Topology – week 7

Math3061

Daniel Tubbenhauer, University of Sydney

Technicalities

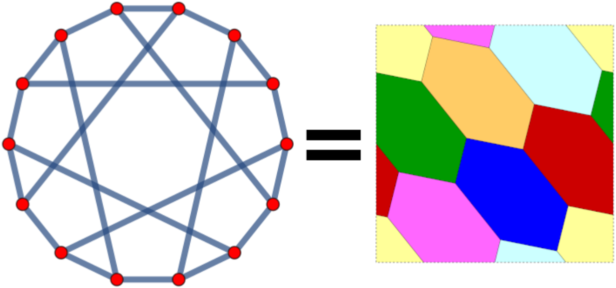
Lecturer Daniel Tubbenhauer

Office hour Zoom (https://uni-sydney.zoom.us/j/89436493625) Monday 4:30pm-5:30pm or by appointment (an informal email suffices)

Contact daniel.tubbenhauer@sydney.edu.au

Web www.dtubbenhauer.com/teaching.html

- I apologize in advances for any typos or other errors on these slides!
- I'd be grateful for any corrections that you send to me or post on Ed



Unit outline

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- Surfaces
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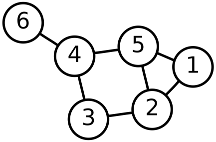

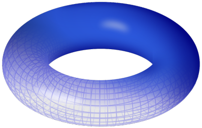

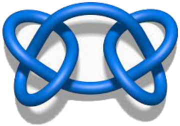

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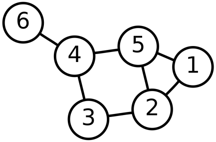

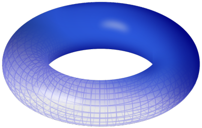

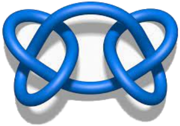

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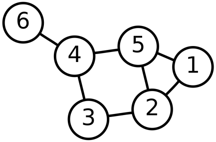

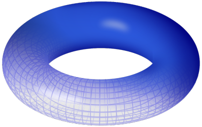

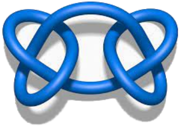

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- In topology we are allowed to bend and stretch
- We are not allowed to cut, tear or join surfaces together

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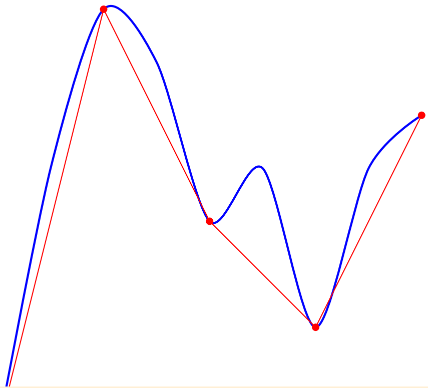

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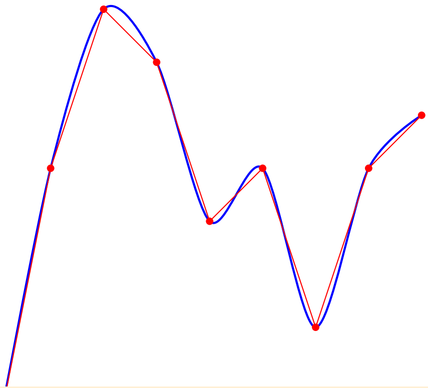
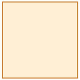


Topological equivalences



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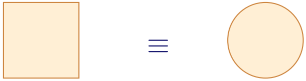

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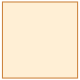
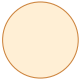
 $\equiv$ 

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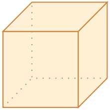
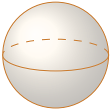
 $\equiv$ 

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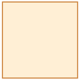
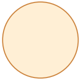
 $\equiv$ 

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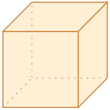
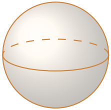
 $\equiv$ 

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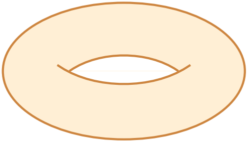
...as well as looking at more exotic surfaces

 =  $\equiv$??

A torus is the same as a coffee mug

Source https://en.wikipedia.org/wiki/Topology

Graphs

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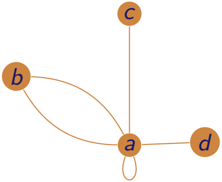

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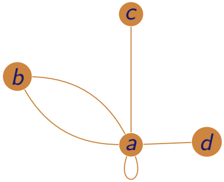

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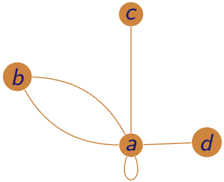

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As shown, we allow loops and duplicate edges

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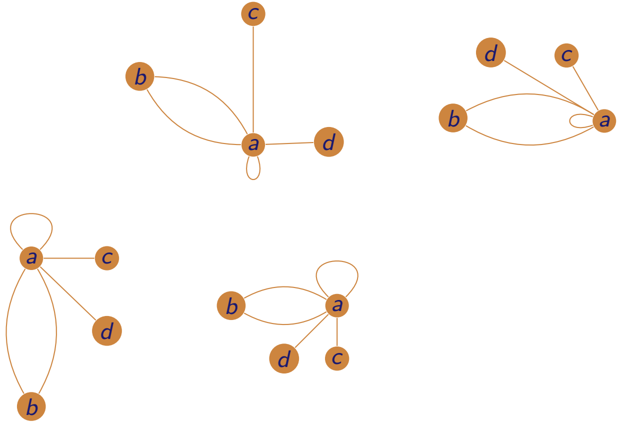
Here are four different ways to draw the same graph



Standard graphs

Path graphs P_n, for $n \geq 1$ (also called line graphs)

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Edge set $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}\}$

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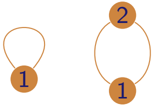

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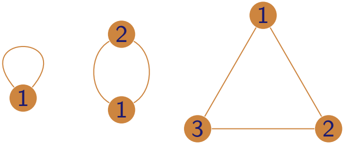

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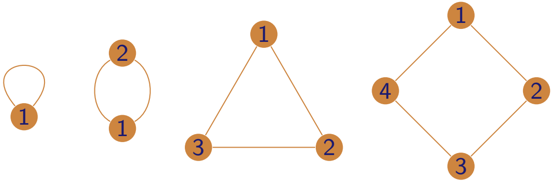

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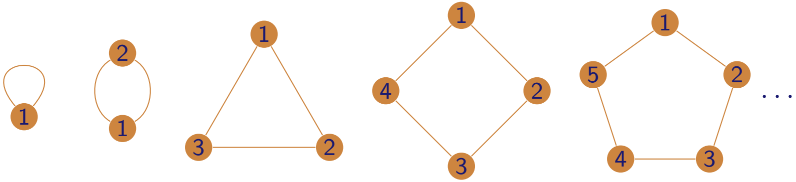

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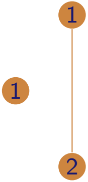

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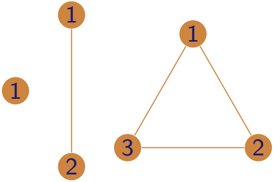

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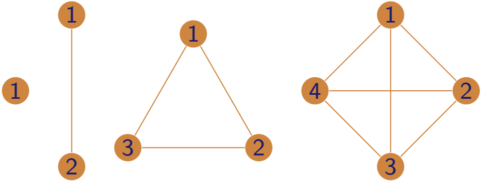

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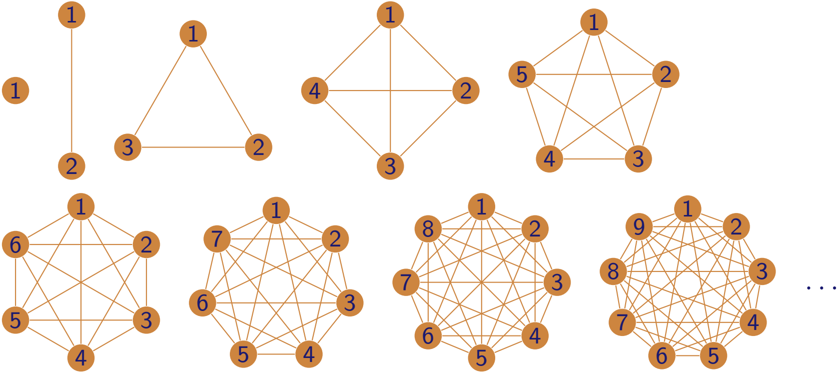

Standard graphs...

Complete bipartite graphs $K_{n,m}$, for $n, m \geq 1$

Vertex set $V = \{1, 2, \ldots, n, 1', 2', \ldots, m'\}$

Edge set $E = \left\{ \{i, j'\} \,\middle|\, 1 \leq i \leq n, 1 \leq j \leq m \right\}$

$K_{2,3}$

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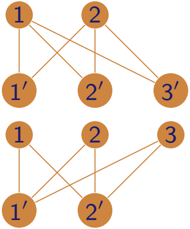

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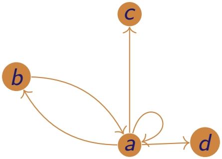

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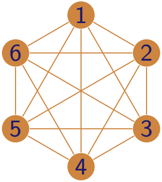

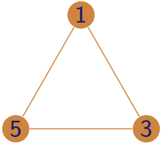

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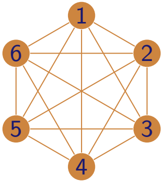

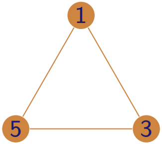

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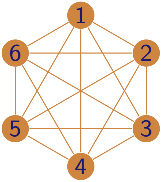

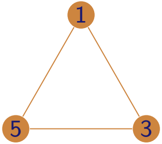

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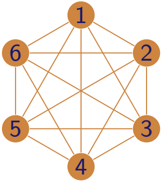

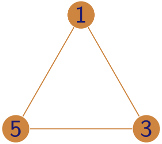

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...but what does it mean for graphs to be "the same "?

Isomorphic graphs

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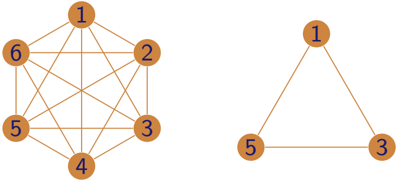

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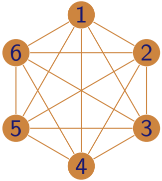

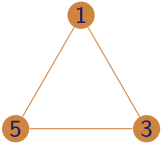

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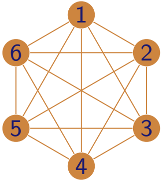

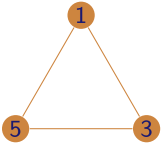

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For example, define f by

$f(1) = 1$,

$f(3) = 2$, and

$f(5) = 3$

Subgraphs of complete graphs

Proposition

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Write $V = \{v_1, v_2, \ldots, v_n\}$.

Let $N = \{1, 2, \ldots, n\}$ be the vertex set of K_n and let

$$E_n = \{\, \{i, j\} \mid 1 \le i < j \le n \,\}$$

be its edge set.

Define $H = (N, E_V)$ to be the subgraph of K_n with

$$E_V = \{\, \{i, j\} \mid \{v_i, v_j\} \in E \,\}.$$

Then the map $f : N \longrightarrow V$ given by $f(i) = v_i \in V$ is a graph isomorphism.

Planar graphs

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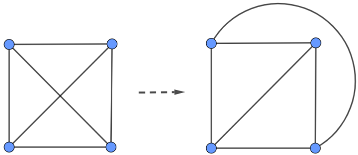
Examples

- Graphs can have planar embeddings and other non-planar realizations

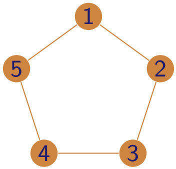
- Every path graph P_n is planar

- Every cyclic graph C_n is planar



Planar graph (K_4)

Planar embedding of K_4



Complete graphs are rarely planar

- K_1 

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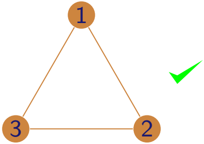
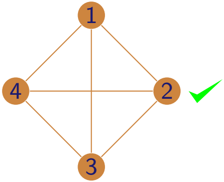
- K_3 

- K_4 

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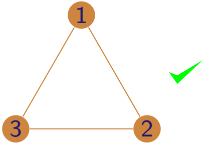
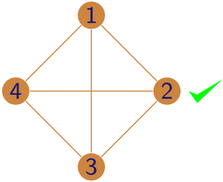
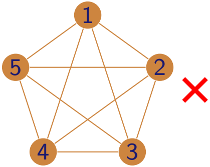
- K_3 
- K_4 
- K_5 

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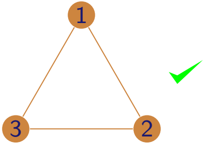
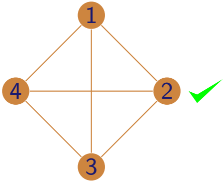
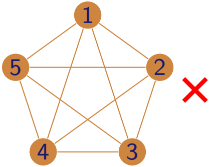
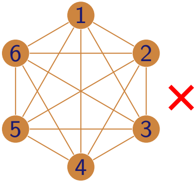
- K_3 

- K_4 

- K_5 

- K_6 

Graph embeddings in $\mathbb{R}^3$

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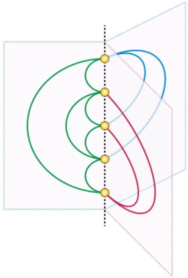
Moral Graphs are "low dimensional" objects

Proof First, loops and duplicate edges are easy to treat, so we ignore them. Next, use a book embedding:

A 3-page embedding of K_5:



In general, one can embed K_n into a book with $\lceil n/2 \rceil$ pages. Since every graph is a subgraph of some K_n, so we are done since books $\subset \mathbb{R}^3$

The degree of a vertex

Let $G = (V, E)$ be a graph. The degree of a vertex $v \in V$ is

$$\deg(v) = \#\Big\{\text{number of edges in } E \text{ that have } v \text{ as an endpoint}\Big\}$$

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Examples

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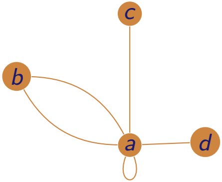
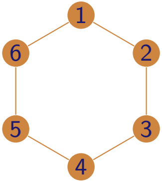
- 

deg(a)=5
- P_n 

deg(4)=2

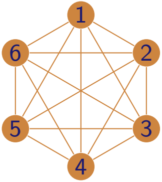
Degrees of vertices in standard graphs; examples

- C_n



deg(4)=2

- K_n



deg(4)=5

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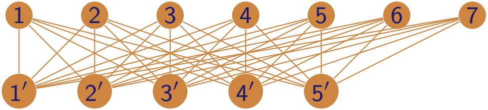

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The handshaking lemma

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Let $G = (V, E)$ be a finite graph. Then

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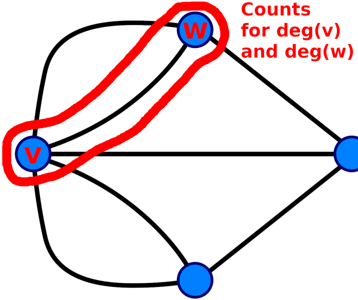
Proof If I shake your hand, then you shake mine: every edges is adjacent to two vertices, hence each edges contributes twice



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Strictly speaking, we would use induction on $|E|$:

There is nothing to show if there is no edge, and if $|E| > 0$ remove any edge e use induction for $E' = E \setminus \{e\}$, and add e using the previous observation

The Euler characteristic of a graph

Let $G = (V, E)$ be a graph. The Euler characteristic of G is the integer

$$\chi(G) = |V| - |E|$$

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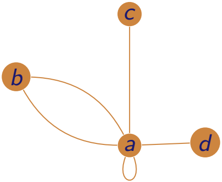
Examples

- 

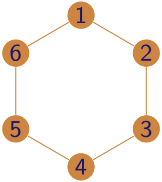
$\chi(G) = -1$
- 

$\chi(G) = -1$
- P_n 

$\chi(G) = 1$

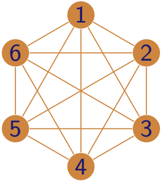
The Euler characteristic of standard graphs

- C_n



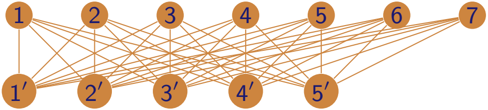
$\chi(G) = 0$

- K_n



$\chi(G) = n - \frac{1}{2}n(n-1) = -\frac{1}{2}n(n-3)$

- $K_{n,m}$



$\chi(G) = n + m - nm$

Subdividing graphs

Let $G = (V, E)$. A subdivision of G is any graph $\dot{G}$ that is obtained from G by successively replacing V with $V \cup \{u\}$, for $u \notin V$, and E with $E \cup \big\{\{v, u\}, \{u, w\}\big\} \setminus \big\{\{v, w\}\big\}$, for an edge $\{v, w\} \in E$

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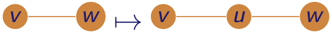

clearly increases V and E by one, so their difference does not change.

Paths in graphs

Let $G = (V, E)$ be a graph and $v, w \in V$. A path in G of length n from v to w is a sequence of vertices $v = v_0, v_1, \ldots, v_n = w$ such that $\{v_i, v_{i+1}\} \in E$, for $0 \leq i < n$.

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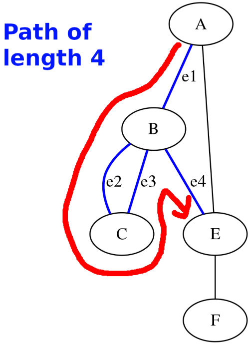

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Example

Not connected, two connected components

Connected examples

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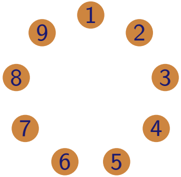
- A fully "disconnected" graph:



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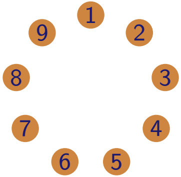

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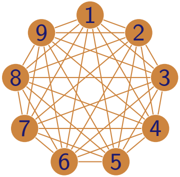

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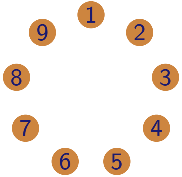

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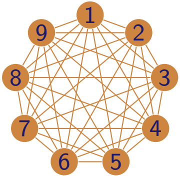

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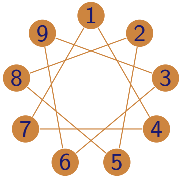

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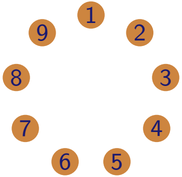

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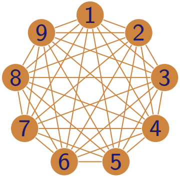

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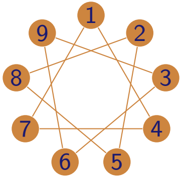

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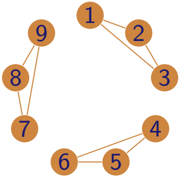

Circuits

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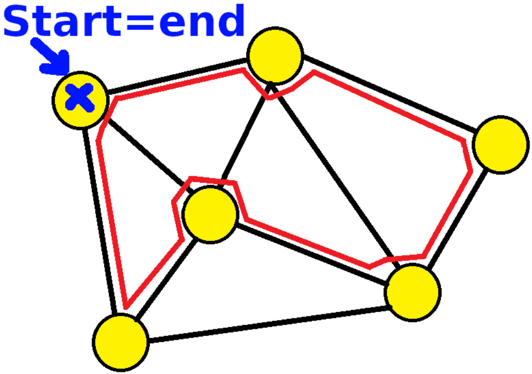

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- We will soon see that the Euler characteristic is closed related to the number of "reduced" circuits in a graph

Contractible circuits

A circuit $v = v_0 \to v_1 \to \cdots \to v_n = v$ is contractible if it contains two consecutive repeated edges $\{v_i, v_{i+1}\} = \{v_{i+1}, v_{i+2}\}$, for some $0 \leq i \leq n-2$

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Observations

- Reduced circuits are "efficient" in the sense that they do not backtrack
- A reduced circuit of length n is not necessarily isomorphic to the cycle graph C_{n+1} because it could, for example, be a figure 8 graph

Leaves and trees

A non-trivial circuit is a reduced circuit of length $n > 0$

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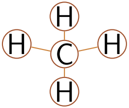
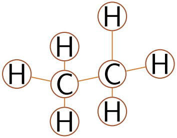
A tree is a connected graph that has no non-trivial circuits

- Saturated hydrocarbons



Methane



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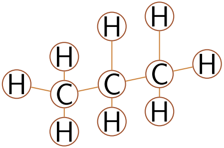

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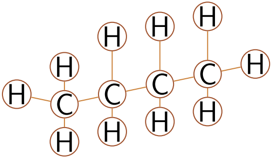

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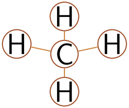

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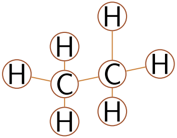

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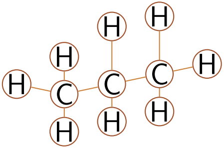

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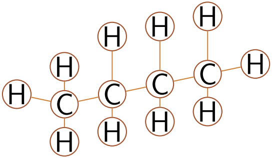

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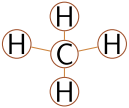

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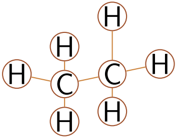

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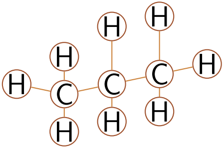

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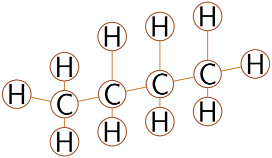
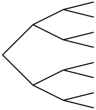
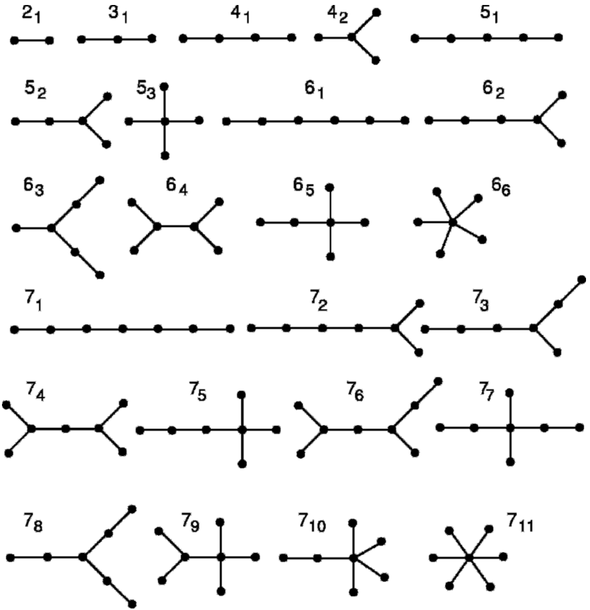


Butane

- A tournament tree



A catalog of small (connected) trees



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Proof Take a longest reduced path P in T, then both endpoints of P are leaves

Why? Say the endpoints are v and w. WLOG suppose v is not a leaf; then v has at least two neighbors and one of them is not in P. (Otherwise we would have a circuit.) Thus one can make P longer. Contradiction

The Euler characteristic of a tree

Theorem

Suppose that T is a tree. Then $\chi(T) = 1$

The Euler characteristic of a tree

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Suppose that T is a tree. Then $\chi(T) = 1$

Proof Argue by induction on the number of edges $|E|$

For $|E|$ small use the previous table.

Otherwise, remove one leave (which exists by the previous statement). The resulting tree has $\chi(T) = 1$, and adding the leave back increases V and E by one, so χ remains constant

Number of edges and vertices in a tree

Corollary

Suppose that $T = (V, E)$ *is a tree. Then* $|V| = |E| + 1$.

Number of edges and vertices in a tree

Corollary

Suppose that $T = (V, E)$ is a tree. Then $|V| = |E| + 1$.

Proof By the previous statement

Spanning trees

Proposition

Suppose that $G = (V, E)$ *is a connected graph.*
Then G *has a subgraph* $T = (V, F)$ *(same vertices) that is a tree*

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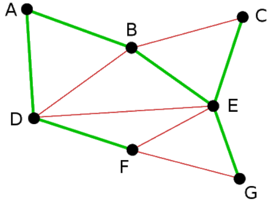
Example



Spanning trees continued

Proposition

Suppose that $G = (V, E)$ is a connected graph.
Then G has a spanning tree $T = (V, F)$ (same vertices)

Proof Remove edges from nontrivial circuit of G to break them; the result is a spanning tree

(Formally, use induction on the number of nontrivial circuit of G)

An upper bound on $\chi(G)$

Corollary

Suppose that G is a connected graph. Then $\chi(G) \leq 1$ with equality if and only if G is a tree.

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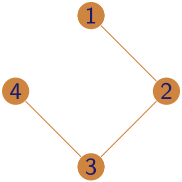
Corollary

Let G be a connected graph. The number of independent cycles (defined via example on the next slide) in G is $1 - \chi(G)$

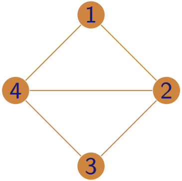
Proof By the previous statements

Independent cycles

Examples



$\chi = 1$, no independent cycles

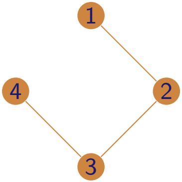


$\chi = -1$, two independent cycles

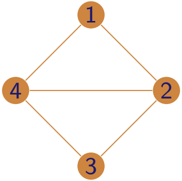
We have $\{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\} =$
$\{\{1,2\},\{2,4\},\{1,4\}\} + \{\{2,3\},\{3,4\},\{2,4\}\} \mod 2$

Independent cycles

Examples



$\chi = 1$, no independent cycles



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Remark It is possible to construct a vector space of "cycles" that has dimension $1 - \chi(G)$, which shows that the number of independent cycles makes sense. This is beyond the scope of this course.