Tutorial 10

Weekly summary and definitions and results for this tutorial

a) Up to homeomorphism, every connected surface can be written *uniquely* in **standard form**

$$S = S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T},$$

where $d, p, t \geqslant 0$ and $tp = 0$.

b) If S and T are surfaces then $\chi(S \# T) = \chi(S) + \chi(T) - 2$.

c) We saw that $\mathbb{K} \cong \mathbb{P}^2 \# \mathbb{P}^2$ and that $\mathbb{T} \# \mathbb{P}^2 \cong \mathbb{K} \# \mathbb{P}^2 \cong \#^3 \mathbb{P}^2$.

d) The surface $S^2 \# \#^t \mathbb{T} \# S^2$ is a sphere with **t-handles**. The surface $S^2 \# \#^p \mathbb{P}^2$ is a sphere with **c-cross caps**. The surface $S^2 \# \#^d \mathbb{D}^2$ is a sphere with **n-punctures** is $\#^n \mathbb{D}^2 \# S^2$.

e) The **word** of a surface S that has a polygonal decomposition with one polygon is the sequence of *directed* edges read in anti-clockwise order around the perimeter of the polygon. Edges pointing anti-clockwise are written as $a, b, c, \dots$ and clockwise edges as $\overline{a}, \overline{b}, \overline{c}, \dots$.

f) A paired edge a a surface is **oriented** if a and $\overline{a}$ both appear in the word; otherwise it is **unoriented**.

g) The standard word for the surface $\#^t$ is $a_1\, b_1\, \overline{a_1}\, \overline{b_1} \dots a_t\, b_t\, \overline{a_t}\, \overline{b_t}$ and the standard ford for $\#^c \mathbb{P}^2$ is $a_1\, a_1\, a_t\, a_t$. Standard words for punctured surfaces are more complicated!

h) The boundary of a surface is a disjoint union of boundary circles. In particular, every free edge is contained in a boundary circle.

i) In the standard form, if $p \neq 0$ then the surface is non-orientable with $p > 0$ **cross caps** and, otherwise, it is orientable with $t \geqslant 0$ **handles**. In all cases, d is the number of punctures.

j) The **Euler characteristic** of the standard surface $S = S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$ is $\chi(S) = 2 - d - p - 2t$.

k) Up to homeomorphism, a surface is uniquely determined by the number of boundary circles (d), its orientability (orientable if $t \neq 0$ and non-orientable if $p \neq 0$), and its Euler characteristic (determines t and p).

Questions to complete *before* the tutorial

1. Express the following surfaces in standard form, compute its Euler characteristic and determine whether the surface is orientable or non-orientable.

a) $S_1 = S^2 \# \#^2 \mathbb{T} \# \mathbb{P}^2 \# \mathbb{K} \# \mathbb{T} \# \mathbb{K}$.
b) $S_2 = \mathbb{A} \# \#^2 \mathbb{M} \# \mathbb{K} \# \#^2 \mathbb{D}^2 \# \#^2 \mathbb{T}$.
c) $S_3 = \#^3 (\mathbb{T} \# \mathbb{A}) = (\mathbb{T} \# \mathbb{A}) \# (\mathbb{T} \# \mathbb{A}) \# (\mathbb{T} \# \mathbb{A})$.

Questions to complete *during* the tutorial

2. The following words each represent a polygonal decomposition with one face of a surface.

$(i)\quad abc\bar{b}d \qquad (ii)\quad abcba \qquad (iii)\quad abc\bar{a}\bar{b} \qquad (iv)\quad abcd\bar{a}\bar{b}\bar{c}\bar{d} \qquad (v)\quad abcd\bar{a}bc\bar{d}.$

Let S be the corresponding surface in each case.

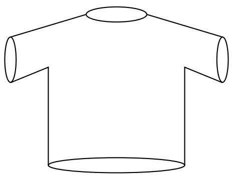
a) Draw a polygon with directed edges labelled by the edge labels $a, b, c, \ldots$.
b) Hence determine the number of vertices in the decomposition.
c) Work out the Euler characteristic $\chi(S)$ of S.
d) Calculate how many boundary circles the surface has.
e) Determine if the surface is orientable or non-orientable.
f) Write down the standard form for S.

3. Determine the standard forms of the connected surfaces corresponding to the following words:

a) $abc\bar{a}\bar{b}\bar{c}$
b) $abcabc$
c) $abc\bar{b}\bar{a}c$.

In each case apply the surgery operations from lectures to rewrite the words as the corresponding *standard word* for the surface.

4. a) Determine the standard form and the Euler characteristic of an "ideal" T-shirt (an ideal T-shirt is a T-shirt that has no thickness).



b) Determine the standard form and the Euler characteristic of the surface of a fully padded T-shirt. That is, a T-shirt that is made by taking two T-shirts with one inside the other and sewing (or gluing), along the boundary circles corresponding to the neck, sleeves and hem.

5. Determine the possible standard forms of the following connected surfaces:

a) A surface with 7 punctures and Euler characteristic -10
b) A surface with 10 punctures and Euler characteristic -7
c) A surface with 10 punctures and Euler characteristic -11
d) A surface with 10 punctures and Euler characteristic -8
e) A surface with 10 punctures and Euler characteristic -10

6. Let S be the surface given by the word $a\,d\,b\,\bar{c}\,\bar{a}\,c\,b\,d$.

a) Draw a polygonal decomposition of S.
b) How many vertices are there in your polygonal decomposition of S?
c) How many boundary circles does S have?
d) Is S orientable? Explain.
e) Compute the Euler characteristic $\chi(S)$ of the surface S.
f) Describe S as a *standard surface* – that is, as a sphere with punctures, handles and cross caps.