Topology – week 9

Math3061

Daniel Tubbenhauer, University of Sydney

Classifying surfaces using invariants

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- The same Euler characteristic
- The same number of boundary circles

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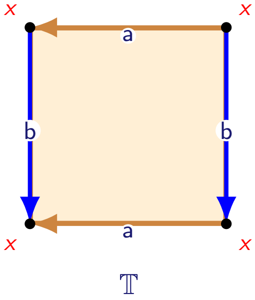

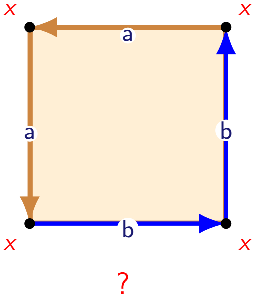

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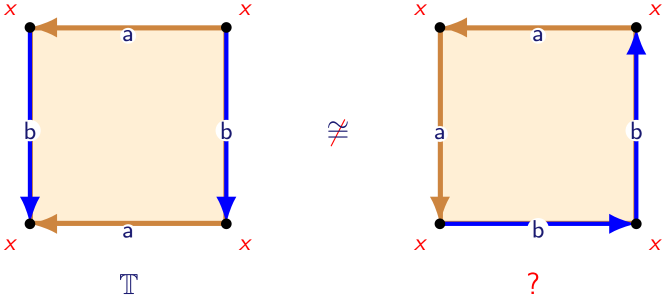

Orientability

Definition

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If S does not contain a Möbius strip it is orientable

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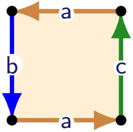
$\mathbb{M} =$ 

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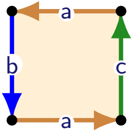
$\mathbb{M} =$ 

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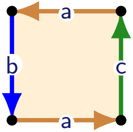
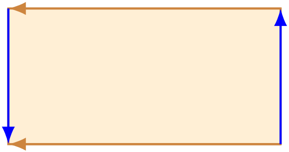
$\mathbb{M} =$ 

- Are S^2, $\mathbb{A}$, $\mathbb{D}^2$, $\mathbb{T}$, $\mathbb{P}^2$, $\mathbb{K}$, ... orientable or non-orientable?
- Can a surface be orientable and non-orientable for different polygonal decompositions? (That would be bad!)

The Klein bottle $\mathbb{K}$

$\mathbb{K} \quad =$ 

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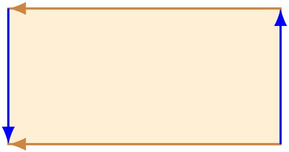
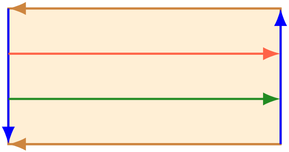
$\mathbb{K} =$  $=$ 

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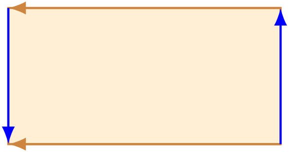

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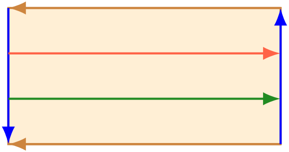

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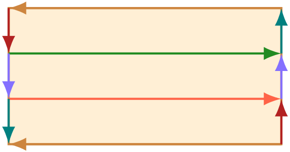

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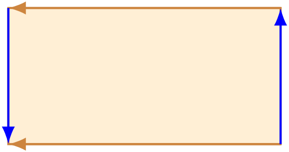

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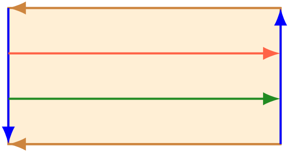

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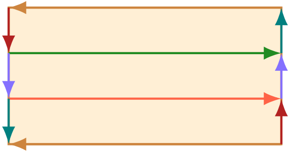

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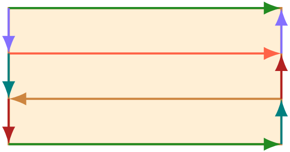

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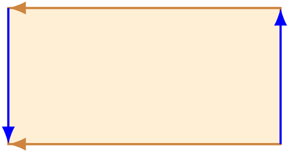

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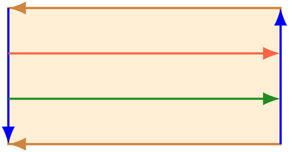

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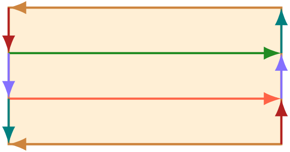

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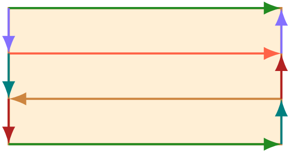

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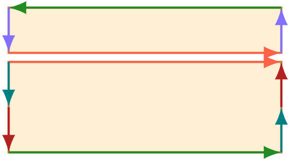

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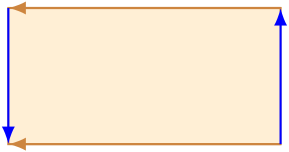

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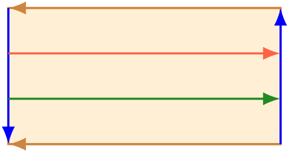

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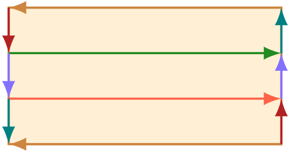

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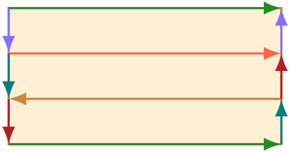

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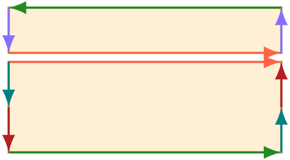

= $\mathbb{M}$

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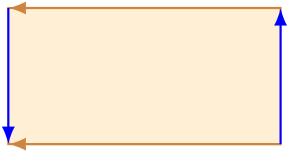

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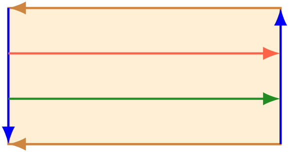

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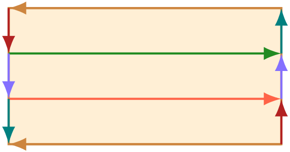

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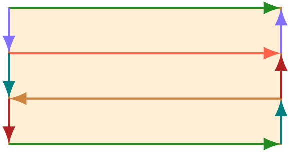

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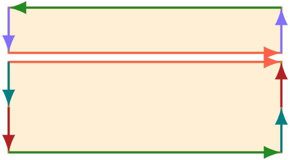

= $\begin{matrix}\mathbb{M}\\ \mathbb{M}\end{matrix}$

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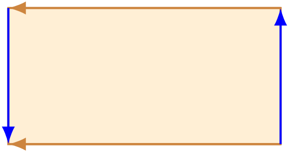
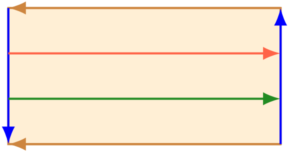
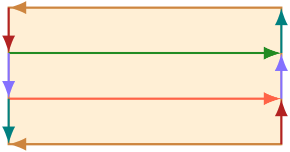
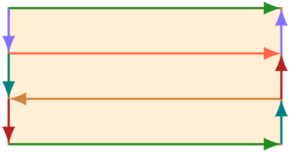
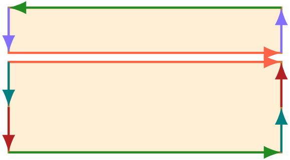
$\mathbb{K}$ =  = 

=  = 

=  = $\mathbb{M}$
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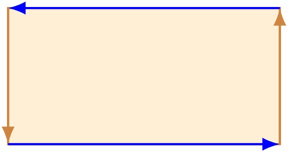
... although it might be more accurate to say that the Klein bottle is a Möbius strip without boundary

The projective plane $\mathbb{P}^2$

$\mathbb{P}^2 \quad =$ 

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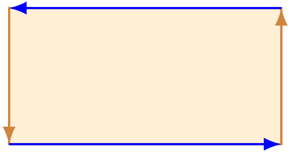
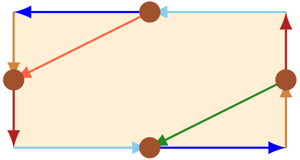
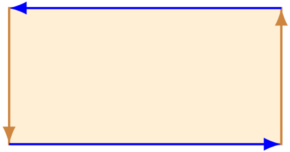
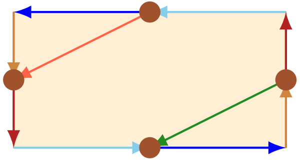
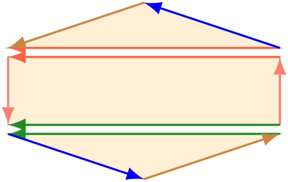
$\mathbb{P}^2 \quad = $  $=$ 

$\mathbb{P}^2 \quad =$  $=$ 

$=$ 

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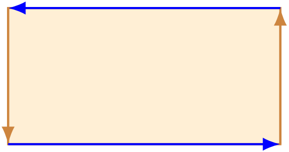

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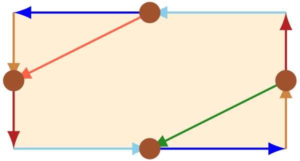

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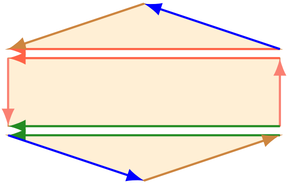

$=$

$\mathbb{D}^2$
$\mathbb{M}$
$\mathbb{D}^2$

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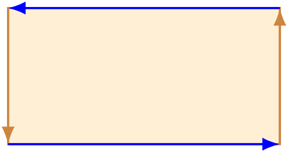

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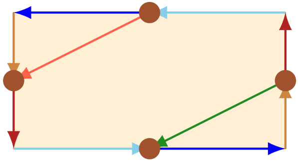

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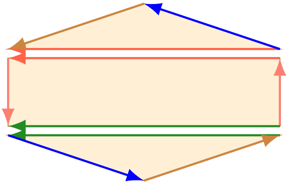

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$\mathbb{D}^2$
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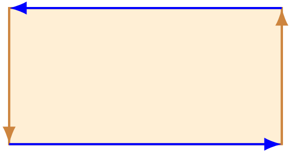
$\Longrightarrow$ The projective plane $\mathbb{P}^2$ is non-orientable

$\mathbb{P}^2 \quad =$



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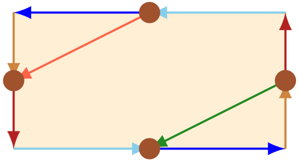

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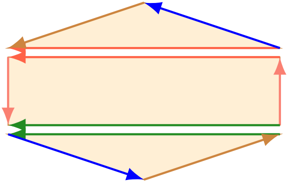

$=$ $\mathbb{D}^2$ $\mathbb{M}$ $\mathbb{D}^2$

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... or maybe $\mathbb{P}^2$ and not $\mathbb{K}$
is a Möbius strip without boundary?

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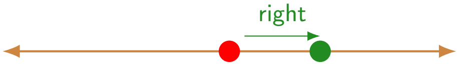
- One dimension $\mathbb{R}$



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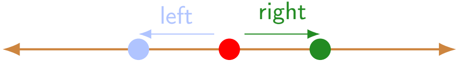

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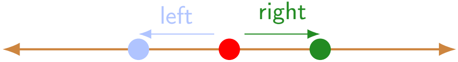
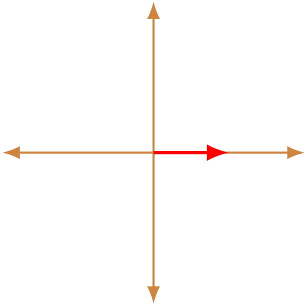


- Two dimensions $\mathbb{R}^2$



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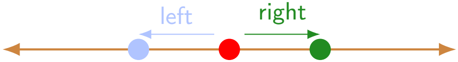

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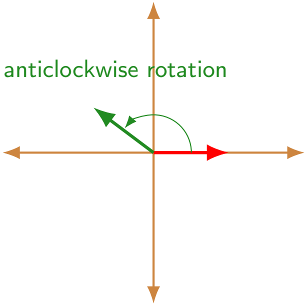

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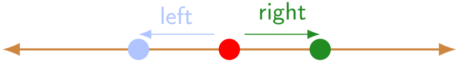

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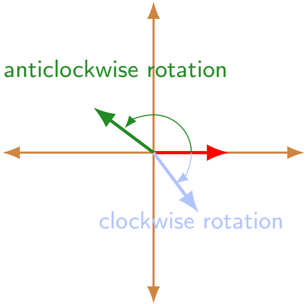

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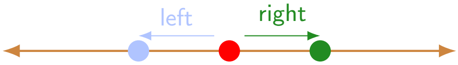

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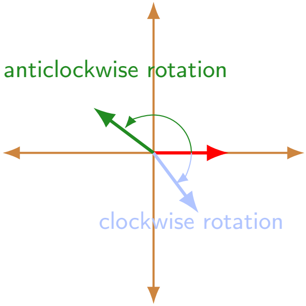

- Three dimensions $\mathbb{R}^3$???

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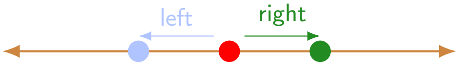
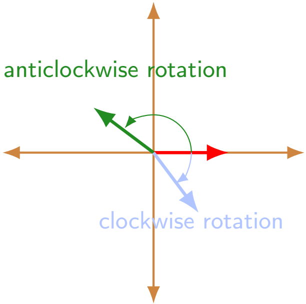


- Two dimensions $\mathbb{R}^2$



- Three dimensions $\mathbb{R}^3$???
- Higher dimensions $\mathbb{R}^n$, for $n \geq 3$???

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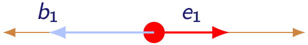

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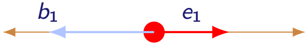

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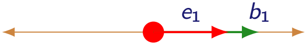

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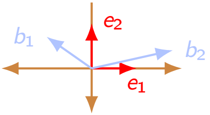

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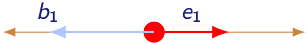

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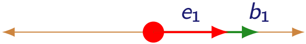

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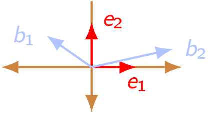

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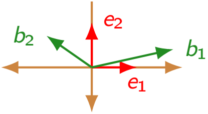

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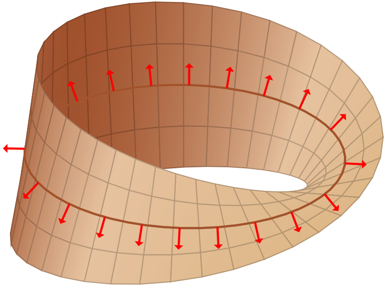

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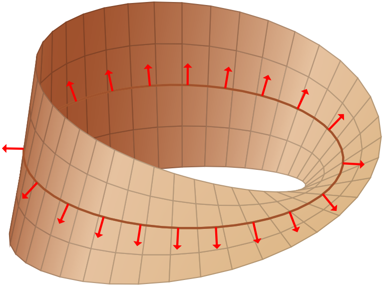

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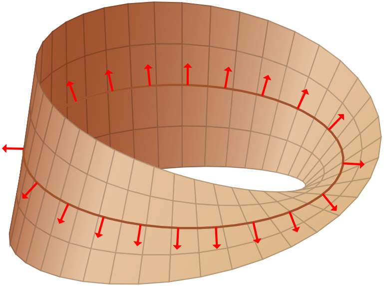

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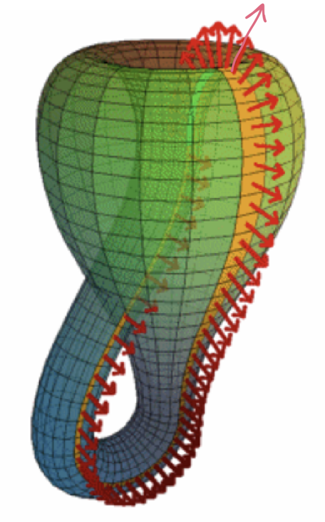
The vector b_3 is always normal to the surface of the Möbius strip. The direction of b_3 can change from pointing outside to inside because the Möbius strip is a surface with a boundary that only has one side



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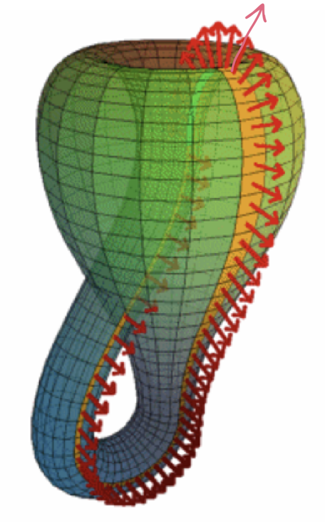

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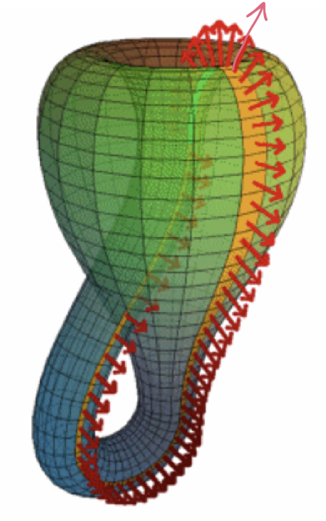

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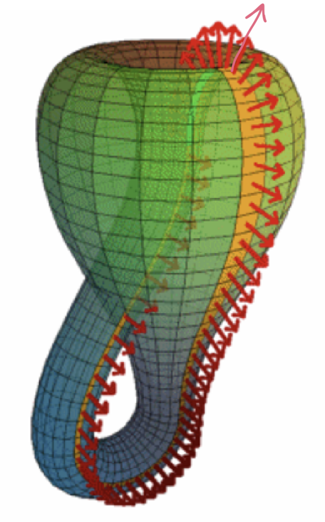

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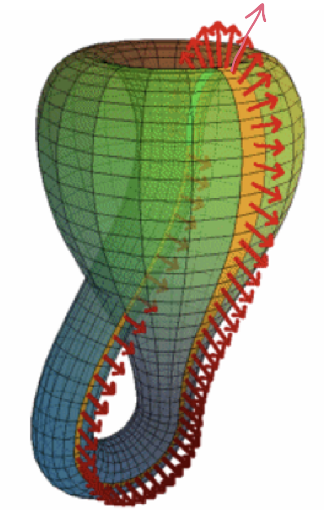

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Warning: this is a drawing of $\mathbb{K}$ in $\mathbb{R}^3$ but it is not the actual Klein bottle! Similarly, the pictures of the sphere S^2 in $\mathbb{R}^3$ are not really the sphere!

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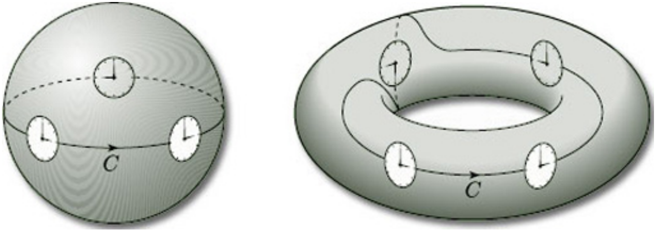

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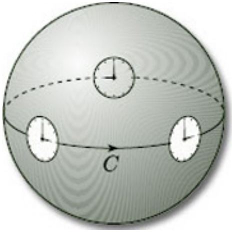

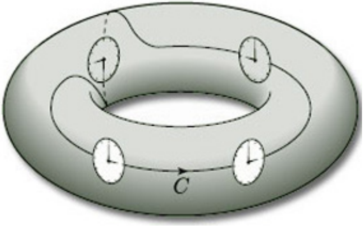

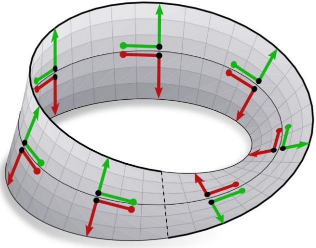

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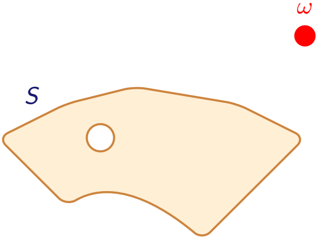

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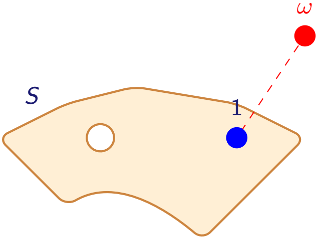

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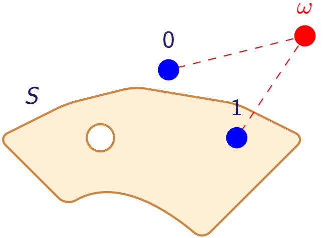

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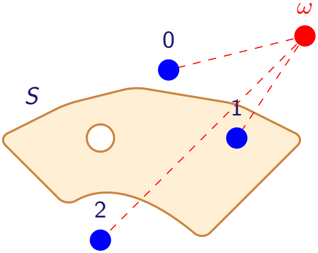

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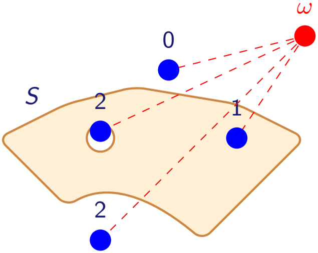

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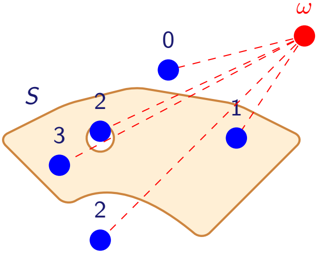

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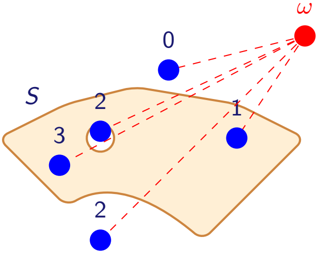

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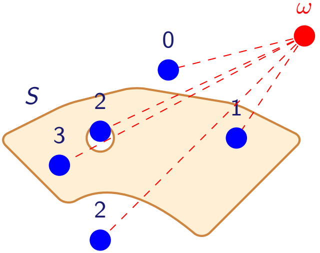

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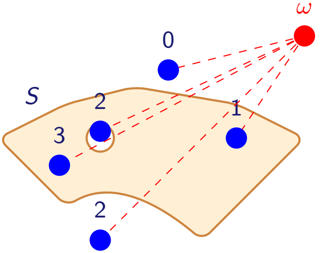

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Notice that since S is a closed surface it does not have boundary, so the "circle" in the picture, which contains a point x with $s(x) = 2$, should be interpreted as a tube through the surface

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Orientable surfaces...

Now suppose that S is non-orientable, so that it contains a Möbius strip $\mathbb{M}$

Pick a point $m \in S$ that is on this Möbius strip and fix an ordered basis $\{b_1, b_2, b_3\}$ with b_1 and b_2 tangential to m and $b_3 = b_1 \times b_2$.

Replacing b_3 with $-b_3$, if necessary, we assume that b_3 points out of S

Now move m, and $B = \{b_1, b_2, b_3\}$, continuously around S

$\implies$ $\det(B)$ changes continuously as m moves around S

By moving m around the Möbius strip in M, we can move m to a point where b_3 now points inside S

$\implies$ By continuity, at some point b_3 must have been in the plane spanned by b_1 and b_2

$\implies$ $\det(B) = 0$ ↯↯↯ since B is linearly independent!

Corollary

Let S be a non-orientable closed surface. Then S does not embed in $\mathbb{R}^3$.

You can't fill a liquid into the Klein bottle

Strictly speaking the liquid is neither in- nor outside

Jordan curve theorem

This argument used to prove theorem can be made rigorous for surfaces with finite polygonal decompositions but for "general surfaces" it is difficult to prove that $\mathbb{R}^3 = S \cup V_{\text{in}} \cup V_{\text{out}}$.

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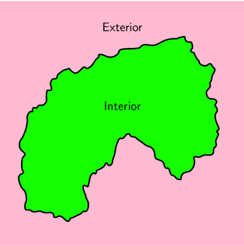

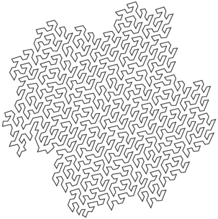

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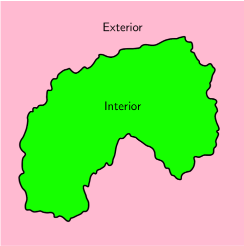
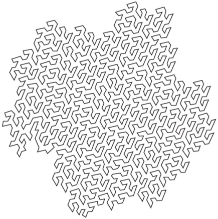
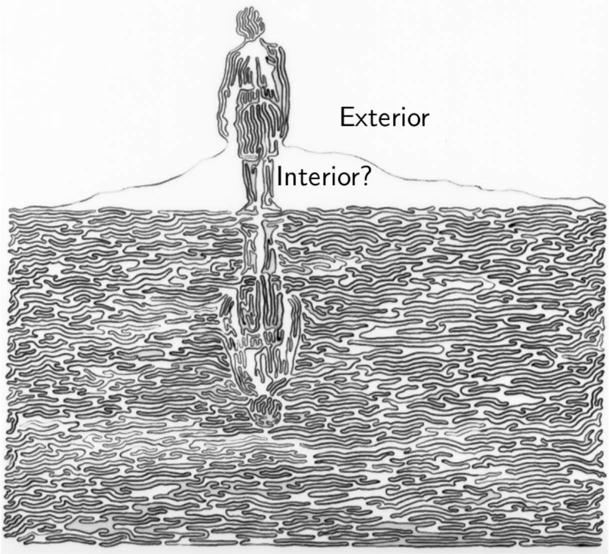




The left is easy, but can you tell for the right what is "in" or "out"?

Jordan curve theorem - 2

The main meat is that one needs to deal with "crazy" curves:



Embedding the projective plane in $\mathbb{R}^4$

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In contrast, every orientable surface embeds in $\mathbb{R}^3$

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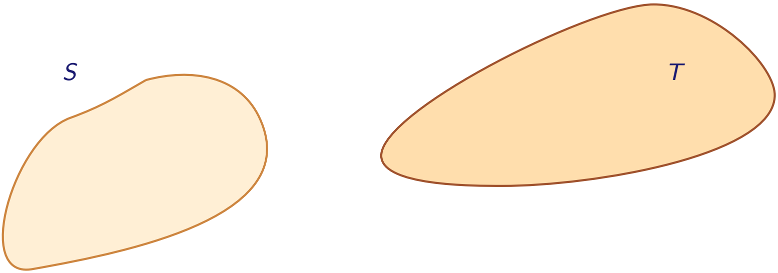

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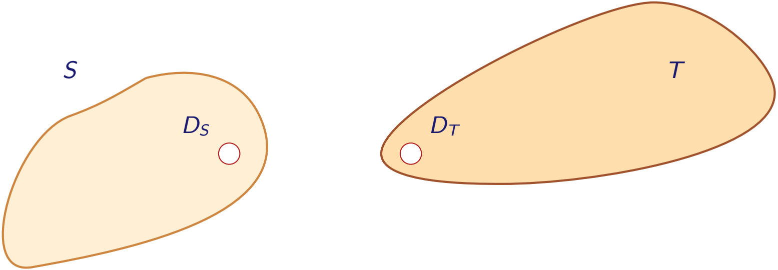

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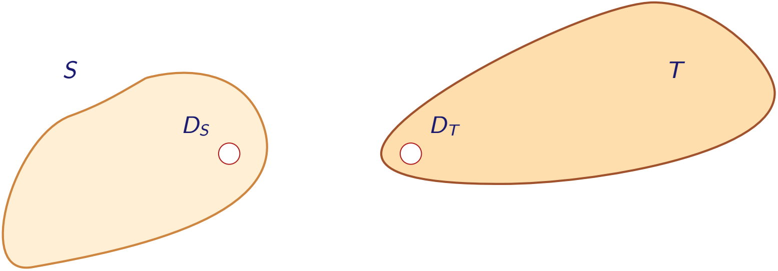

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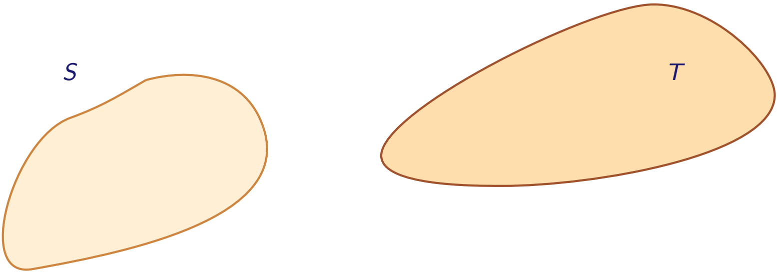

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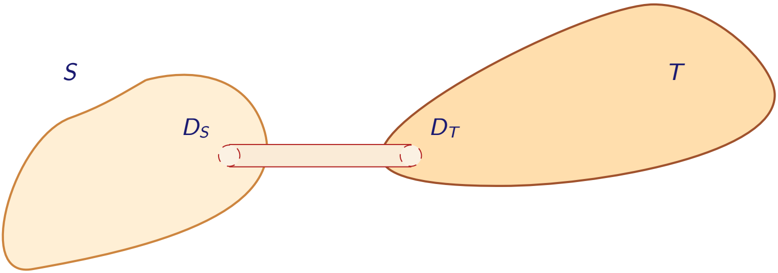

Identifying D_S and D_T is the same as connecting them with a cylinder

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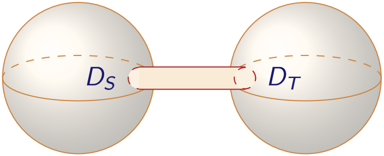

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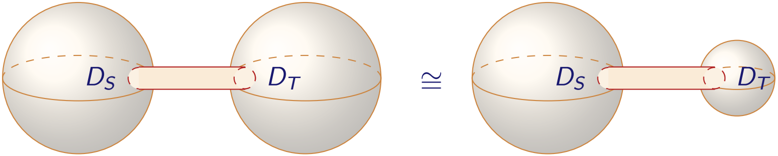

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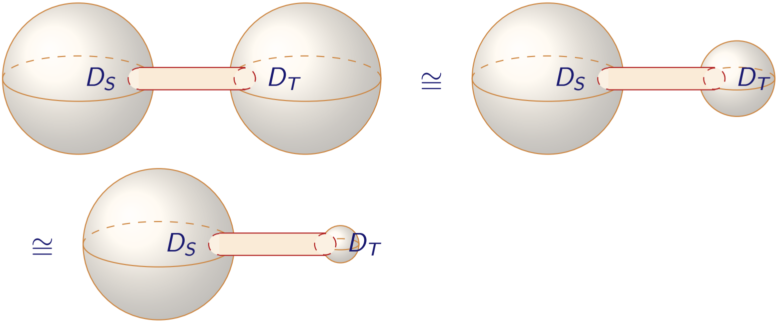

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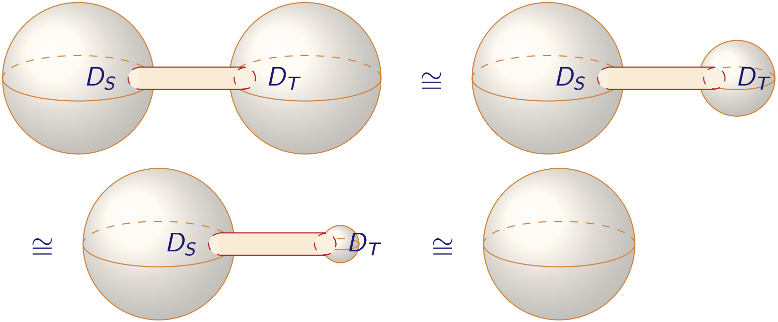

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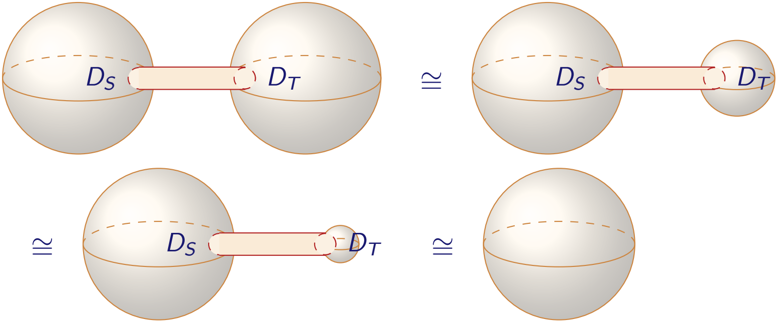

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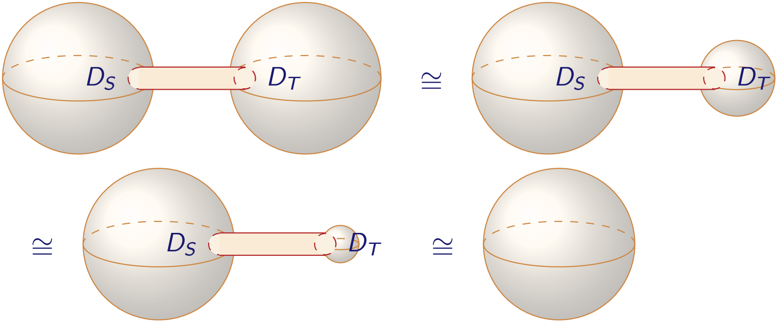

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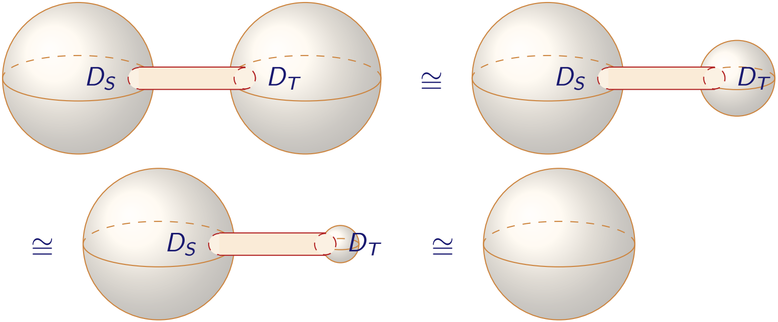

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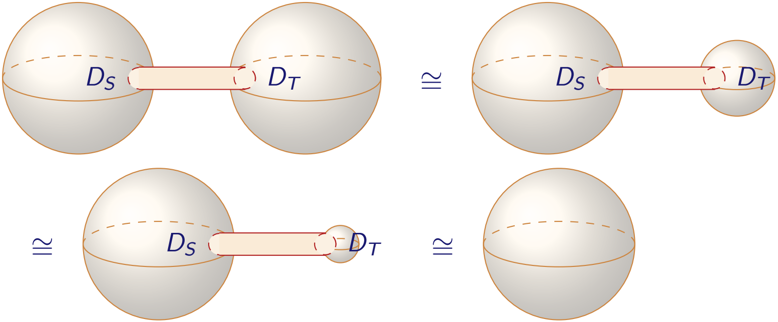

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So S^2 is the unit under the operation $\#$

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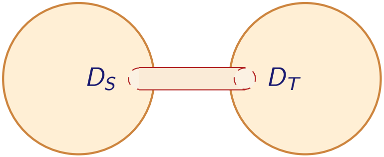

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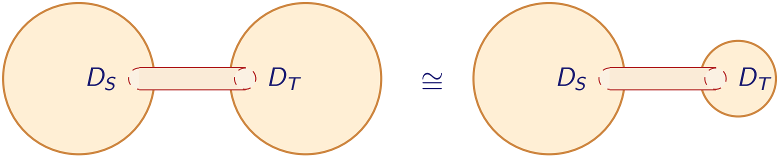

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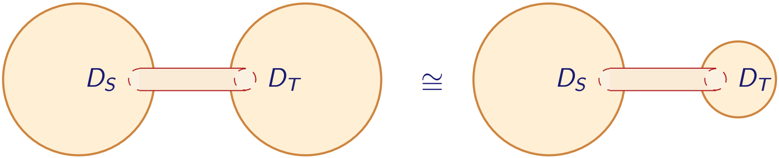

This is not the same as collapsing a sphere, which closes up the hole, because the disk has a boundary!

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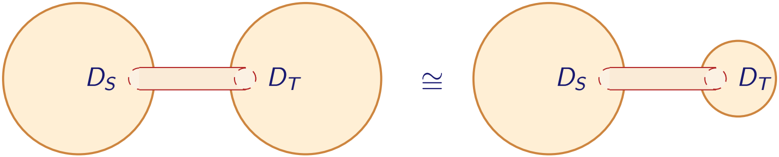

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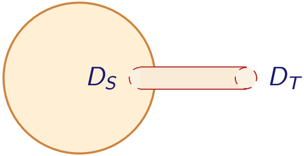

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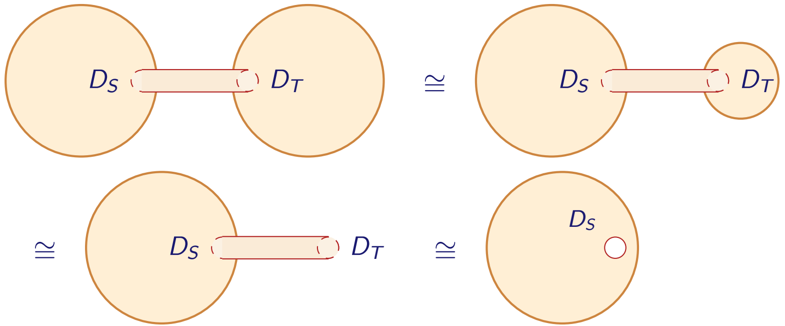

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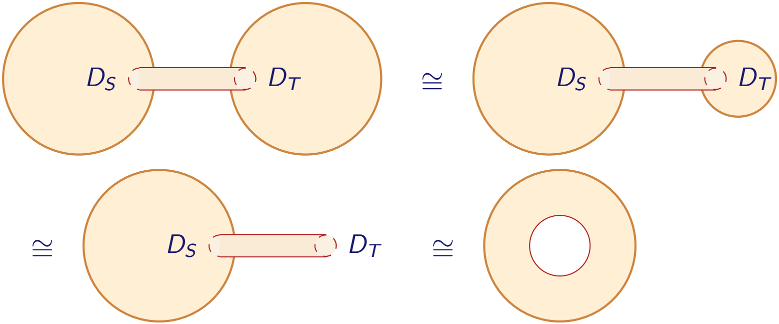

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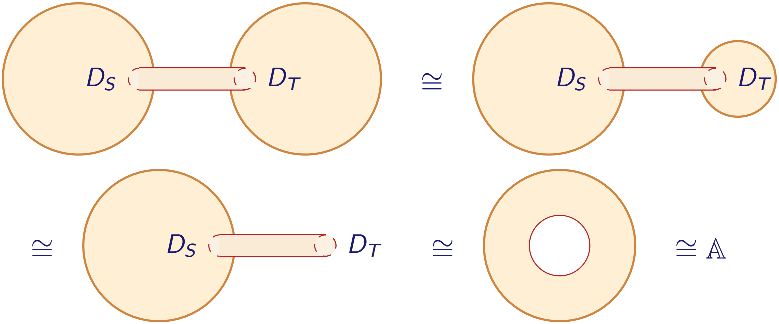

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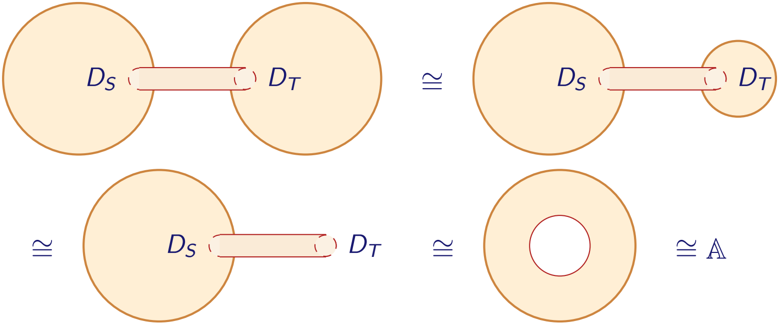

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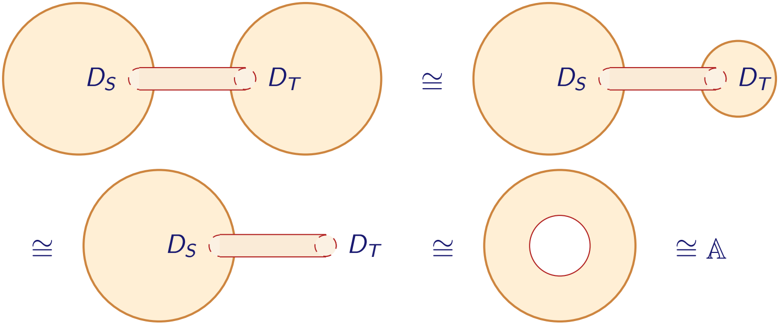

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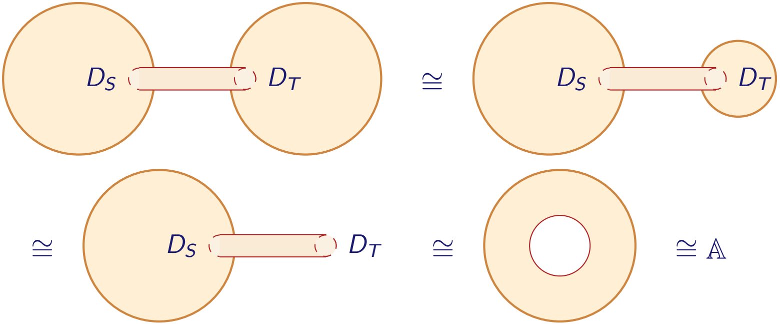

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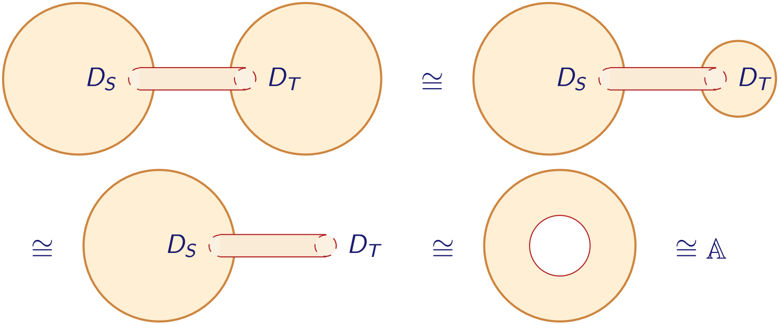

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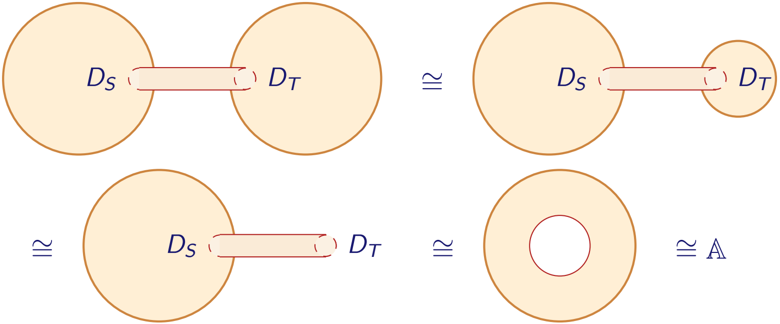

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punctures or, equivalently, T with d additional boundary circles

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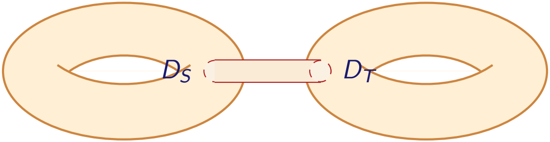

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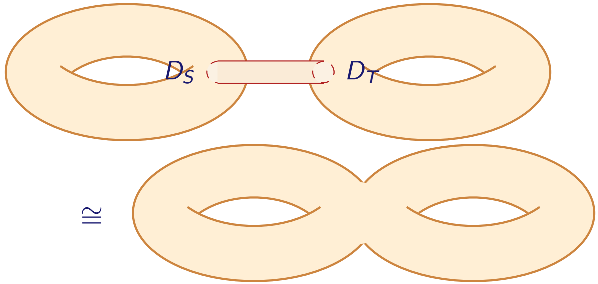

The double torus
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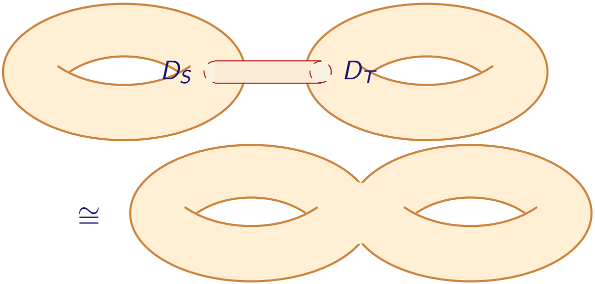

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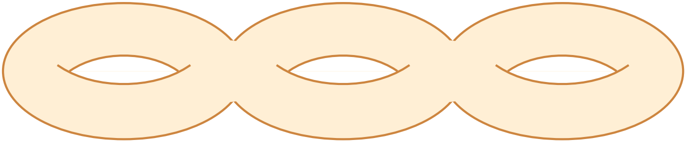
Similarly, there are triple tori $\#^3\mathbb{T}$



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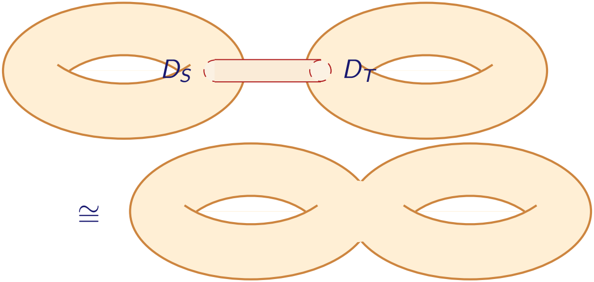

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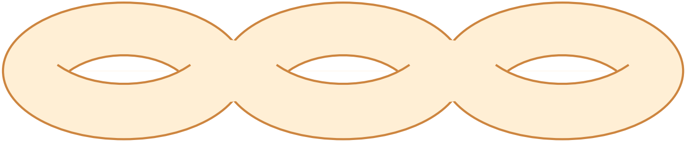
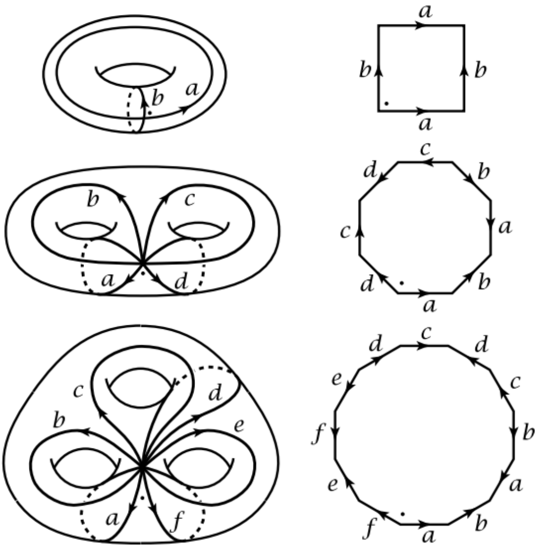


. . . and, more generally, t-tori $\#^t \mathbb{T}$

We already know t-tori



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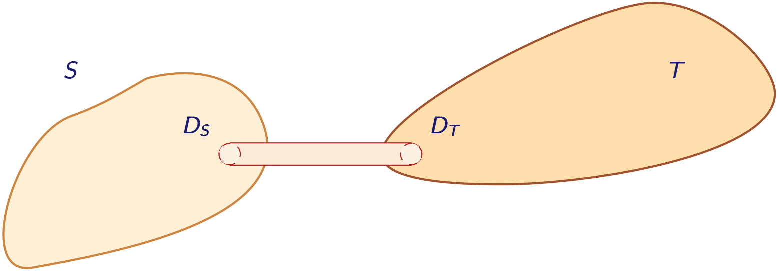

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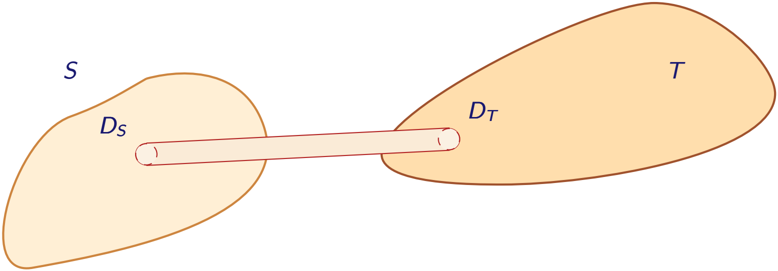

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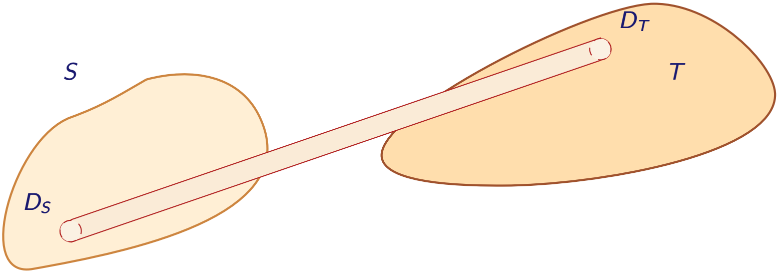

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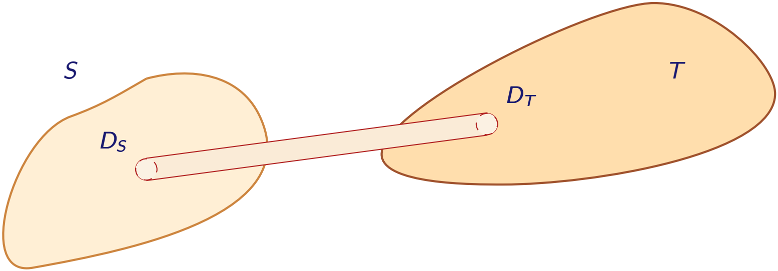

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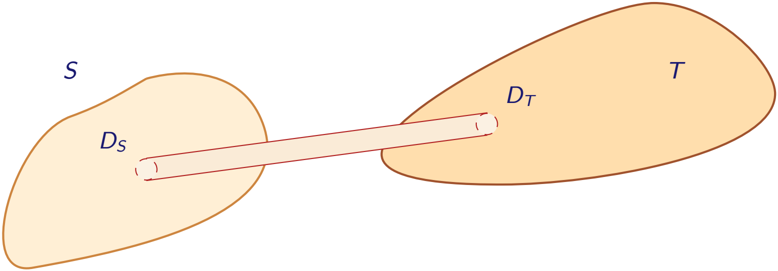

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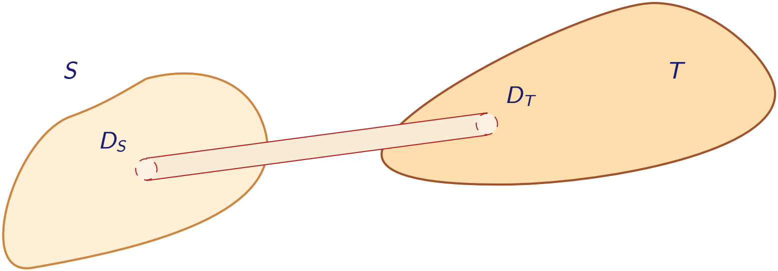

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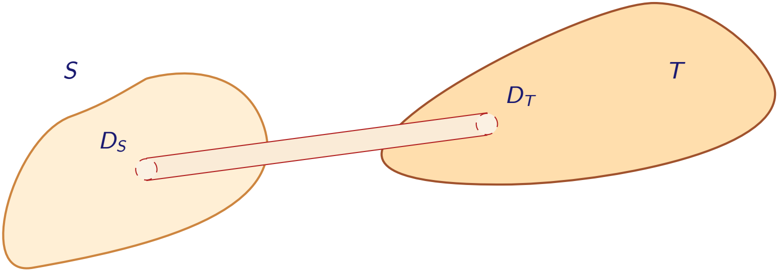

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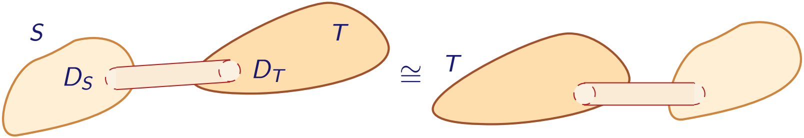

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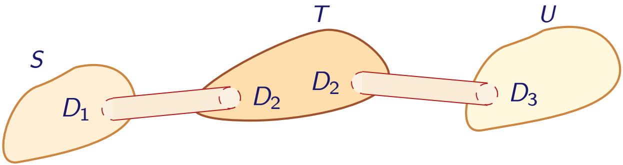

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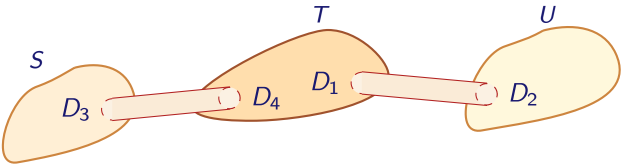

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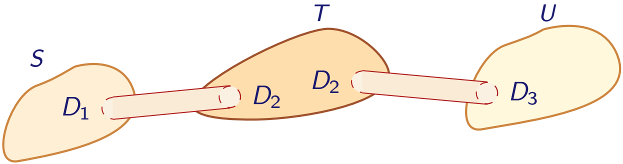

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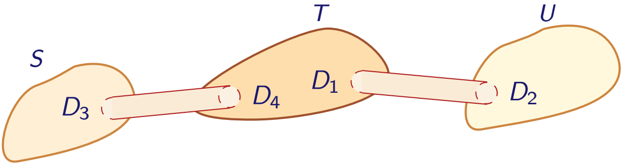

In these diagrams, D_1 and D_2 are cut first and then D_3 and D_4

$\implies$ $\#$ is a "surface addition or multiplication"

Connected sums of Euler characteristic

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Let S and T be surfaces with polygonal decompositions. Then

$$\chi(S \# T) = \chi(S) + \chi(T) - 2$$

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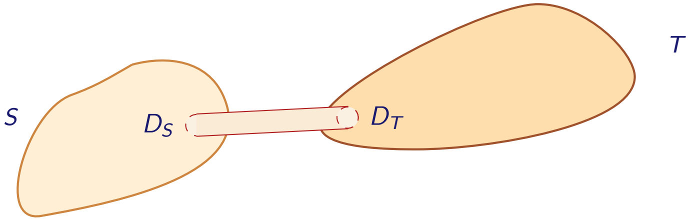

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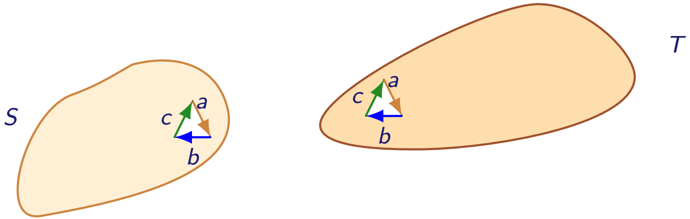

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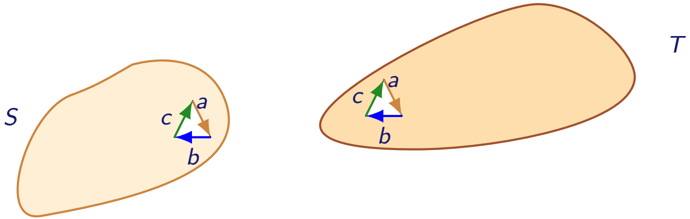

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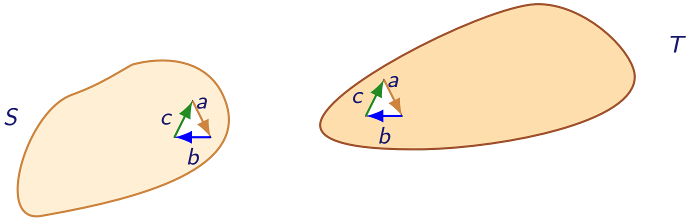

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Moral The -2 comes from cutting out two disks

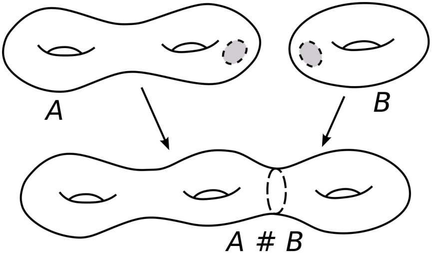
Examples

- If S is any surface then $S \cong S \# S^2$

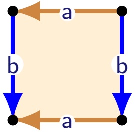
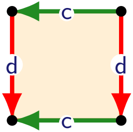
$$\implies \quad \chi(S) = \chi(S) + \underbrace{\chi(S^2)}_{=2} - 2 = \chi(S)$$

- $\mathbb{A} \cong \mathbb{D}^2 \# \mathbb{D}^2 \quad \implies \quad \chi(\mathbb{A}) = \chi(\mathbb{D}^2) + \chi(\mathbb{D}^2) - 2 = 1 + 1 - 2 = 0$

- $\chi(\mathbb{T} \# \mathbb{T} \# \mathbb{T}) = (\chi(\mathbb{T}) + \chi(\mathbb{T}) - 2) + \chi(\mathbb{T}) - 2 = -4$



Connected sums and polygonal decompositions

$\mathbb{T} \# \mathbb{T} =$  $\#$ 

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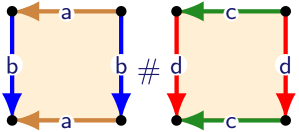

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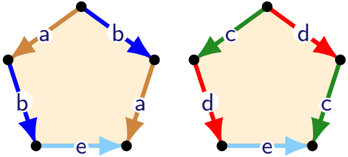

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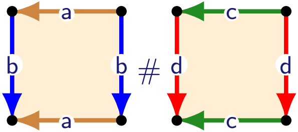

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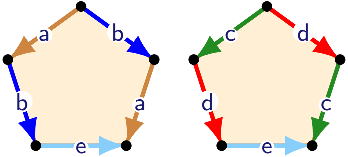

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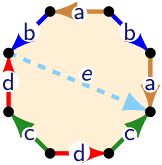

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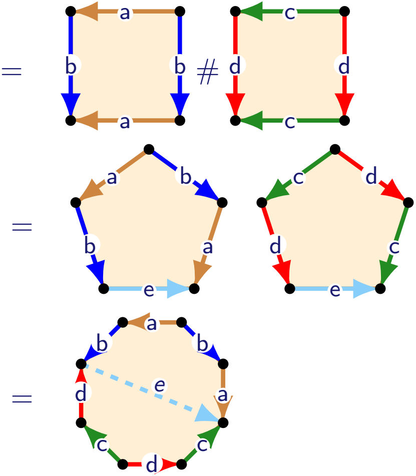
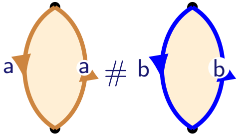


$\Longrightarrow$ For surfaces without a boundary you can cut the disks anywhere!

- What is $\mathbb{P}^2 \# \mathbb{P}^2$?

$\mathbb{P}^2 \# \mathbb{P}^2 \quad \cong$ 

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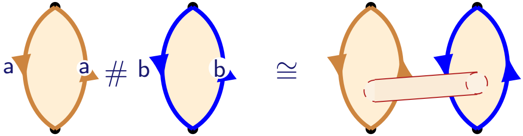

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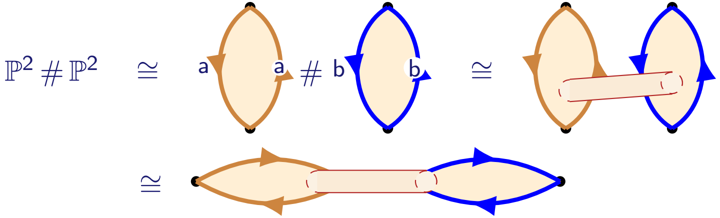

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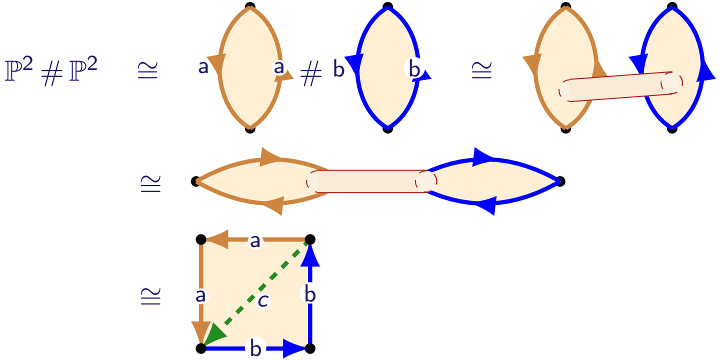

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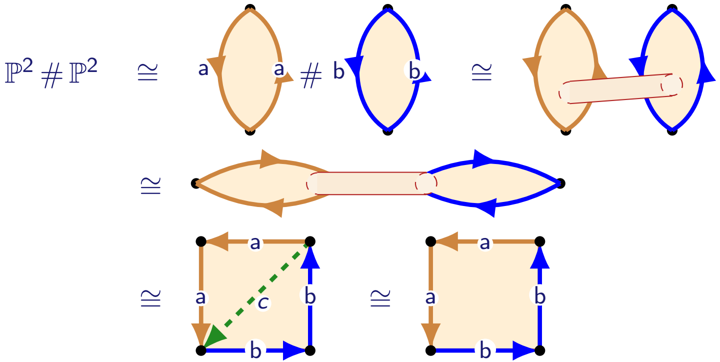

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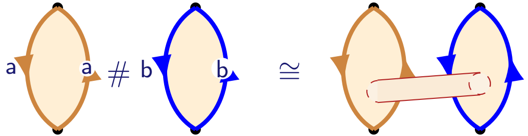

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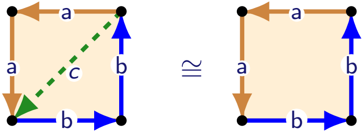
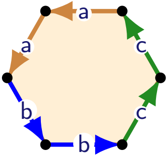


Similarly, $\#^3\mathbb{P}^2 \cong$



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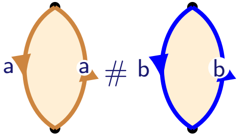

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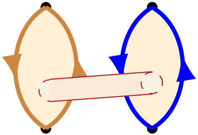

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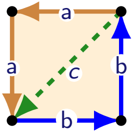

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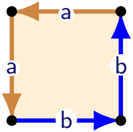

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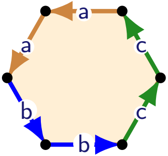

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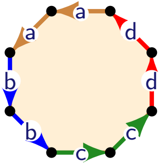

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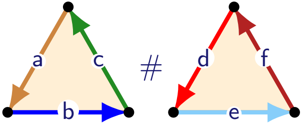
Connected sums and polygonal decompositions...

$\mathbb{D}^2 \# \mathbb{D}^2 \quad \cong$ 

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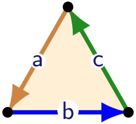
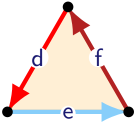
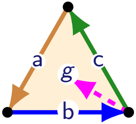
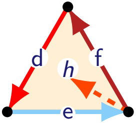
$\mathbb{D}^2 \# \mathbb{D}^2 \quad \cong$  $\#$  $\cong$  $\#$ 

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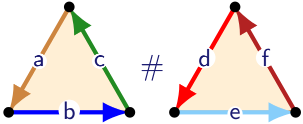

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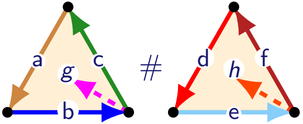

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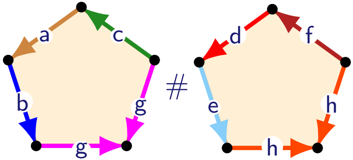

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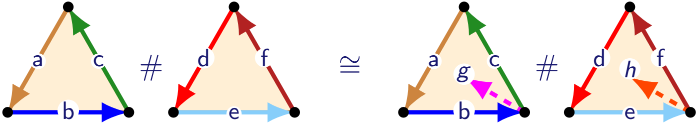

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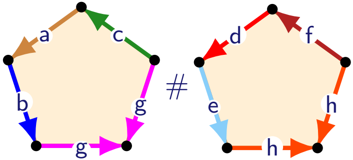

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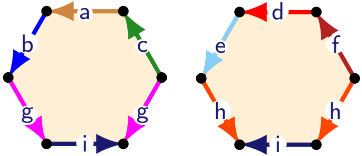

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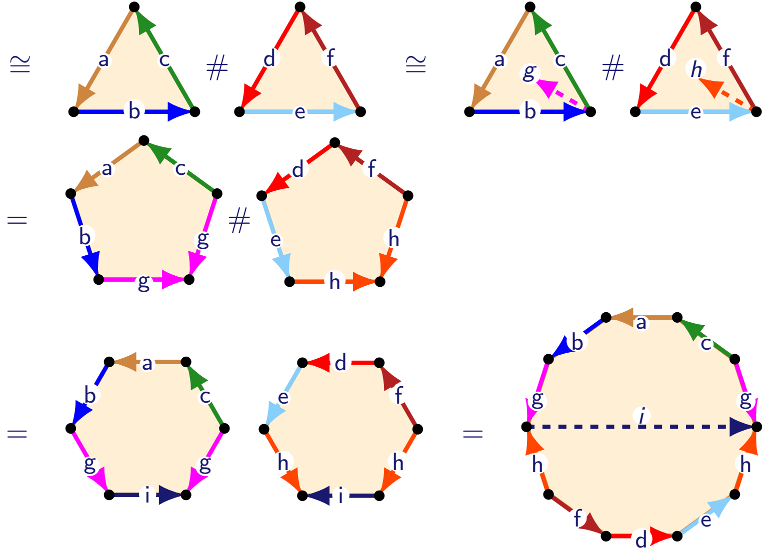

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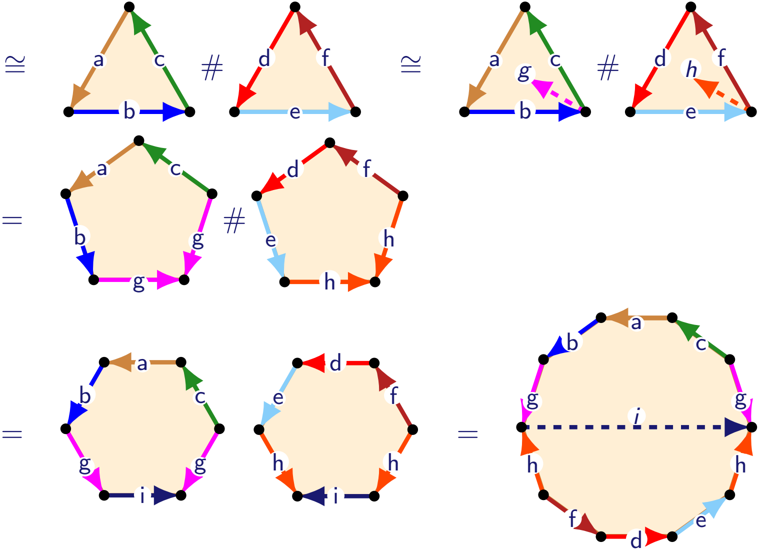

$\Longrightarrow$ For surfaces with a boundary, you can cut into the interior, if necessary, to form the connected sum

Surgery

We have already seen that it is possible to change one polygonal decomposition into another using surgery

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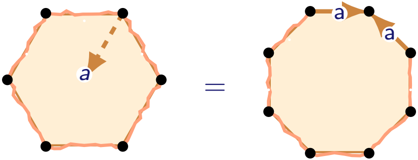

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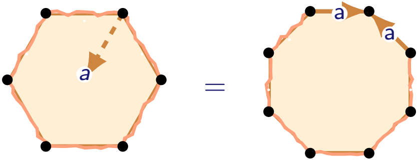

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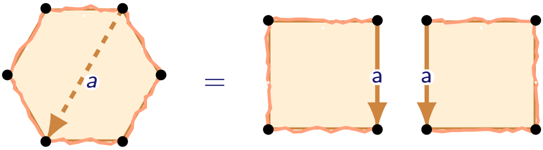

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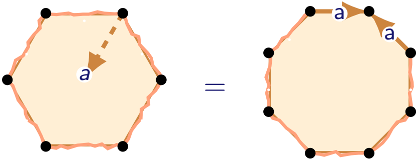

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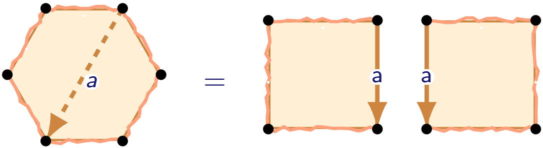

Perhaps surprisingly, these two operations and subdivision are all that we need

Surgery on the Möbius strip

Lemma

$\mathbb{M} \cong \mathbb{D}^2 \# \mathbb{P}^2$ $\qquad$ (= a *punctured projective plane*)

Proof

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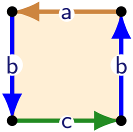
$\mathbb{M} \quad =$ 

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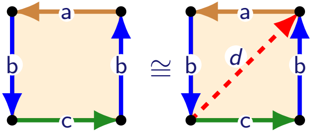
$\mathbb{M} =$ 

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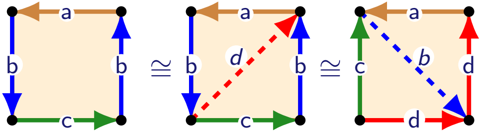
$\mathbb{M} =$ 

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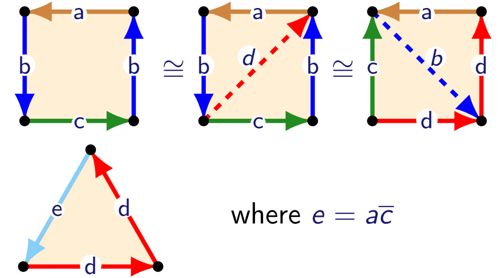

where $e = a\bar{c}$

Surgery on the Möbius strip

Lemma

$\mathbb{M} \cong \mathbb{D}^2 \# \mathbb{P}^2$ (= a *punctured projective plane*)

Proof

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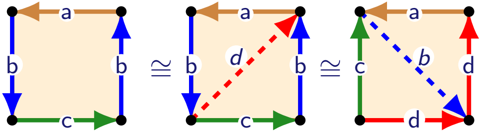
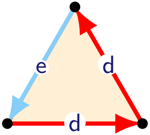


$\cong$



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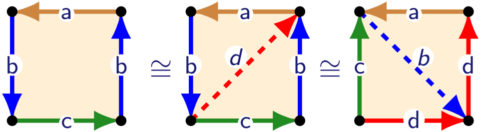

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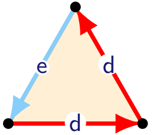

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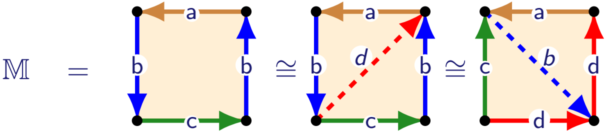
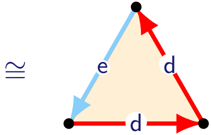
Proof





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$\Longrightarrow$ Every non-orientable surface contains the projective plane

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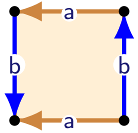

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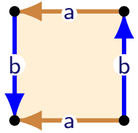
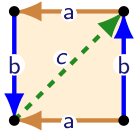
$\mathbb{K} =$  $=$ 

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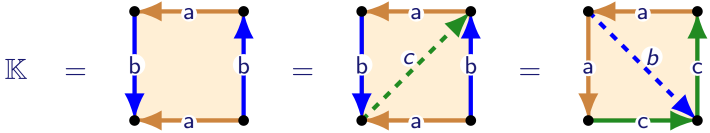

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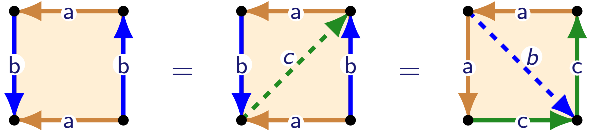

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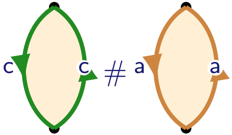

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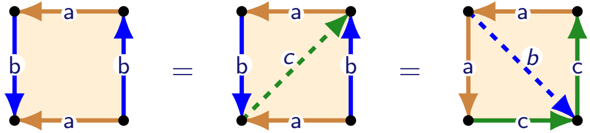

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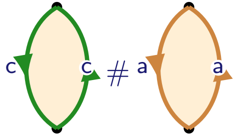

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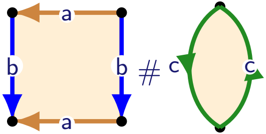
$\mathbb{T} \# \mathbb{P}^2 \quad =$ 

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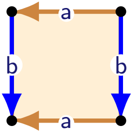

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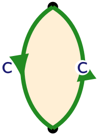

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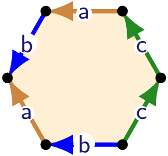

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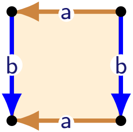

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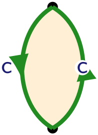

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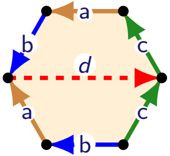

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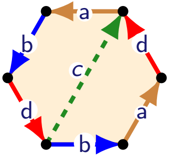

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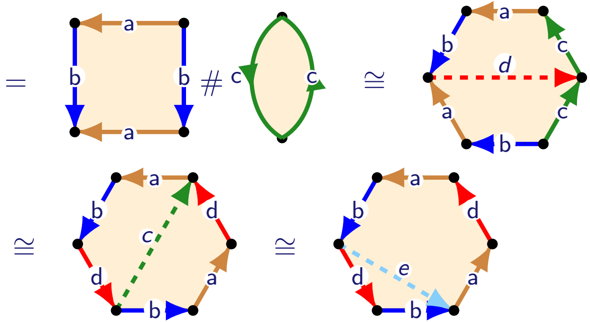

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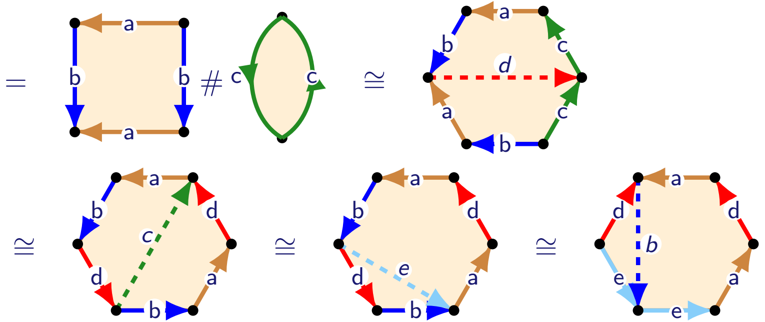
$\mathbb{T} \# \mathbb{P}^2 =$ 

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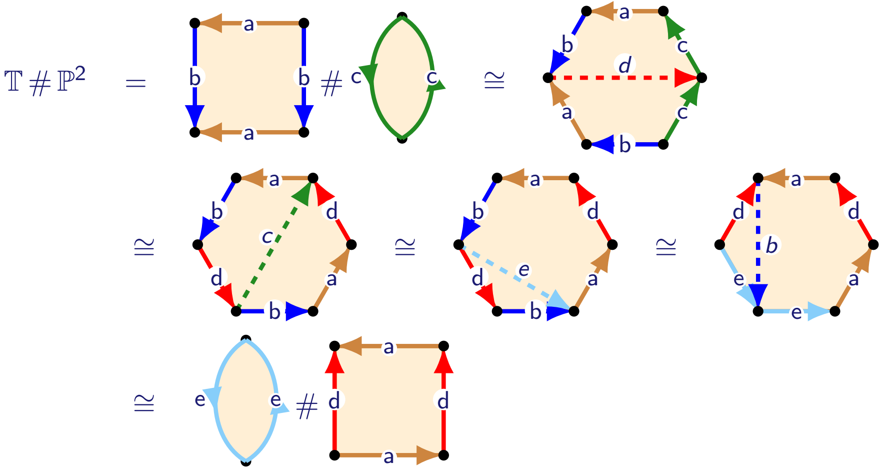

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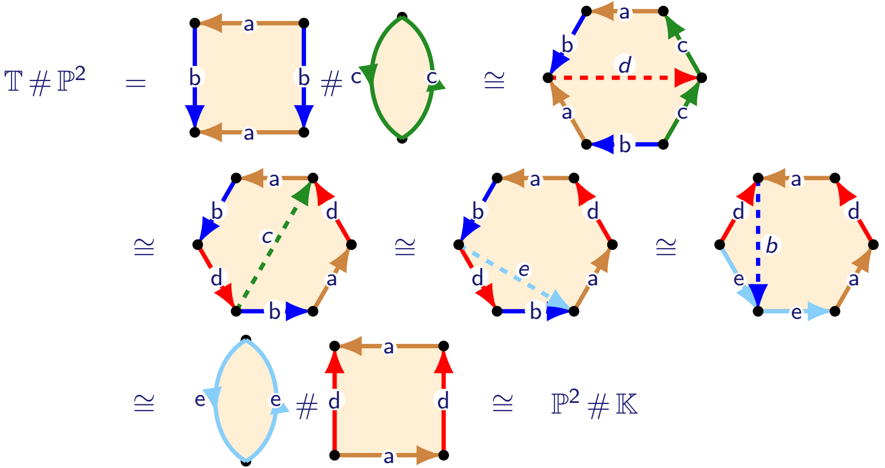

Projective planes dominate

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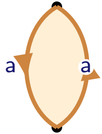
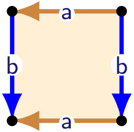
Why? $\mathbb{T}$ embeds in $\mathbb{R}^3$ but $\mathbb{K}$ does not!

Oriented and unoriented edges

Compare: $\mathbb{P}^2 =$ 

and $\mathbb{T} =$ 

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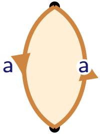
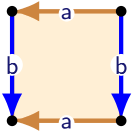
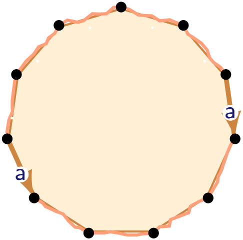
Compare: $\mathbb{P}^2 =$ 

and $\mathbb{T} =$ 

Paired edges on a polygon are oriented if they point in opposite directions and unoriented if they point in the same direction



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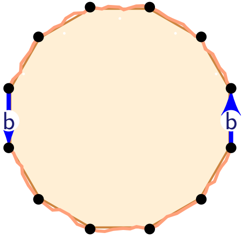

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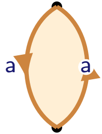
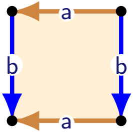
Compare: $\mathbb{P}^2 =$ 

and $\mathbb{T} =$ 

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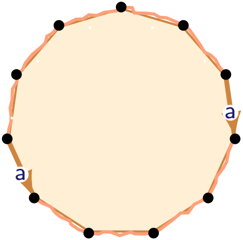

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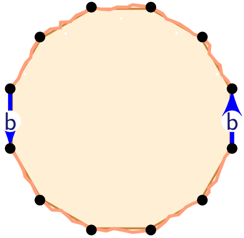

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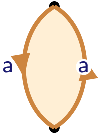
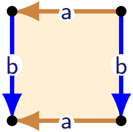
Compare: $\mathbb{P}^2 =$ 

and $\mathbb{T} =$ 

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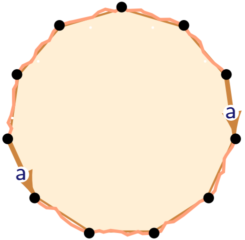

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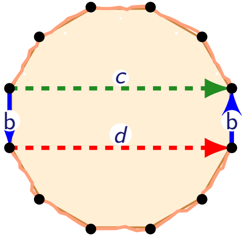

$\mathbb{M}$

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Classification of connected surfaces

Theorem

Let S be a connected surface. Then there exist non-negative integers d, p and t such that

1. $S \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$
2. *the boundary of S is the disjoint union of d circles*
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Moreover, we can assume that $pt = 0$, in which case S is uniquely determined up to homeomorphism by (d, p, t)

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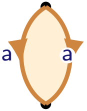
$S =$ 

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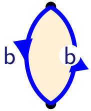
$S =$ 

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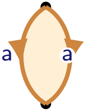
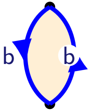
$S =$ 

$\cong S^2$ or $S =$ 

$\cong \mathbb{P}^2$

$\implies$ The theorem is true in this case

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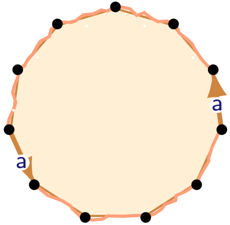

Now suppose that S has at least two edges and that the theorem is true whenever all surfaces that have a polygonal decomposition with one face and fewer edges

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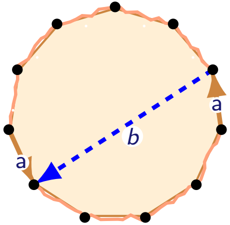

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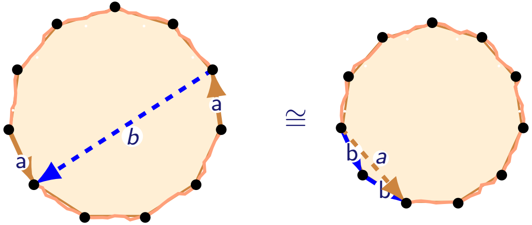

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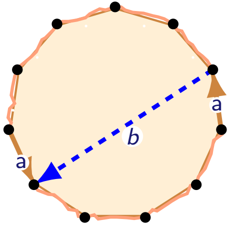

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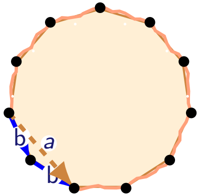

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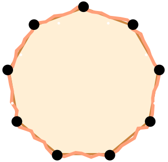

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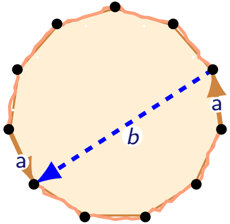

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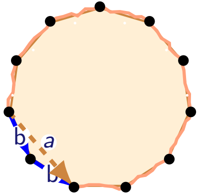

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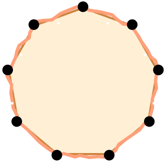

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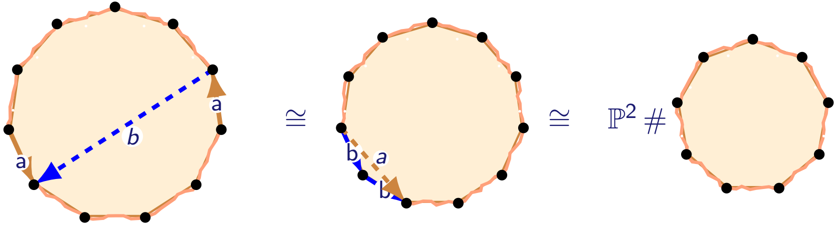

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By induction, $T \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$ since T has fewer edges

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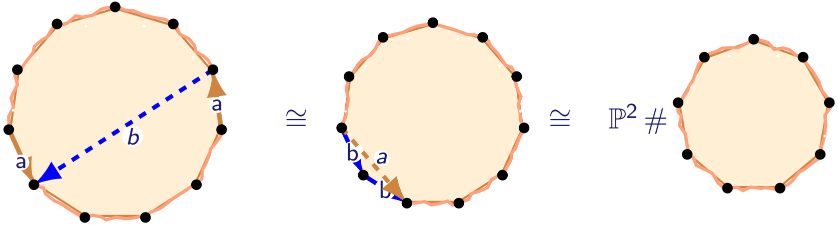

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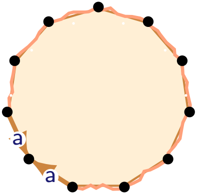
$\implies \quad S \cong S^2 \# \#^d \mathbb{D}^2 \# \#^{p+1} \mathbb{P}^2 \# \#^t \mathbb{T}$ as required

Case II: All paired edges in S are oriented

If S has adjacent oriented edges then

$S \cong$ 

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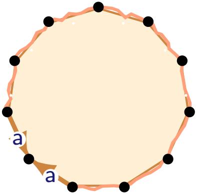
$S \cong$ 

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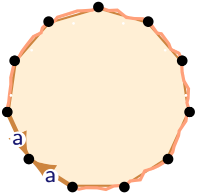
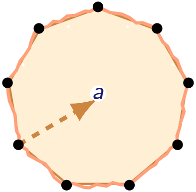
$S \cong$  $\cong$ 

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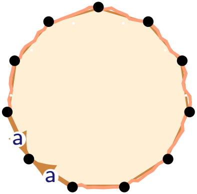
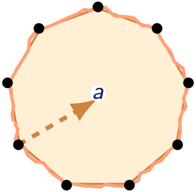
$S \cong$  $\cong$  $\cong T$

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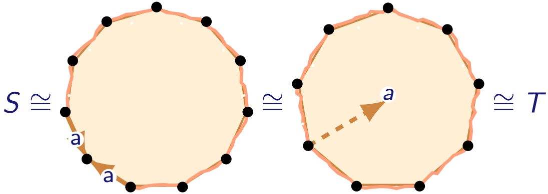

$\Longrightarrow \quad S \cong T \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$ by induction

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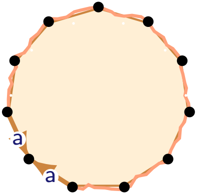
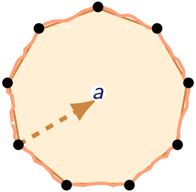
$S \cong$  $\cong$  $\cong T$

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Hence, we can assume that the paired edges are not adjacent

Proof of the classification theorem...

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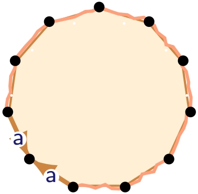
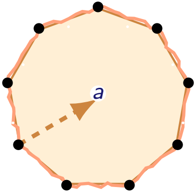
$S \cong$  $\cong$  $\cong T$

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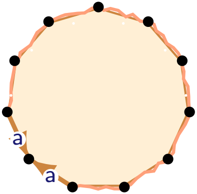
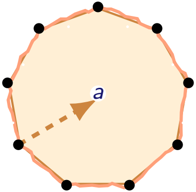
$S \cong$  $\cong$  $\cong T$

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Fix an (oriented) paired edge a such that the number of edges between the two copies of a is minimal

Proof of the classification theorem...

Case IIa: All edges on one side of a are free

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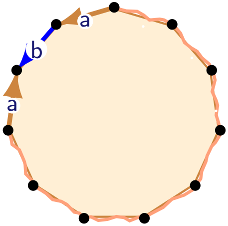
$\Longrightarrow \quad S \quad \cong$ 

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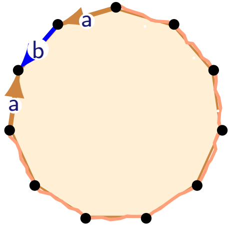
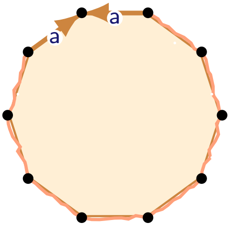
$\implies \quad S \quad \cong$ 

$\cong \quad \mathbb{D}^2 \#$ 

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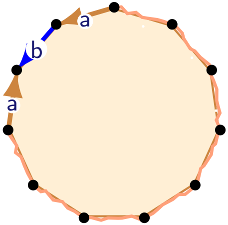
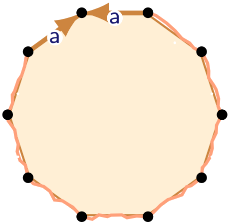
$\Longrightarrow \quad S \quad \cong$  $\cong \quad \mathbb{D}^2 \#$ 

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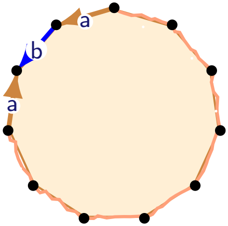

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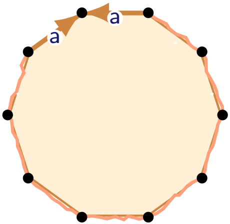

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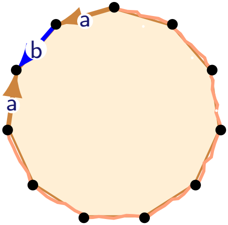
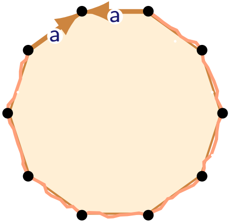
$\Longrightarrow \quad S \quad \cong$ 

$\cong \quad \mathbb{D}^2 \#$ 

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By induction, $T \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$

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Proof of the classification theorem...

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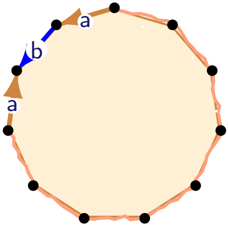
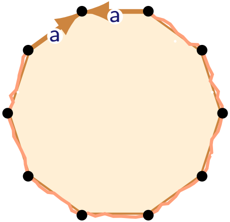
$\Longrightarrow \quad S \quad \cong$ 

$\cong \quad \mathbb{D}^2 \#$ 

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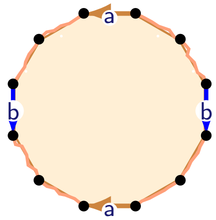
Hence, we can assume that there are paired edges on both sides of a

Case IIb: There are paired edges on both sides of a

Proof of the classification theorem...

Case IIb: There are paired edges on both sides of a

The number of edges between the ends of a is minimal, so

$\Longrightarrow \quad S \quad \cong$ 

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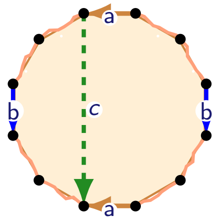

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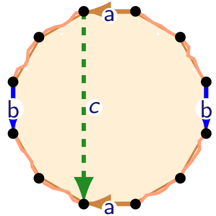
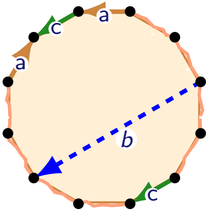
$\implies S \cong$  $\cong$ 

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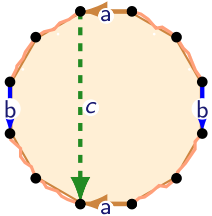

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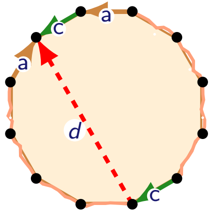

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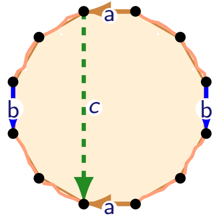

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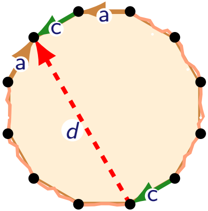

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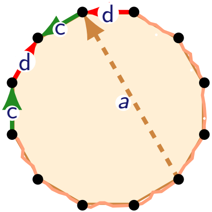

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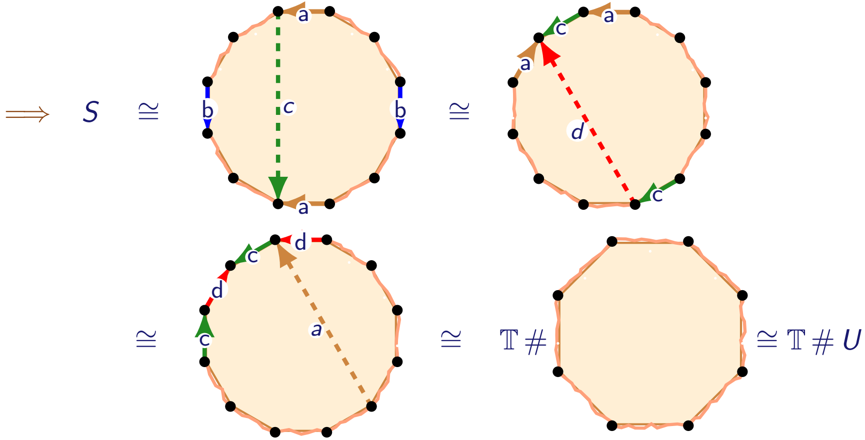

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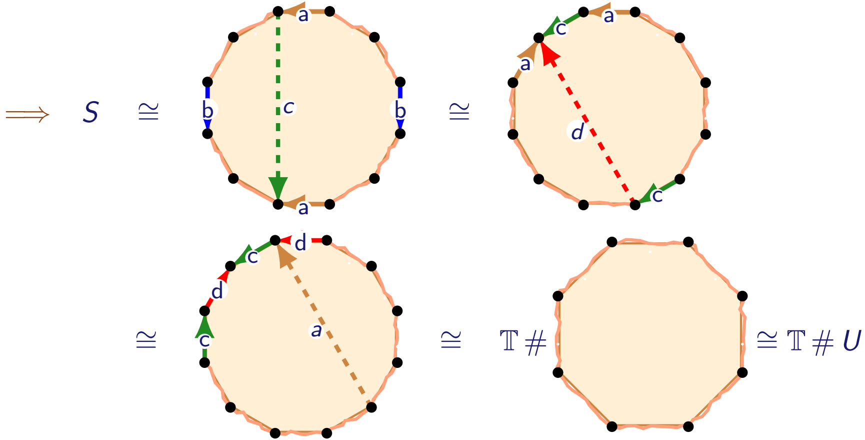

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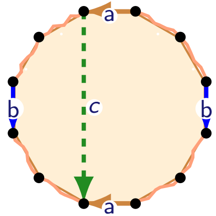
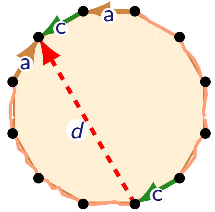
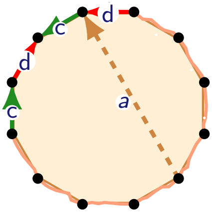
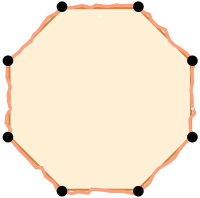
$\implies \quad S \quad \cong$ 

$\cong$ 

$\cong$ 

$\cong \quad \mathbb{T} \#$  $\cong \mathbb{T} \# U$

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$\implies \quad S \cong \mathbb{D}^2 \# U \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^{t+1} \mathbb{T}$

Proof of the classification theorem...

We have now proved that every surface can be written in the form

$$S \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$$

for non-negative integers d, p and t

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On the other hand, $S \cong S^2 \# \#^d \mathbb{D}^2 \# \#^t \mathbb{T} \hookrightarrow \mathbb{R}^3$ is orientable if $p = 0$

$\Longrightarrow$ S is orientable if and only if $p = 0$

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Hence, we now know all surfaces up to homeomorphism!

Orientability

Corollary

A surface S is non-orientable if and only if its polygonal decomposition contains an unoriented edge

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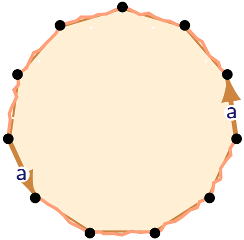
Proof Any unoriented edge gives a Möbius band inside S:



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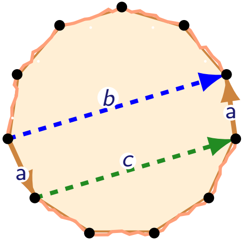

Conversely, $S = S^2 \# \#^d \mathbb{D}^2 \# \#^t \mathbb{T}$ embeds in $\mathbb{R}^3$, so it is orientable. Hence, a polygonal decomposition of S can only contain oriented edges

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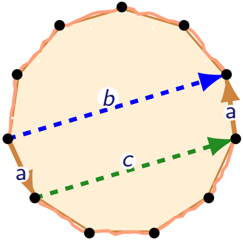

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It is now not hard to find an explicit polygonal decomposition of

$$S = S^2 \# \#^d \mathbb{D}^2 \# \#^t \mathbb{T}$$

and check that surgery cannot create unoriented edges in S

Standard forms

Theorem

Let S be a connected surface. Then there exist non-negative integers d, p and t with $pt = 0$ such that

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The standard form of a surface that is not connected has each component in standard form

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Conversely, these three characteristics of S determine (d, p, t)

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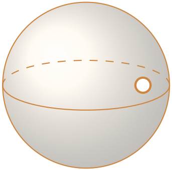
$S^2 \# \mathbb{D}^2 \quad =$ 

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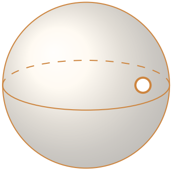

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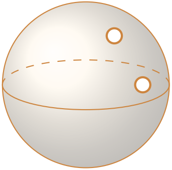

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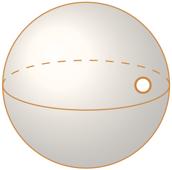

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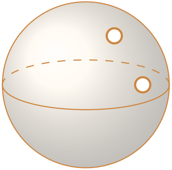

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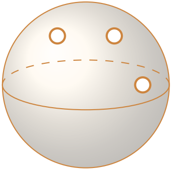

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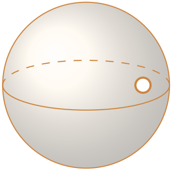

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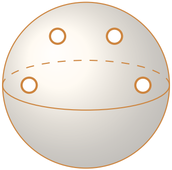
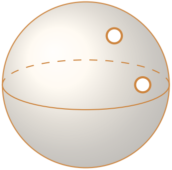
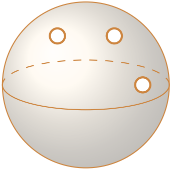


$S^2 \# \#^2 \mathbb{D}^2 =$



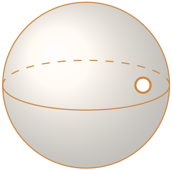
$S^2 \# \#^3 \mathbb{D}^2 =$



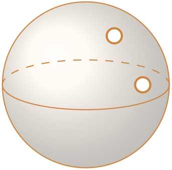
Spheres with punctures

- $S^2 \# \#^d \mathbb{D}^2$ is a sphere with d punctures

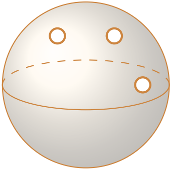
$S^2 \# \mathbb{D}^2 =$



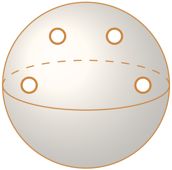
$S^2 \# \#^2 \mathbb{D}^2 =$



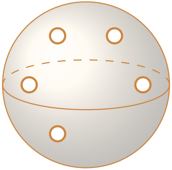
$S^2 \# \#^3 \mathbb{D}^2 =$



$S^2 \# \#^4 \mathbb{D}^2 =$

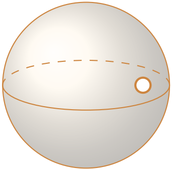


$S^2 \# \#^5 \mathbb{D}^2 =$

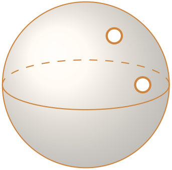


- $S^2 \# \#^d \mathbb{D}^2$ is a sphere with d punctures

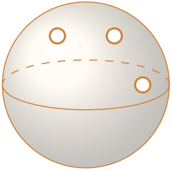
$S^2 \# \mathbb{D}^2 =$



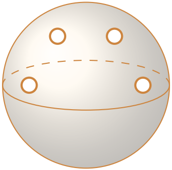
$S^2 \# \#^2 \mathbb{D}^2 =$



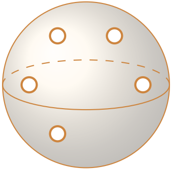
$S^2 \# \#^3 \mathbb{D}^2 =$



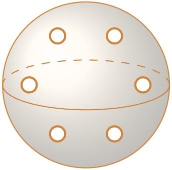
$S^2 \# \#^4 \mathbb{D}^2 =$



$S^2 \# \#^5 \mathbb{D}^2 =$

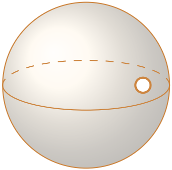


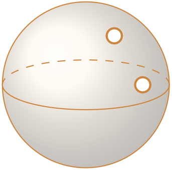
$S^2 \# \#^6 \mathbb{D}^2 =$

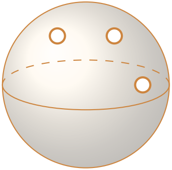


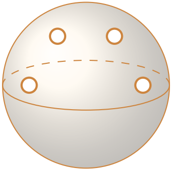
Spheres with punctures

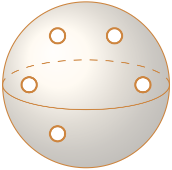
- $S^2 \# \#^d \mathbb{D}^2$ is a sphere with d punctures

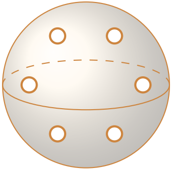
$S^2 \# \mathbb{D}^2 =$ 

$S^2 \# \#^2 \mathbb{D}^2 =$ 

$S^2 \# \#^3 \mathbb{D}^2 =$ 

$S^2 \# \#^4 \mathbb{D}^2 =$ 

$S^2 \# \#^5 \mathbb{D}^2 =$ 

$S^2 \# \#^6 \mathbb{D}^2 =$ 

More generally, $S \# \#^d \mathbb{D}^2$ is S with d punctures

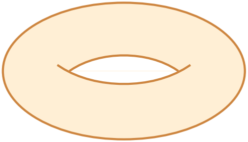
A spheres with zero and one puncture

Spheres with handles

- $S^2 \# \#^t \mathbb{T}$ is a sphere with t handles

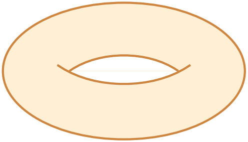
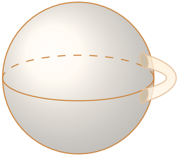
Spheres with handles

- $S^2 \# \#^t \mathbb{T}$ is a sphere with t handles

$S^2 \# \mathbb{T} \quad \cong \quad \mathbb{T} \quad \cong$ 

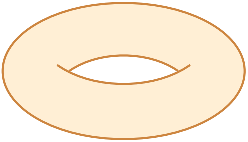
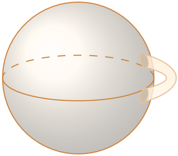
Spheres with handles

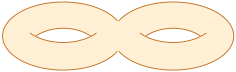
- $S^2 \# \#^t \mathbb{T}$ is a sphere with t handles

$S^2 \# \mathbb{T} \quad \cong \quad \mathbb{T} \quad \cong$  $\cong$ 

Spheres with handles

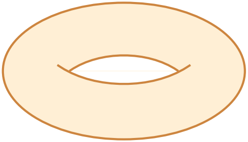
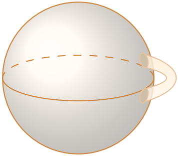
- $S^2 \# \#^t \mathbb{T}$ is a sphere with t handles

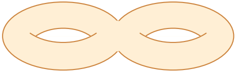
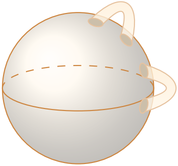
$S^2 \# \mathbb{T} \quad \cong \quad \mathbb{T} \quad \cong$  $\cong$ 

$S^2 \# \#^2 \mathbb{T} \quad \cong \quad \#^2 \mathbb{T} \quad \cong$ 

Spheres with handles

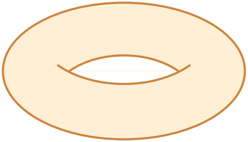
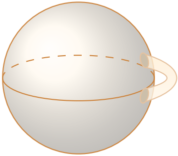
- $S^2 \# \#^t \mathbb{T}$ is a sphere with t handles

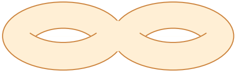
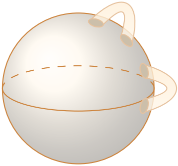
$S^2 \# \mathbb{T} \quad \cong \quad \mathbb{T} \quad \cong$  $\cong$ 

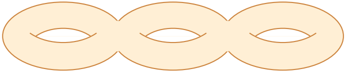
$S^2 \# \#^2 \mathbb{T} \quad \cong \quad \#^2 \mathbb{T} \quad \cong$  $\cong$ 

Spheres with handles

- $S^2 \# \#^t \mathbb{T}$ is a sphere with t handles

$S^2 \# \mathbb{T} \quad \cong \quad \mathbb{T} \quad \cong$  $\cong$ 

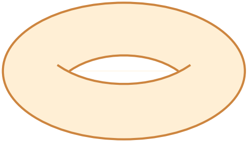
$S^2 \# \#^2 \mathbb{T} \quad \cong \quad \#^2 \mathbb{T} \quad \cong$  $\cong$ 

$S^2 \# \#^3 \mathbb{T} \quad \cong$ 

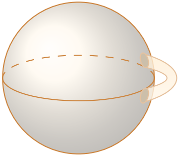
Spheres with handles

- $S^2 \# \#^t \mathbb{T}$ is a sphere with t handles

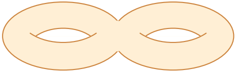
$S^2 \# \mathbb{T} \cong \mathbb{T} \cong$



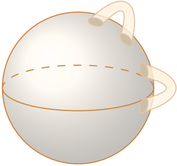
$\cong$



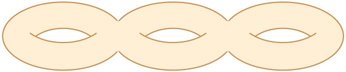
$S^2 \# \#^2 \mathbb{T} \cong \#^2 \mathbb{T} \cong$



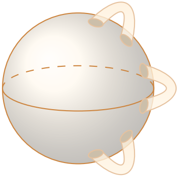
$\cong$



$S^2 \# \#^3 \mathbb{T} \cong$



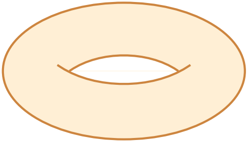
$\cong$



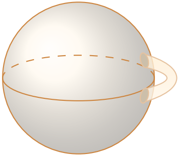
Spheres with handles

- $S^2 \# \#^t \mathbb{T}$ is a sphere with t handles

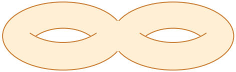
$S^2 \# \mathbb{T} \cong \mathbb{T} \cong$



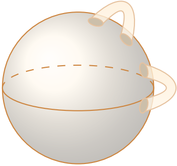
$\cong$



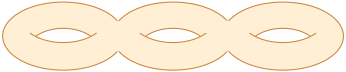
$S^2 \# \#^2 \mathbb{T} \cong \#^2 \mathbb{T} \cong$



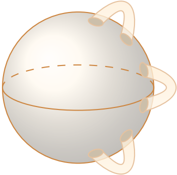
$\cong$



$S^2 \# \#^3 \mathbb{T} \cong$

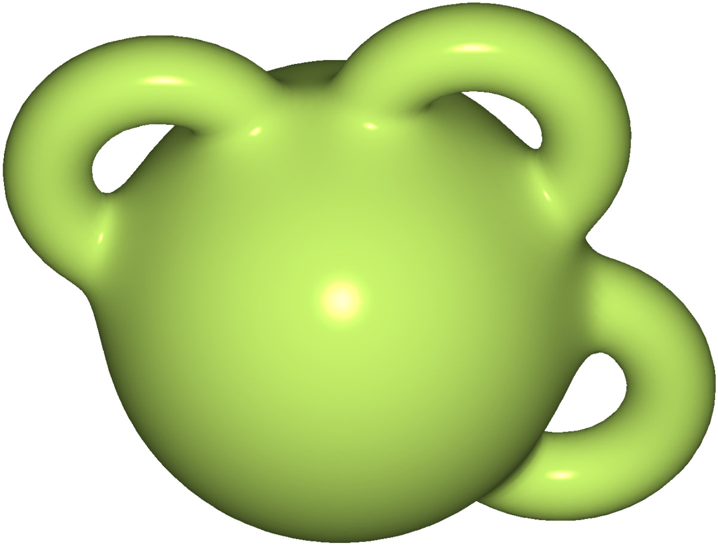


$\cong$



Continuing like this constructs a sphere with t-handles $\#^t \mathbb{T}$

Handle decomposition



Sphere with cross-caps

- $S^2 \# \#^p \mathbb{P}^2$ is a sphere with p cross-caps

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A cross-cap is what you get when you sew a Möbius strip onto the sphere

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This shape lives in $\mathbb{R}^4$, so difficult to visualize but Wikipedia draws it as:

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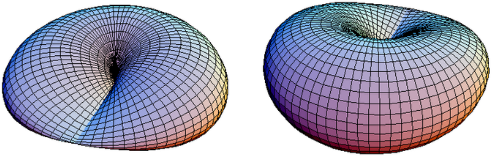

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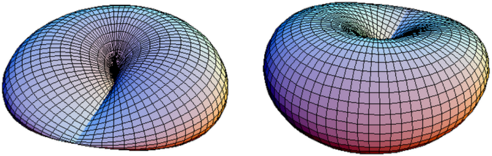
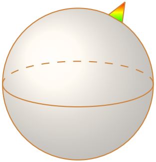


In $\mathbb{R}^3$ this surface self-intersects. We draw surfaces with cross caps as:

$S^2 \# \#^1 \mathbb{P}^2 \quad \cong$



Sphere with cross-caps

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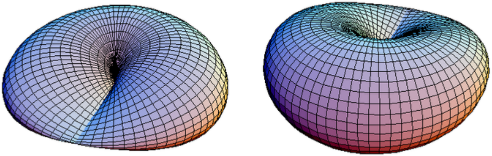
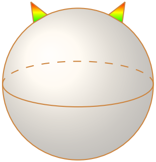


In $\mathbb{R}^3$ this surface self-intersects. We draw surfaces with cross caps as:

$S^2 \# \#^2 \mathbb{P}^2 \quad \cong$



Sphere with cross-caps

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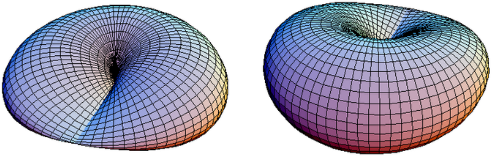

In $\mathbb{R}^3$ this surface self-intersects. We draw surfaces with cross caps as:

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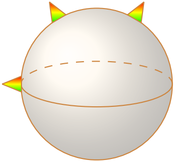

Sphere with cross-caps

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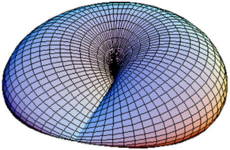
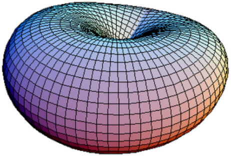
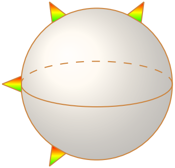




In $\mathbb{R}^3$ this surface self-intersects. We draw surfaces with cross caps as:

$$S^2 \# \#^4 \mathbb{P}^2 \quad \cong$$



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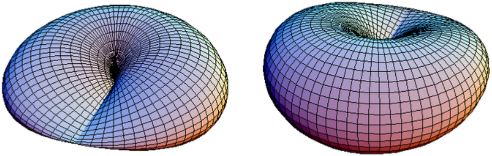
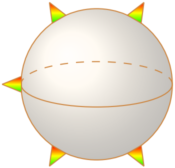


In $\mathbb{R}^3$ this surface self-intersects. We draw surfaces with cross caps as:

$S^2 \# \#^5 \mathbb{P}^2 \quad \cong$

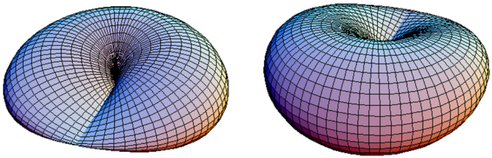


Sphere with cross-caps

- $S^2 \# \#^p \mathbb{P}^2$ is a sphere with p cross-caps

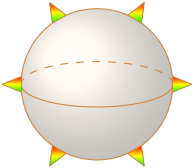
A cross-cap is what you get when you sew a Möbius strip onto the sphere

This shape lives in $\mathbb{R}^4$, so difficult to visualize but Wikipedia draws it as:



In $\mathbb{R}^3$ this surface self-intersects. We draw surfaces with cross caps as:

$$S^2 \# \#^6 \mathbb{P}^2 \quad \cong$$



What do standard surfaces look like?

We can combine the pictures above to draw all of the standard surfaces:

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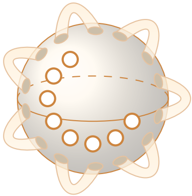
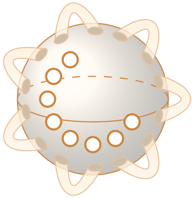
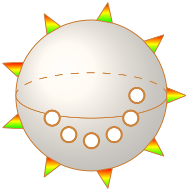
$$\#^8\mathbb{D}^2 \# \#^7\mathbb{T} \quad \cong$$



We can combine the pictures above to draw all of the standard surfaces:

$\#^8\mathbb{D}^2 \# \#^7\mathbb{T} \quad \cong$ 

$\#^6\mathbb{D}^2 \# \#^9\mathbb{P}^2 \quad \cong$ 

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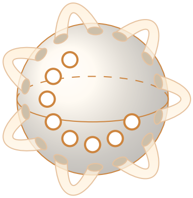
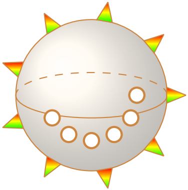
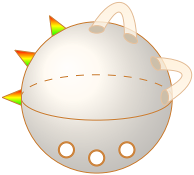


$$\#^6\mathbb{D}^2 \# \#^9\mathbb{P}^2 \quad \cong$$



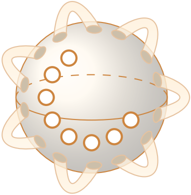
$$\#^3\mathbb{D}^2 \# \#^2\mathbb{T} \# \#^3\mathbb{P}^2 \quad \cong$$



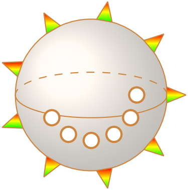
What do standard surfaces look like?

We can combine the pictures above to draw all of the standard surfaces:

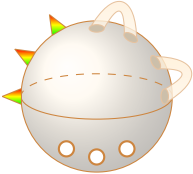
$\#^8\mathbb{D}^2 \# \#^7\mathbb{T} \quad \cong$



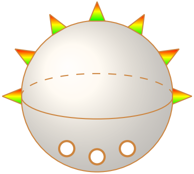
$\#^6\mathbb{D}^2 \# \#^9\mathbb{P}^2 \quad \cong$



$\#^3\mathbb{D}^2 \# \#^2\mathbb{T} \# \#^3\mathbb{P}^2 \quad \cong$



$\cong$



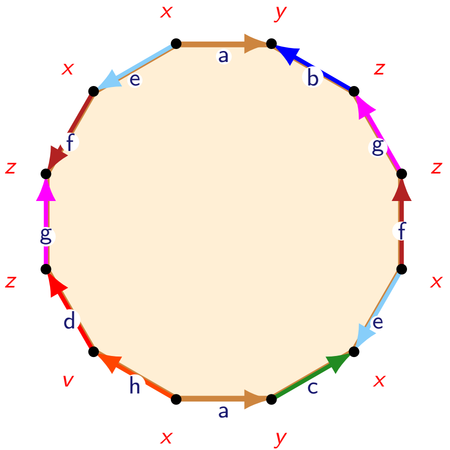
Putting a surface in standard form

Given a polygonal decomposition for a surface we can put it in standard form by:

- Find all of the vertices (identified edges implicitly identify vertices)
- Count the number d of boundary circles
- S is orientable $(p = 0)$ if all edges are oriented otherwise it is non-orientable $(t = 0)$
- Compute $\chi(S) = 2 - d - p - 2t$ to determine the missing variable, which is t if S is orientable and or p if non-orientable

Example 1

What is the surface with the below polygonal decomposition?

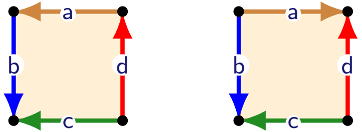


$a\,c\,\overline{e}\,f\,g\,b\,\overline{a}\,e\,f\,\overline{g}\,\overline{dh}$ (overline=opposite direction)

$\Longrightarrow$ This is $\#^1\mathbb{D}^2 \# \#^0\mathbb{T} \# \#^4\mathbb{P}^2$

Example 2

What is the standard form of the surface with polygonal decomposition?



Example 2

What is the standard form of the surface with polygonal decomposition?