhauer, University of Sydney

Words for surfaces

A polygonal decomposition for a surface that has one face can be encoded in a word

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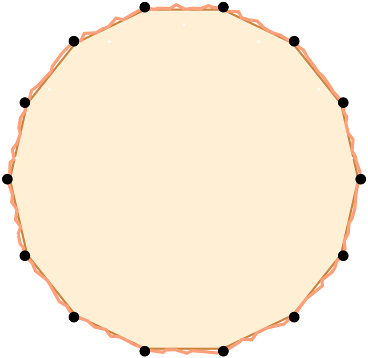

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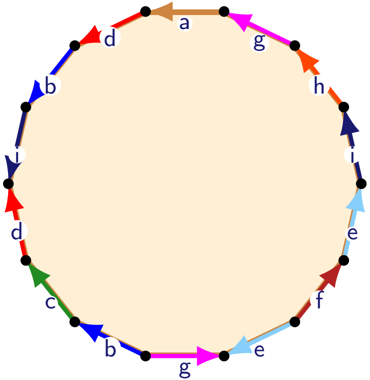

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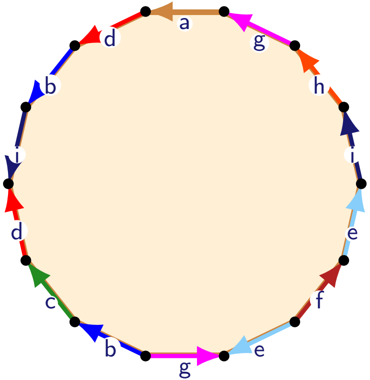

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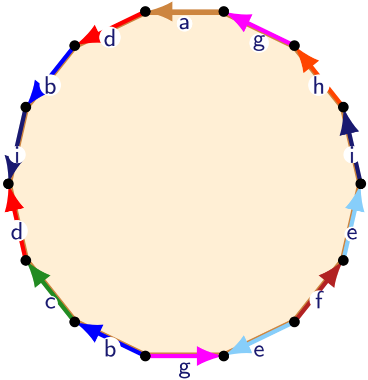
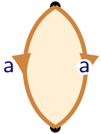


- write x for an edge pointing anticlockwise
- write $\overline{x}$ for an edge pointing clockwise
- We always read the word in anticlockwise order

Words for basic surfaces

- $S^2 =$ 

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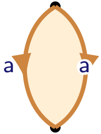
- $S^2 =$ 

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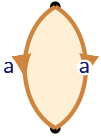
- $S^2 =$ 

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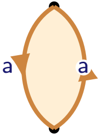
- $\mathbb{P}^2 =$ 

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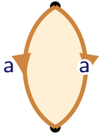
- $S^2 =$ 

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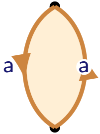
- $\mathbb{P}^2 =$ 

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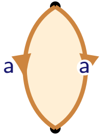
- $S^2 =$ 

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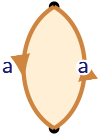
- $\mathbb{P}^2 =$ 

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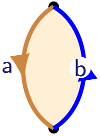
- $\mathbb{D}^2 =$ 

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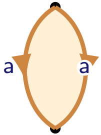
- $S^2 =$ 

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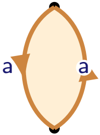
- $\mathbb{P}^2 =$ 

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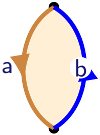
- $\mathbb{D}^2 =$ 

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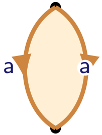
- $S^2 =$ 

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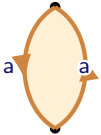
- $\mathbb{P}^2 =$ 

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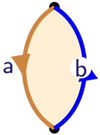
- $\mathbb{D}^2 =$ 

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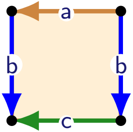
- $\mathbb{A} =$ 

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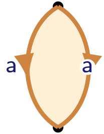
- $S^2 =$ 

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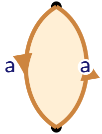
- $\mathbb{P}^2 =$ 

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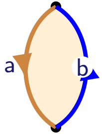
- $\mathbb{D}^2 =$ 

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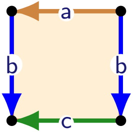
- $\mathbb{A} =$ 

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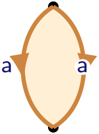
- $S^2 =$ 

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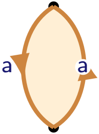
- $\mathbb{P}^2 =$ 

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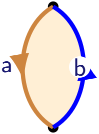
- $\mathbb{D}^2 =$ 

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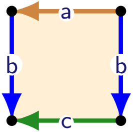
- $\mathbb{A} =$ 

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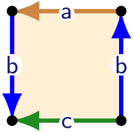
- $\mathbb{M} =$ 

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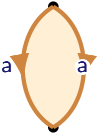
- $S^2 =$ 

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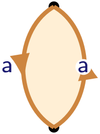
- $\mathbb{P}^2 =$ 

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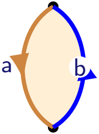
- $\mathbb{D}^2 =$ 

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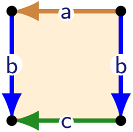
- $\mathbb{A} =$ 

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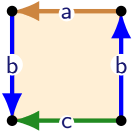
- $\mathbb{M} =$ 

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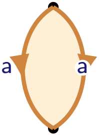
- $S^2 =$ 

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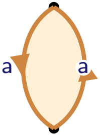
- $\mathbb{P}^2 =$ 

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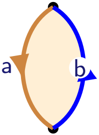
- $\mathbb{D}^2 =$ 

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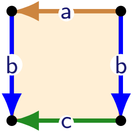
- $\mathbb{A} =$ 

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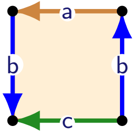
- $\mathbb{M} =$ 

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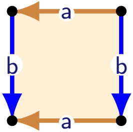
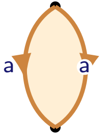
- $\mathbb{T} =$ 

- $S^2 =$ 

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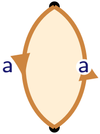
- $\mathbb{P}^2 =$ 

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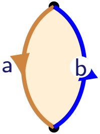
- $\mathbb{D}^2 =$ 

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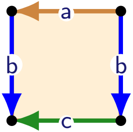
- $\mathbb{A} =$ 

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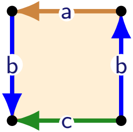
- $\mathbb{M} =$ 

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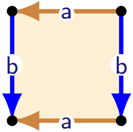
- $\mathbb{T} =$ 

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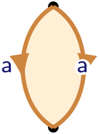
Words for basic surfaces

- $S^2 =$ 

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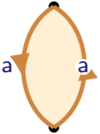
- $\mathbb{P}^2 =$ 

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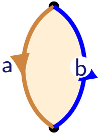
- $\mathbb{D}^2 =$ 

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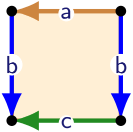
- $\mathbb{A} =$ 

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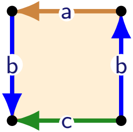
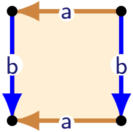
- $\mathbb{M} =$ 

$= a\ b\ \bar{c}\ b$
- $\mathbb{T} =$ 

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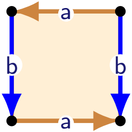
- $\mathbb{K} =$ 

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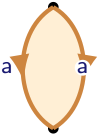
- $S^2 =$ 

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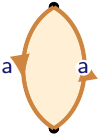
- $\mathbb{P}^2 =$ 

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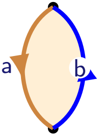
- $\mathbb{D}^2 =$ 

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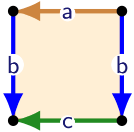
- $\mathbb{A} =$ 

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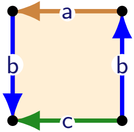
- $\mathbb{M} =$ 

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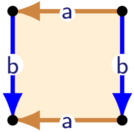
- $\mathbb{T} =$ 

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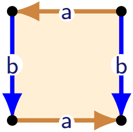
- $\mathbb{K} =$ 

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Properties of words

- Words encode orientability
  - Orientable: $\ldots a \ldots \overline{a} \ldots$ or $\ldots \overline{a} \ldots a \ldots$
  - Non-orientable: $\ldots a \ldots a \ldots$ or $\ldots \overline{a} \ldots \overline{a} \ldots$

Properties of words

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- The word of a surface is well-defined only up to cyclic permutation and reversing the direction of any edge

Properties of words

- Words encode orientability
  - Orientable: $\dots a \dots \overline{a} \dots$ or $\dots \overline{a} \dots a \dots$
  - Non-orientable: $\dots a \dots a \dots$ or $\dots \overline{a} \dots \overline{a} \dots$

- Words give a compact and easily readable way of describing surfaces
- Words can be read in clockwise or anticlockwise order
  (we always read in anticlockwise order)
- The word of a surface is well-defined only up to cyclic permutation and reversing the direction of any edge

Example The following words are all words for the torus $\mathbb{T}$:

$$a\ b\ \overline{a}\ \overline{b} \qquad b\ \overline{a}\ \overline{b}\ a \qquad \overline{a}\ \overline{b}\ a\ b \qquad \overline{b}\ a\ b\ \overline{a}$$

$$a\ \overline{b}\ \overline{a}\ b \qquad \overline{b}\ \overline{a}\ b\ a \qquad \overline{a}\ b\ a\ \overline{b} \qquad b\ a\ \overline{b}\ \overline{a}$$

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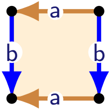
$$\begin{array}{cccc} a\,b\,\overline{a}\,\overline{b} & b\,\overline{a}\,\overline{b}\,a & \overline{a}\,\overline{b}\,a\,b & \overline{b}\,a\,b\,\overline{a} \\ a\,\overline{b}\,\overline{a}\,b & \overline{b}\,\overline{a}\,b\,a & \overline{a}\,b\,a\,\overline{b} & b\,a\,\overline{b}\,\overline{a} \end{array}$$

- The word of a surface can be used to give generators and relations for the first homotopy group of the surface — this generalises independent cycles and are beyond the scope of this unit

- Connected sums of tori: $\#^t \mathbb{T}$
  - $\mathbb{T} =$  $= a\ b\ \overline{a}\ \overline{b}$

Standard words for closed orientable surfaces

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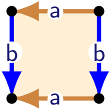
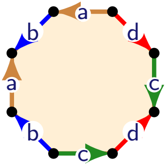
  - $\mathbb{T} =$  $= a\ b\ \overline{a}\ \overline{b}$

  - $\#^2\mathbb{T} =$  $= a\ b\ \overline{a}\ \overline{b}\ c\ d\ \overline{c}\ \overline{d}$

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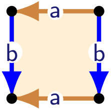
  - $\mathbb{T} =$ 

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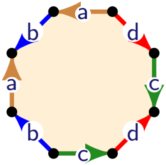
  - $\#^2\mathbb{T} =$ 

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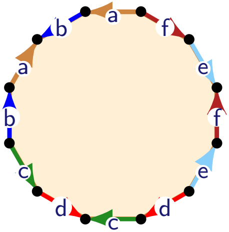
  - $\#^3\mathbb{T} =$ 

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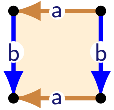
  - $\mathbb{T} =$ 

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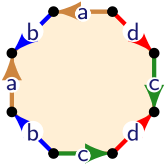
  - $\#^2\mathbb{T} =$ 

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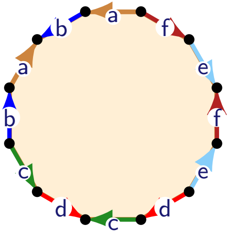
  - $\#^3\mathbb{T} =$ 

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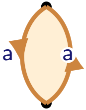
  - ... $\#^t\mathbb{T} = a_1\ b_1\ \overline{a_1}\ \overline{b_1}\ a_2\ b_2\ \overline{a_2}\ \overline{b_2}\ \ldots\ a_t\ b_t \overline{a_t}\ \overline{b_t}$

Words for closed non-orientable surfaces

- Connected sums of projective planes $\#^p\mathbb{P}^2$
  - $\mathbb{P}^2 =$  $= a\ a$

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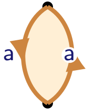
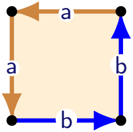
  - $\mathbb{P}^2 =$ 

$= a\ a$
  - $\#^2\mathbb{P}^2 =$ 

$= a\ a\ b\ b$

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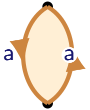
  - $\mathbb{P}^2 =$ 

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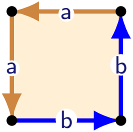
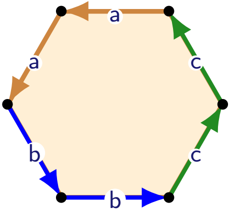
  - $\#^2\mathbb{P}^2 =$ 

$= a\ a\ b\ b$
  - $\#^3\mathbb{P}^2 =$ 

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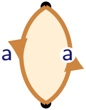
  - $\mathbb{P}^2 =$ 

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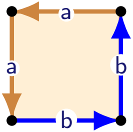
  - $\#^2\mathbb{P}^2 =$ 

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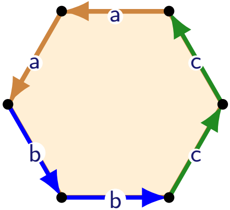
  - $\#^3\mathbb{P}^2 =$ 

$= a\ a\ b\ b\ c\ c$
  - ... $\#^p\mathbb{P}^2 = a_1\ a_1\ a_2\ a_2\ \dots\ a_p\ a_p$

Standard words for surfaces with boundary

- $\#^d \mathbb{D}^2$

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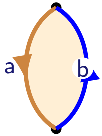
  - $\mathbb{D}^2 =$  $= a\ b$

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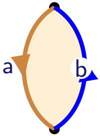
  - $\mathbb{D}^2 =$ 

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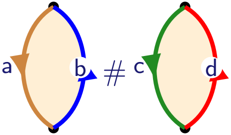
  - $\#^2 \mathbb{D}^2 =$ 

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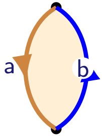
  - $\mathbb{D}^2 =$ 

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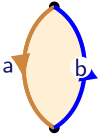
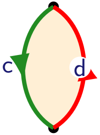
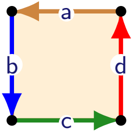
  - $\#^2\mathbb{D}^2 =$ 

$\#$ 

$\stackrel{?}{\cong}$ 

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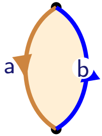
  - $\mathbb{D}^2 =$ 

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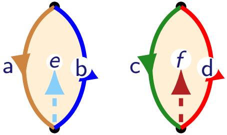
  - $\#^2 \mathbb{D}^2 =$ 

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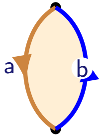
  - $\mathbb{D}^2 =$ 

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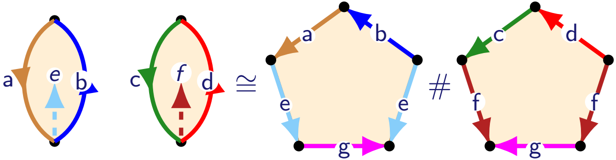
  - $\#^2 \mathbb{D}^2 =$ 

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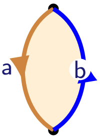
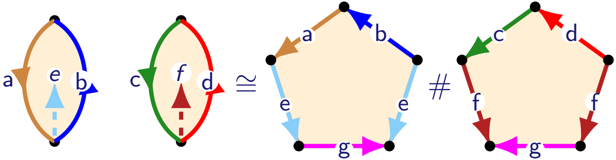
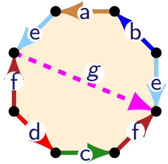
  - $\mathbb{D}^2 =$  $= a\ b$

  - $\#^2 \mathbb{D}^2 =$ 

$\cong$ 

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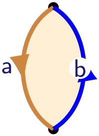
  - $\mathbb{D}^2 =$ 

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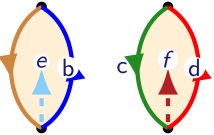
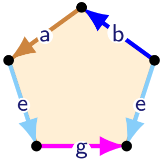
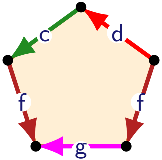
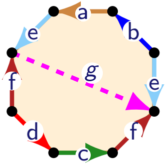
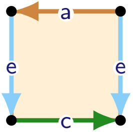
  - $\#^2\mathbb{D}^2 =$ 

$\cong$ 

$\#$ 

$\cong$ 

$\cong$ 

$= a\ e\ c\ \overline{e}$

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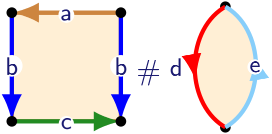
- $\#^3\mathbb{D}^2 = \#^2\mathbb{D}^2 \# \mathbb{D}^2 \cong$ 

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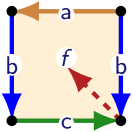
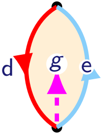
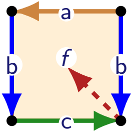
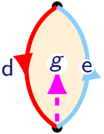
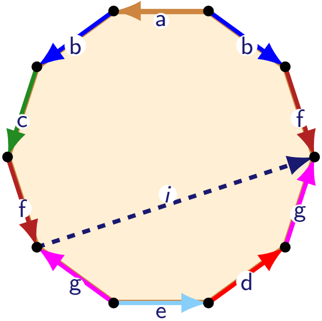
- $\#^3\mathbb{D}^2 = \#^2\mathbb{D}^2 \# \mathbb{D}^2 \cong$ 

$\#$ 

- $\#^3\mathbb{D}^2 = \#^2\mathbb{D}^2 \# \mathbb{D}^2 \cong$ 

$\#$ 

$\cong$ 

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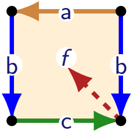
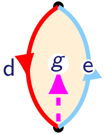
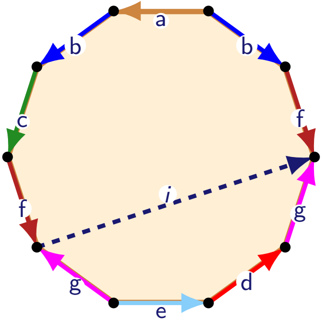
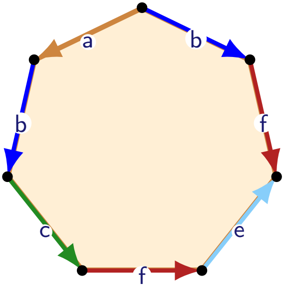
- $\#^3\mathbb{D}^2 = \#^2\mathbb{D}^2 \# \mathbb{D}^2 \cong$ 

$\#$ 

$\cong$ 

$\cong$ 

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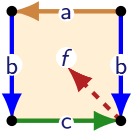
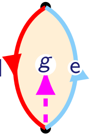
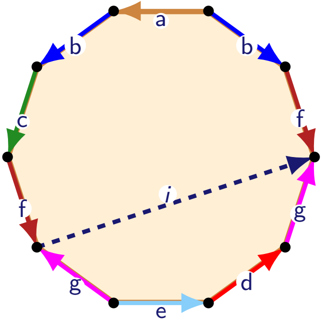
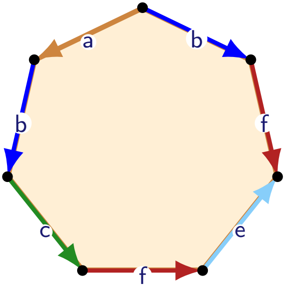
- $\#^3\mathbb{D}^2 = \#^2\mathbb{D}^2 \, \# \, \mathbb{D}^2 \cong$ 

$\#$ 

$\cong$ 

$\cong$ 

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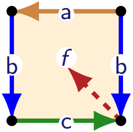
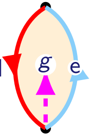
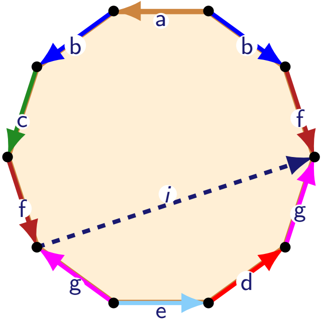
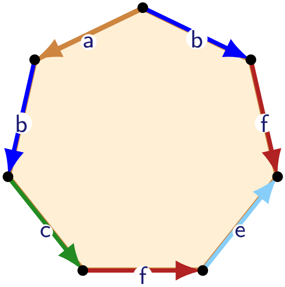
- $\#^3\mathbb{D}^2 = \#^2\mathbb{D}^2 \,\#\, \mathbb{D}^2 \cong$ 

$\#$ 

  $\cong$ 

$\cong$ 

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- $\#^d\mathbb{D}^2 = a_1\ b_1\ a_2\ b_2\ \ldots\ b_{d-1} a_d\ \overline{b}_{d-1}\ \ldots \overline{b}_2\ \overline{b}_1$

Words to surfaces

What standard surface is given by the word $a\ d\ b\ \overline{c}\ e\ \overline{c}\ \overline{f}\ d\ h\ g\ e\ \overline{a}\ h$?

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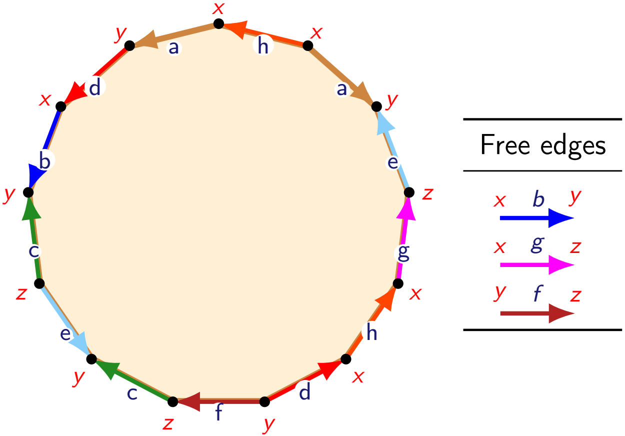

$\implies$ $d = 1$ and $\chi(S) = 3 - 8 + 1 = -4$

$\implies$ $S \cong \mathbb{D}^2 \# \#^5 \mathbb{P}^2$

$\implies$ $S = a\ b\ b\ c\ c\ d\ d\ e\ e\ f\ f$

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Answer Yes and no!

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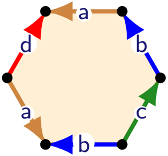

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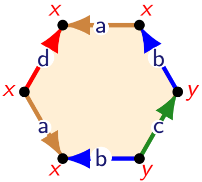

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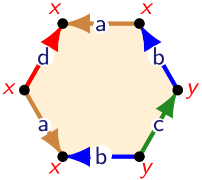

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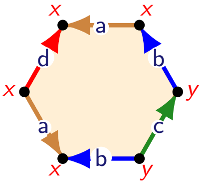

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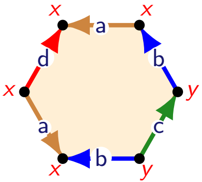

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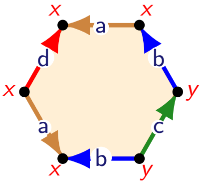

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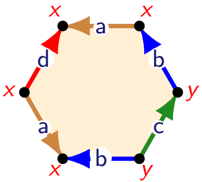

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The vertex-degree equation holds using either identified or non-identified edges and vertices because in both cases the degree of a vertex is defined to be the number of incident edges to the vertex

The surface degree-vertex equation

Proposition

Let $S = (V, E, F)$ *be a surface with polygonal decomposition. Then*

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Therefore, we have two degree-vertex equations:

- The graph degree-vertex equation where we do not identify edges and vertices in S
- The surface degree-vertex equation where we do identify edges and vertices in S

The degree of a face

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Let $f \in F$ be a face of S. The degree of f is

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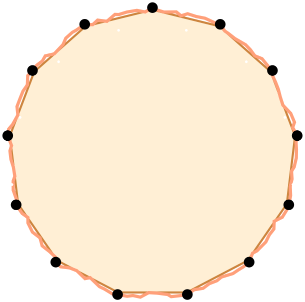

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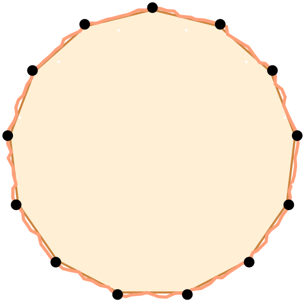

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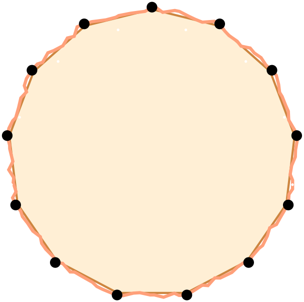

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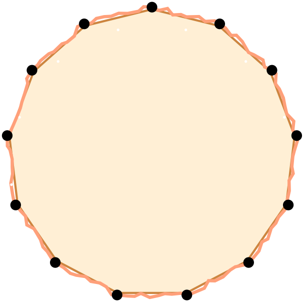

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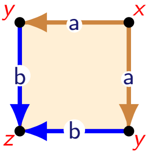
Question How are $\sum \deg(f)$ and $2|E|$ related?

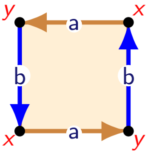
Face degrees of basic surfaces

In all cases deg(face) $= 4$ as there are 4 non-identified edges

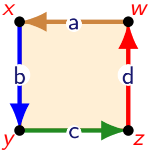
- Sphere

$S^2 \cong$ 

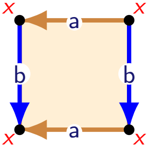
- Projective plane

$\mathbb{P}^2 \cong$ 

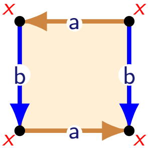
- Disk

$\mathbb{D}^2 \cong$ 

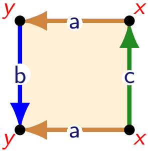
- Torus

$\mathbb{T} \cong$ 

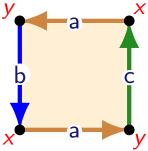
- Klein bottle

$\mathbb{K} \cong$ 

- Annulus

$\mathbb{A} \cong$ 

- Möbius band

$\mathbb{M} \cong$ 

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Remark To use this formula we need to know the number of identified edges in the polygonal decomposition

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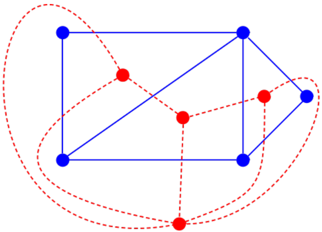
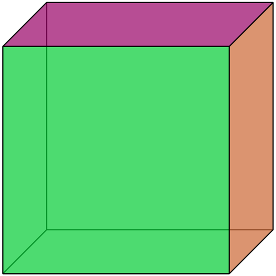


The dual of the cube



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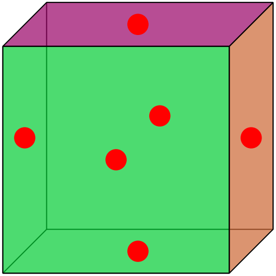

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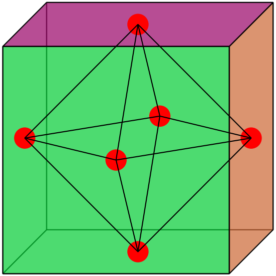

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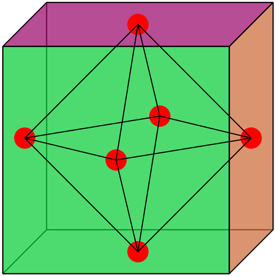

$\Longrightarrow$ the dual surface to the cube is the octahedron

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Example

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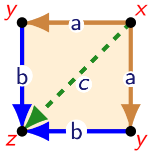

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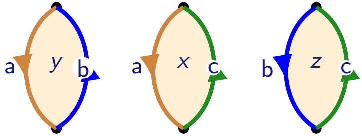
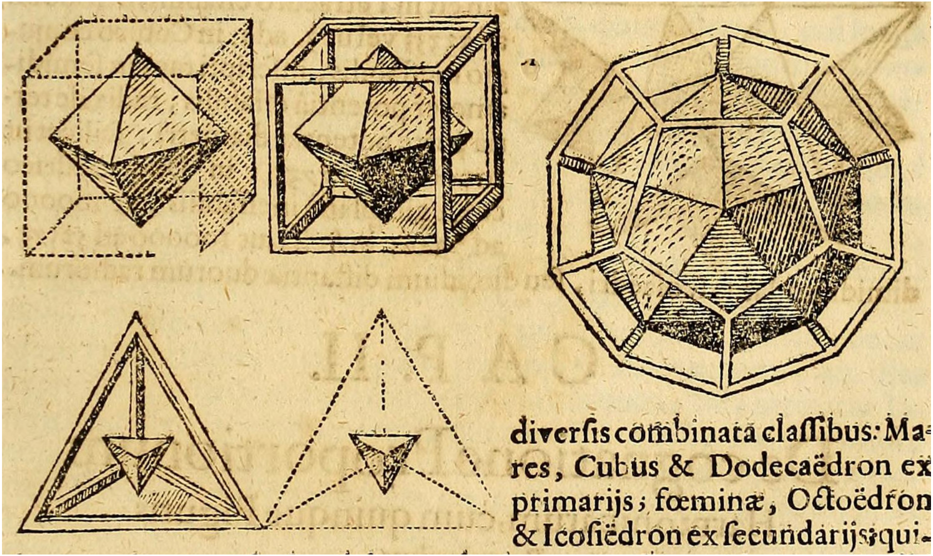


We will see better examples when we look at Platonic solids

Kepler's Harmonices Mundi



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such that:

- The map f is injective
- If $e = \{v, w\} \in E$ then $\mathrm{p}(e) \in \mathscr{P}(S)$ is an injective path from $f(v)$ to $f(w)$

Graphs on surfaces

Recall that a graph G is planar if it can be drawn in $\mathbb{R}^2$ without edge crossings

Let S be a surface and $x, y \in S$. A path from x to y on S is a continuous map $p\colon [0,1] \longrightarrow S$ such that $p(0) = x$ and $p(1) = y$

Let $\mathscr{P}(S)$ be the set of all paths on S

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- If $e, e' \in E$ then the paths $F(e)$ and $F(e')$ can intersect only at the images of their endpoints

Planar graphs

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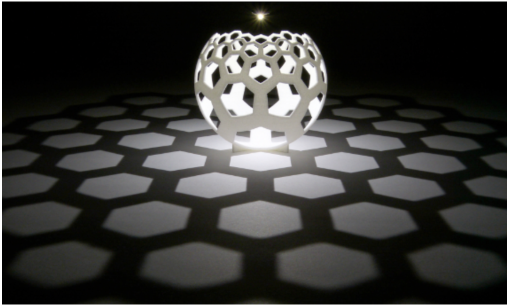
Proof Stereographic projection! (Move G away from ∞.)



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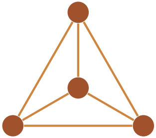
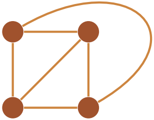
K_4 =  = 

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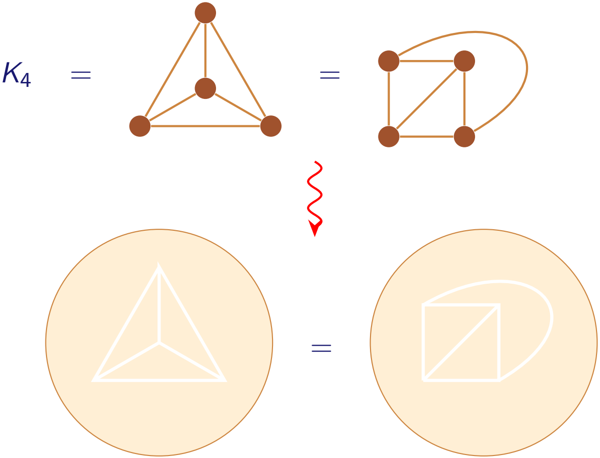

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Remark The argument cheats slightly because we are implicitly assuming that the edges are "nice" curves. This allows us to side-step issues connected with the Jordan curve theorem

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Case 1 G is a tree

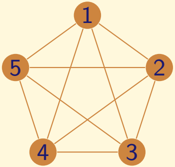
Combine $|V| - |E| = 1$ (previous lectures) and that there is only one face

Case 2 G is not a tree

By $\chi(S^2) = 2$ and the previous theorem

Planarity of K_5

Proposition

The complete graph $K_5 =$  *is not planar*

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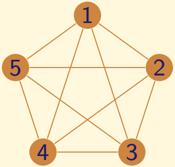
The complete graph $K_5 =$  *is not planar*

Proof Assume that K_5 is planar with $|F|$ faces

We have $|V| = 5$ and $|E| = 10$, so $2 = |V| - |E| + |F| \implies |F| = 7$

Let's count the number of faces in this polygonal decomposition differently

- The faces correspond to cycles in K_5
- Every face has at least 3 edges, so by the degree-face equation

$$\implies 2|E| = \sum_{f \in F} \deg(f) \geq 3|F|$$

$$\implies 2|E| = 20 \geq 21 = 3|F| \quad ⚡⚡⚡$$

Hence, the complete graph K_5 is not planar

Planarity of complete graphs

Corollary

The complete graph K_n is planar if and only if $1 \leq n \leq 4$

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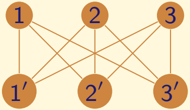
Proof

K_5 sits in K_n for $n \geq 5$, and the previous theorem applies

Planarity of bipartite graphs

Proposition

The bipartite graph $K_{3,3} =$ 

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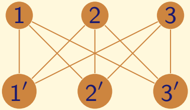
The bipartite graph $K_{3,3} =$ 

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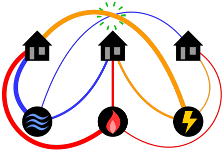
Proof Tutorials



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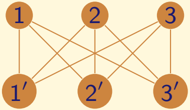
The bipartite graph $K_{3,3} =$ 

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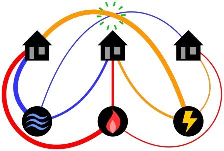

Theorem (Kuratowski)

Let G be a graph. Then G if planar if and only if it has no subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$

The proof is out of the scope of this unit!

Platonic solids

A Platonic solid is a surface that has a polygonal decomposition that is constructed using regular n-gons of the same shape and size such that the same number of polygons meet at every vertex

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Examples

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	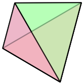	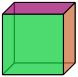	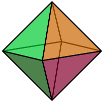	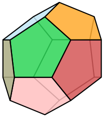	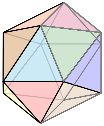
n	3	4	3	5	3
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	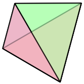	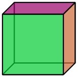	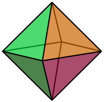	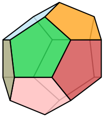	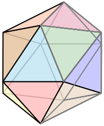
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	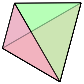	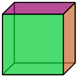	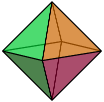	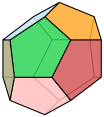	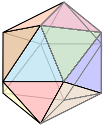
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- Can we understand them as polygonal decompositions of the sphere?

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The equations above give:

$$|E| = \left(\frac{1}{p} + \frac{1}{n} - \frac{1}{2}\right)^{-1}, \ |V| = \frac{2|E|}{p} \text{ and } |F| = \frac{2|E|}{n}$$

Classification of Platonic solids

Theorem

The complete list of Platonic solids is:

p	n	$\frac{1}{p}+\frac{1}{n}$	$e=\left(\frac{1}{p}+\frac{1}{n}-\frac{1}{2}\right)^{-1}$	$v=\frac{2e}{p}$	$f=\frac{2e}{n}$	*Platonic solid*
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3	5	$\frac{8}{15}$	30	20	12	*Dodecahedron*
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5	3	$\frac{8}{15}$	30	12	20	*Isosahedron*

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3	5	$\frac{8}{15}$	30	20	12	*Dodecahedron*
4	3	$\frac{7}{12}$	12	6	8	*Octahedron*
5	3	$\frac{8}{15}$	30	12	20	*Isosahedron*

Proof Since $\frac{1}{p}+\frac{1}{n}>\frac{1}{2}$ and $p, n \geq 3$ we get $n<6$ since $\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$
Case-by-case we then get the above values for p, n as the only possible values for Platonic solids.

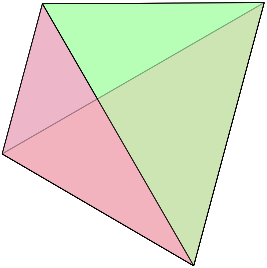
To prove existence we need to actually construct them

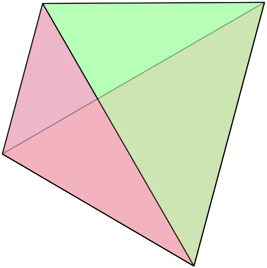
Classification of Platonic solids

Proof Continued Their construction is well-known:

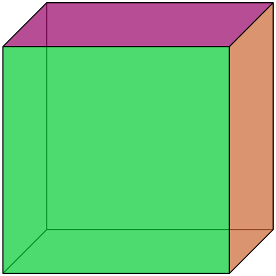
Dual tetrahedron = tetrahedron

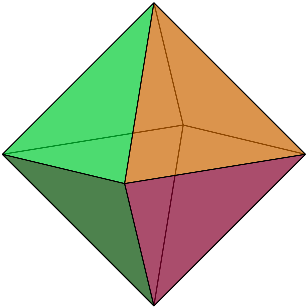
There is a symmetry in the Platonic solids given by $(p, n) \leftrightarrow (n, p)$. This corresponds to taking the dual surface



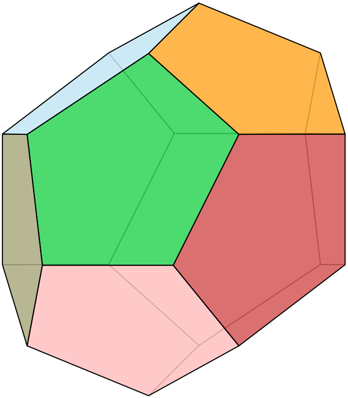


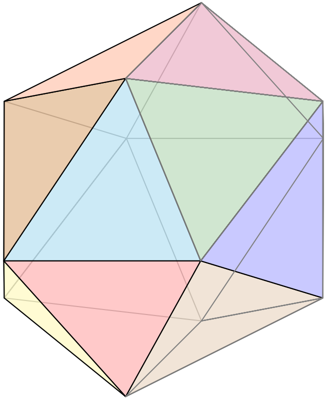
Cube and octahedron





Dodecahedron and icosahedron





Platonic soccer balls

Here are two dodecahedral decompositions of S^2

Soccer ball

Example A ball is made by gluing together triangles and octagons so that each octagon is connected to four non-touching triangles. Determine the number of octagons and triangles used

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Let there be $|V|$ vertices, $|E|$ edges and $|F|$ faces

Write $|F| = o + t$, where $o = \#$octagons and $t = \#$triangles

$$\implies \quad 2 = |V| - |E| + o + t$$

We have:

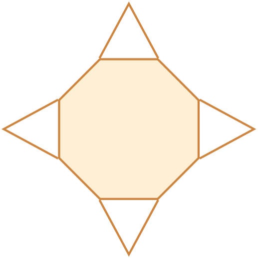
- vertex-degree equation: $3|V| = 2|E|$
- face-degree equation: $2|E| = 3t + 8o$
- Every octagon meets 4 triangles,

$\implies \quad 3t = 4o \quad \implies \quad 2|E| = 12o$

$\implies \quad 2 = o\left(4 - 6 + 1 + \frac{4}{3}\right) = \frac{o}{3}$

$\implies \quad o = 6$ and $t = 8$

$\implies \quad |E| = 36$ and $|V| = 24$



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As with the Platonic solids, we have only shown that if such a surfaces exists then there are 6 octagons, 8 triangles, 24 vertices and 36 edges but we have not shown that such a surface exists!

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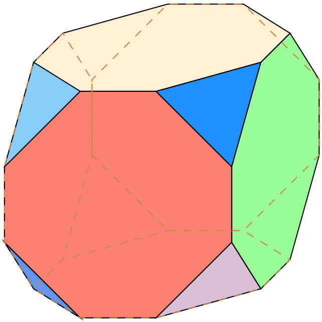
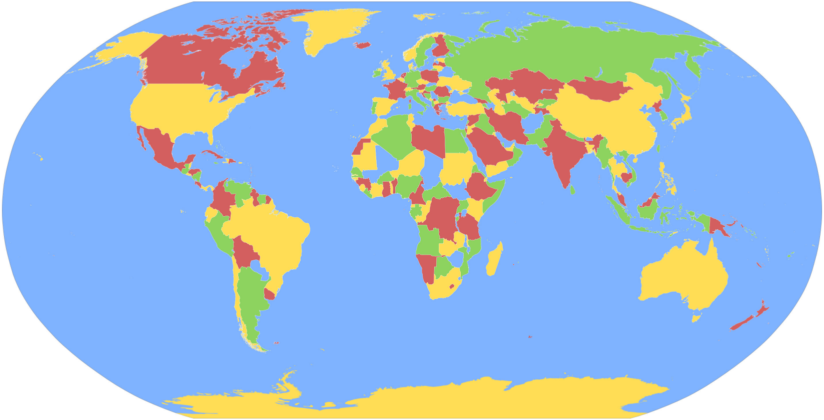
In fact, this surface does exist and it can be constructed by cutting triangular corners off a cube



Coloring maps

Question

How many different colors do you need to color a map so that adjacent countries have different colors?



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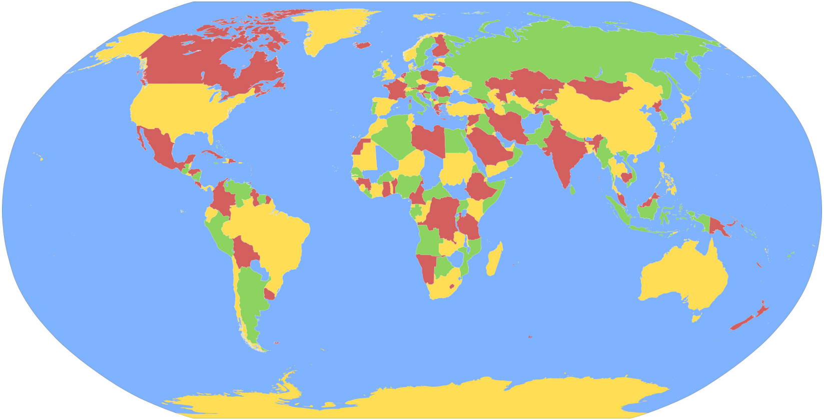

A map is a polygonal decomposition. The answer to this question involves the same ideas we used to understand Platonic solids

Chromatic number of (connected – assumed from now on) surfaces

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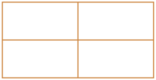

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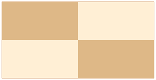

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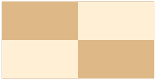

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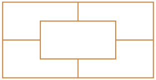

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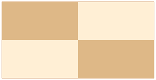

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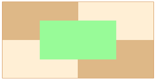

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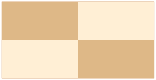

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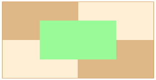

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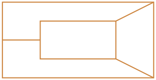

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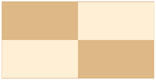

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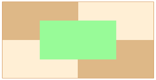

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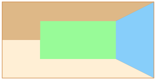

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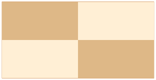

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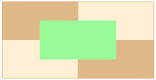

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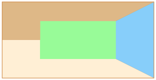

$C_P(\mathbb{D}^2) = 4$

$\implies \quad C(\mathbb{D}^2) \geq 4$

For maps of the world we are most interested in $C(\mathbb{D}^2) = C(S^2)$

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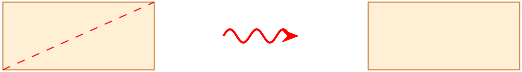

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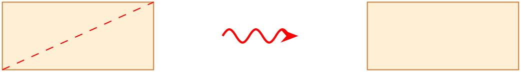

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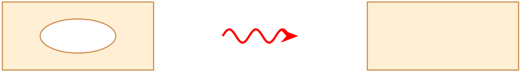

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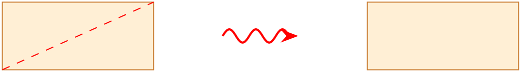

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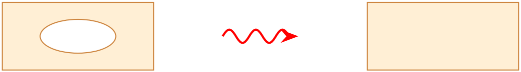

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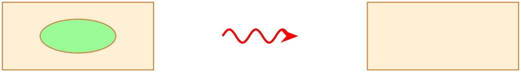

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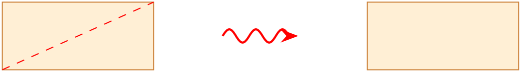

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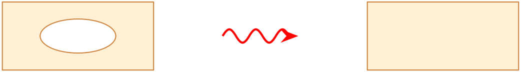

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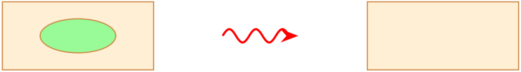

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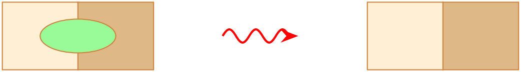

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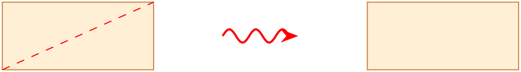

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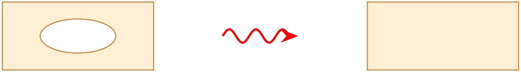

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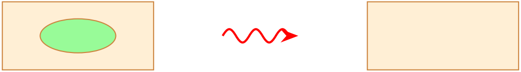

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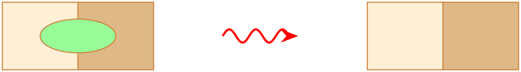

These assumptions are purely for convenience because, in each case, we can colour these maps using the same number of colours

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The basic idea is to use the Euler characteristic and the degree-vertex and degree-face equations to understand colourings

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Remark For a Platonic solid that is made from n-gons with p polygons meeting at each vertex we have $\partial_V = p$ and $\partial_F = n$

Bounding the face degree

Lemma

Suppose that M is a map on a closed surface S. Then

$$\partial_F = \left(1 - \frac{\chi(S)}{|F|}\right) \Big/ \left(\frac{1}{2} - \frac{1}{\partial_V}\right)$$

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Proof This is a simple calculation with the Euler characteristic:

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Proof This is a simple calculation with the Euler characteristic:

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Maps on sphere and projective planes

Corollary

Let M *be a map on* S^2 *or* $\mathbb{P}^2$*. Then* $\partial_F < 6$

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Let M be a map on S^2 or $\mathbb{P}^2$. Then $\partial_F < 6$

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Hence the result follows since $\chi(S^2) = 2$ and $\chi(\mathbb{P}^2) = 1$

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2. If the average face degree $\partial_F < 6$ then there must be at least one face f with $\deg(f) \leq 5$
   This observation will be important when we prove the Five color theorem (not quite the four color theorem)