# Tutorial 10 — Solutions

*Weekly summary and definitions and results for this tutorial*

a) Up to homeomorphism, every connected surface can be written *uniquely* in **standard form**

$$S = S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T},$$

where $d, p, t \geqslant 0$ and $tp = 0$.

b) If $S$ and $T$ are surfaces then $\chi(S \# T) = \chi(S) + \chi(T) - 2$.

c) We saw that $\mathbb{K} \cong \mathbb{P}^2 \# \mathbb{P}^2$ and that $\mathbb{T} \# \mathbb{P}^2 \cong \mathbb{K} \# \mathbb{P}^2 \cong \#^3 \mathbb{P}^2$.

d) The surface $S^2 \# \#^t \mathbb{T} \# S^2$ is a sphere with ***t*-handles**. The surface $S^2 \# \#^p \mathbb{P}^2$ is a sphere with ***c*-cross caps**. The surface $S^2 \# \#^d \mathbb{D}^2$ is a sphere with ***n*-punctures** is $\#^n \mathbb{D}^2 \# S^2$.

e) The **word** of a surface $S$ that has a polygonal decomposition with one polygon is the sequence of *directed* edges read in anti-clockwise order around the perimeter of the polygon. Edges pointing anti-clockwise are written as $a, b, c, \dots$ and clockwise edges as $\overline{a}, \overline{b}, \overline{c}, \dots$.

f) A paired edge $a$ a surface is **oriented** if $a$ and $\overline{a}$ both appear in the word; otherwise it is **unoriented**.

g) The standard word for the surface $\#^t$ is $a_1\, b_1\, \overline{a_1}\, \overline{b_1} \dots a_t\, b_t\, \overline{a_t}\, \overline{b_t}$ and the standard ford for $\#^c \mathbb{P}^2$ is $a_1\, a_1\, a_t\, a_t$. Standard words for punctured surfaces are more complicated!

h) The boundary of a surface is a disjoint union of boundary circles. In particular, every free edge is contained in a boundary circle.

i) In the standard form, if $p \neq 0$ then the surface is non-orientable with $p > 0$ **cross caps** and, otherwise, it is orientable with $t \geqslant 0$ **handles**. In all cases, $d$ is the number of punctures.

j) The **Euler characteristic** of the standard surface $S = S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$ is $\chi(S) = 2 - d - p - 2t$.

k) Up to homeomorphism, a surface is uniquely determined by the number of boundary circles ($d$), its orientability (orientable if $t \neq 0$ and non-orientable if $p \neq 0$), and its Euler characteristic (determines $t$ and $p$).

## Questions to complete *before* the tutorial

**1.** Express the following surfaces in standard form, compute its Euler characteristic and determine whether the surface is orientable or non-orientable.

a) $S_1 = S^2 \# \#^2 \mathbb{T} \# \mathbb{P}^2 \# \mathbb{K} \# \mathbb{T} \# \mathbb{K}$.
b) $S_2 = \mathbb{A} \# \#^2 \mathbb{M} \# \mathbb{K} \# \#^2 \mathbb{D}^2 \# \#^2 \mathbb{T}$.
c) $S_3 = \#^3(\mathbb{T} \# \mathbb{A}) = (\mathbb{T} \# \mathbb{A}) \#(\mathbb{T} \# \mathbb{A}) \#(\mathbb{T} \# \mathbb{A})$.

*Solution*

a) The surface $S_1$ contains a projective plane $\mathbb{P}^2$, so it is not orientable. Further, if $S^2$ if any surface then $S^2 = S^2 \# S^2$, so $S_1 = \#^3\,\mathbb{T} \#^2 \mathbb{K} \# \mathbb{P}^2$. In lectures we saw that $\mathbb{T} \# \mathbb{P}^2 \cong \#^3 \mathbb{P}^2$ and that $\mathbb{K} \cong \#^2 \mathbb{P}^2$. Therefore, $S_1 \cong \#^{11} \mathbb{P}^2$.
We could also determine this surface by computing the Euler characteristic. In lectures we saw that $\chi(A \# B) = \chi(A) + \chi(B) - 2$, so $\chi(S_1) = 3\chi(\mathbb{T}) + 2\chi(\mathbb{K}) + \chi(\mathbb{P}^2) - 10 = -9$ because $\chi(\mathbb{T}) = 0 = \chi(\mathbb{K})$ and $\chi(\mathbb{P}^2) = 1$ by Tutorial 8. As $S_1$ has no holes (all edges are paired), it follows that $S_1 = \#^{11}\,\mathbb{P}^2$. That is, $S_1$ is a sphere with 11 cross caps attached. In particular, $S_1$ is non-orientable.

b) From lectures we know that $\mathbb{A} \cong \#^2 \mathbb{D}^2$, $\mathbb{M} \cong \mathbb{D}^2 \# \mathbb{P}^2$, $\mathbb{K} \cong \#^2 \mathbb{P}^2$ and $\mathbb{T} \# \mathbb{P}^2 \cong \#^3 \mathbb{P}^2$. Hence, we deduce that $S_2 \cong \#^6 \mathbb{D}^2 \# \#^8 \mathbb{P}^2$. This surface is non-orientable with its standard form being a sphere with 8 cross caps and 6 punctures.
As an alternative to using our knowledge of the standard surfaces, as in part (a),

$$\chi(S_2) = \chi(\mathbb{A}) + 2\chi(\mathbb{M}) + \chi(\mathbb{K}) + 2\chi(\mathbb{D}^2) + 2\chi(\mathbb{T}) - 14 = 2 - 14 = -12.$$

As in the previous paragraph we count that $S_2$ has 6 punctures (there are two in each Möbius band and two other discs), so the number of cross caps is $c = 2 - 6 - \chi(S_2) = 8$. Hence, this surface is homeomorphic to $\#^6 \mathbb{D}^2 \# \#^8 \mathbb{P}^2$ as before.

c) As $\mathbb{T} \# \mathbb{A} \cong \mathbb{T} \# \#^2 \mathbb{D}^2$ we see immediately that $S_3 \cong \#^3 \mathbb{T} \# \#^6 \mathbb{D}^2$. This is the orientable surface with three handles and six punctures.

## Questions to complete *during* the tutorial

**2.** The following words each represent a polygonal decomposition with one face of a surface.

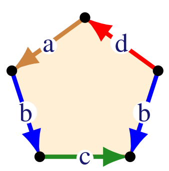
$$(i) \quad abc\bar{b}d \qquad (ii) \quad abcba \qquad (iii) \quad abc\bar{a}\bar{b} \qquad (iv) \quad abcd\bar{a}\bar{b}\bar{c}\bar{d} \qquad (v) \quad abcd\bar{a}bc\bar{d}.$$

Let $S$ be the corresponding surface in each case.

a) Draw a polygon with directed edges labelled by the edge labels $a, b, c, \ldots$ .
b) Hence determine the number of vertices in the decomposition.
c) Work out the Euler characteristic $\chi(S)$ of $S$.
d) Calculate how many boundary circles the surface has.
e) Determine if the surface is orientable or non-orientable.
f) Write down the standard form for $S$.

*Solution*

$(i)$ The only paired edge are $b$ and $\bar{b}$ and the remaining edges $a, c$ and $d$ are boundary edges. Therefore, this is an oriented surface with boundary. The polygon for this surface has 3 vertices, so the Euler characteristic of this surface is $3 - 4 + 1 = 0$. The edges $a, c$ and $d$ are unpaired, so there are at most three boundary circles. As the unpaired edges $a$ and $d$ are consecutive,there are 2 boundary circles. Hence $p = t = 0$, so this surface is the twice punctured sphere. That is, it is the annulus $\mathbb{A} = \mathbb{D}^2 \# \mathbb{D}^2$.



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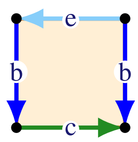
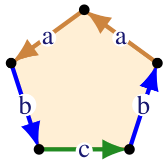
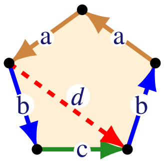


$\cong \quad \mathbb{D}^2 \# \mathbb{D}^2 \quad \cong \quad \mathbb{A}.$

(*ii*) The surface $abcba$ has non-empty boundary, because $c$ is not paired, and it is non-orientable, because the edges $a$ and $b$ are both unoriented. Drawing the polygonal decomposition,




shows that there is only one vertex. Therefore, the Euler characteristic of this surface is $1-3+1=-1$. It follows from the classification of surfaces in lectures, using orientability and the Euler characteristic, that this surface is $\#^2\mathbb{P}^2\,\#\,\mathbb{D}^2$. We can also see this directly using surgery:



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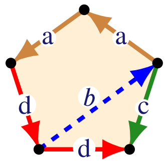
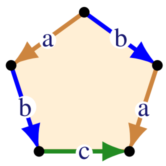


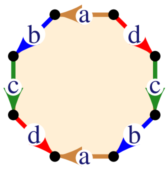
$\cong \quad \#^2\mathbb{P}^2\,\#\,\mathbb{D}^2 \quad \cong \quad \mathbb{K}\,\#\,\mathbb{D}^2.$

(*iii*) The surface $abc\bar{a}\bar{b} = c\bar{a}\bar{b}ab$ has two paired edges $a$ and $\bar{a}$, and $b$ and $\bar{b}$ and one unpaired edge $c$. Therefore, it is an oriented surface with one boundary circle and polygonal decomposition



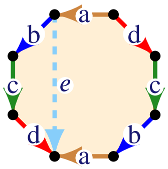

This polygonal decomposition has only one vertex, so the Euler characteristic of $c\bar{a}\bar{b}ab$ is $1-3+1=-1$. This surface is $\mathbb{T}\,\#\,\mathbb{D}^2$, the once punctured torus.

(*iv*) The surface $abcd\bar{a}\bar{b}\bar{c}\bar{d}$ is orientable and has no boundary because each edge is paired with its opposite. Drawing the polygonal decomposition,




shows that this decomposition has only one vertex. Therefore, the Euler characteristic of this surface is $1-4+1=-2$. Therefore, this surface is the double torus, or the sphere with two handles, $\#^2\mathbb{T}$.

Again, we can see this directly using surgery:



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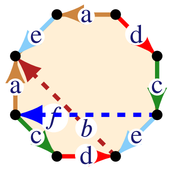

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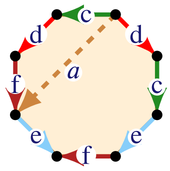
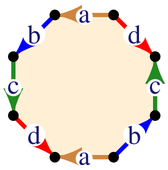
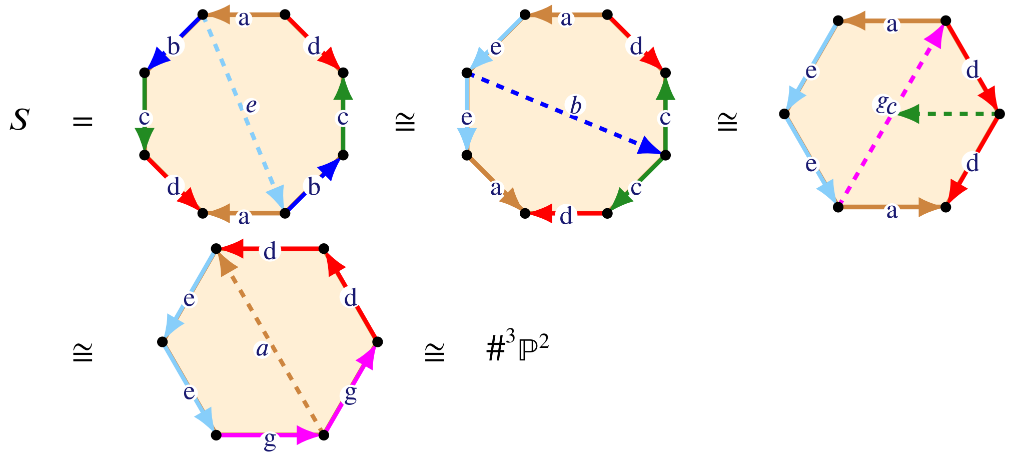



For the first isomorphism, cut along $e$ and the glue along $b$. Next, cut along $f$ and glue along $a$. The last polygonal decomposition is the standard form for $\#^2\mathbb{T}$.

(*v*) The surface $S = abcd\bar{a}bc\bar{d}$ has no boundary because each edge is paired. The surface is non-orientable because the edges $b$ and $c$ are both unoriented. The polygonal decomposition of this surface is:



This polygon has two vertices, so the Euler characteristic of $abcd\bar{a}bc\bar{d}$ is $2-4+1=-1$. Hence, this surface is $\#^3\mathbb{P}^2$, the sphere with 3 cross caps. We show this directly using surgery:




For the second homeomorphism we have contracted the edge sequence $\bar{c}c$.

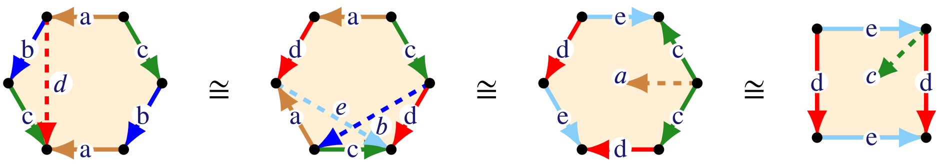
**3.** Determine the standard forms of the connected surfaces corresponding to the following words:

a) $abc\bar{a}\bar{b}\bar{c}$
b) $abcabc$
c) $abc\bar{b}\bar{a}c$.

In each case apply the surgery operations from lectures to rewrite the words as the corresponding *standard word* for the surface.

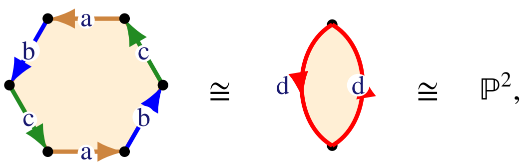
*Solution* The point of this question was for you to practice any techniques used in class over the last couple of lectures. (You should be prepared to experiment a little!) In each case every letter used is repeated (sometimes with inversion) and so there are no boundary circles.

a) The word $abc\bar{a}\bar{b}\bar{c}$ has no unpaired letters and all edges are occur in oriented pairs $\dots x \dots \bar{x} \dots$, so this represents an orientable surface without boundary. Hence, this surface is of the form $\#^t\mathbb{T}$, for some $t \geqslant 0$. In fact, since the minimal polygon for a torus has two edges it follows that $t \in \{0,1\}$, so it is not hard to see that this surface is the torus $\mathbb{T}$. We now show this using surgery:



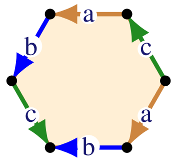

where in the last homeomorphism we have folded in and removed the oriented edges $\bar{c}c$. Hence, this surface is homeomorphic to the torus $\mathbb{T}$ as claimed. Hence, the standard word for this surface is $a_1b_1\bar{a}_1\bar{b}_1$.

b) The word $abcabc$ has three unoriented edges, however, we can replace $abc$ with a single edge to see that this surface is the projective plane $\mathbb{P}^2$:




where $d = abc$. Notice that this is the standard 2-polygonal decomposition of $\mathbb{P}^2$. In particular, the standard word is $a_1a_1$.

c) The surface $abc\bar{b}\bar{a}c$ has no boundary and is non-orientable since all edges are paired and $c$ is unoriented. Notice that the edge $ab$ can be combined into a single edge, so using surgery this surface becomes:



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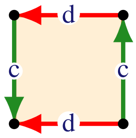
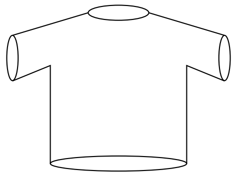


$\cong \mathbb{K}$.

Hence, this surface is the Klein bottle $\mathbb{K} \cong \#^2\mathbb{P}^2$.

**4.** a) Determine the standard form and the Euler characteristic of an "ideal" $T$-shirt (an ideal $T$-shirt is a $T$-shirt that has no thickness).



b) Determine the standard form and the Euler characteristic of the surface of a fully padded T-shirt. That is, a T-shirt that is made by taking two T-shirts with one inside the other and sewing (or gluing), along the boundary circles corresponding to the neck, sleeves and hem.

*Solution*

a) The T-shirt surface is the four times punctured sphere. The four boundary circles of this surface are the neck, two sleeve holes and the hem. So the T-shirt surface has standard form $S^2 \# \#^4 \mathbb{D}^2$. Therefore, the Euler characteristic of the surface corresponding to the T-shirt is

$$\chi(S^2 \# \#^4 \mathbb{D}^2) = \chi(S^2) + 4 \times 1 - 4 \times 2 = -2.$$

Alternatively, you could not that each puncture of a surface reduces the Euler characteristic by 1.

b) The padded T-shirt surface is the union of two surfaces $A$ and $B$, where $A$ and $B$ are both T-shirts and where their intersection $A \cap B$ is the disjoint union of four circles. Now apply the easily checked identity $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$. From the first part of the question we have $\chi(A) = \chi(B) = -2$. Further, $\chi(A \cap B) = 0$ because $A \cap B$ is the disjoint union of four circles, each of which has Euler characteristic zero. Hence, the padded T-shirt surface has Euler characteristic $-4$.

By construction, the padded T-shirt is a closed surface that embeds in $\mathbb{R}^3$, so it is orientable (we saw lectures that any surface that embeds in $\mathbb{R}^3$ is orientable). Therefore, the standard form of the padded T-shirt is of the form $S^2 \# \#^t \mathbb{T}$, for some $t \geqslant 0$. That is, the padded T-shirt is a sphere with $t$ handles. By Tutorial 9, $\chi(S^2 \# \#^t \mathbb{T}) = 2 - 2t$, so $-4 = 2 - 2t$ so that $t = 3$. Hence, the padded T-shirt is homeomorphic to $S^2 \# \#^3 \mathbb{T} \cong \#^3 \mathbb{T}$, the sphere with 3 handles.

**5.** Determine the possible standard forms of the following connected surfaces:

a) A surface with 7 punctures and Euler characteristic $-10$
b) A surface with 10 punctures and Euler characteristic $-7$
c) A surface with 10 punctures and Euler characteristic $-11$
d) A surface with 10 punctures and Euler characteristic $-8$
e) A surface with 10 punctures and Euler characteristic $-10$

*Solution*

a) As the connected surface $S$ has 7 punctures it is either

$$S_1 = S^2 \# \#^7 \mathbb{D}^2 \# \# \#^p \mathbb{P}^2 \quad \text{or} \quad S_2 = \# \#^7 \mathbb{D}^2 \# \#^t \mathbb{T},$$

where either $p > 0$ or $t > 0$. As $\chi(S_1) = 2 - 7 - p$ and $\chi(S_2) = 2 - 7 - 2t$ and $\chi(S) = -10$ we see that $S$ cannot be $S_2$ because the Euler characteristic of $S_2$ is always odd. Hence, $S = S_1$ and $p$, the number of cross caps, is given by $-10 = -5 - p$, so that $p = 5$. That is, $S = \#^7 \mathbb{D}^2 \#\#\#^5 \mathbb{P}^2$ is a sphere with 7 punctures and 5 cross caps.

b) As the surface $S$ has 10 punctures it is either

$$S_1 = S^2 \#\#^{10}\mathbb{D}^2 \#\#\#^p \mathbb{P}^2 \quad \text{or} \quad S_2 = \#\#^{10}\mathbb{D}^2 \#\#^t \mathbb{T},$$

where either $p > 0$ or $t > 0$. As $\chi(S_1) = 2 - 10 - p$ and $\chi(S_2) = 2 - 10 - 2t$ and $\chi(S) = -7$ we see that $S$ cannot be $S_2$ because the Euler characteristic of $S_2$ is always even. Hence, $S = S_1$ and $p$, the number of cross caps, is given by $-7 = -8 - p$, so that $p = -1$. This is impossible, so there is no such surface.

c) As the surface $S$ has 10 punctures it is either

$$S_1 = S^2 \#\#^{10}\mathbb{D}^2 \#\#\#^p \mathbb{P}^2 \quad \text{or} \quad S_2 = \#\#^{10}\mathbb{D}^2 \#\#^t \mathbb{T},$$

where either $p > 0$ or $t > 0$. As $\chi(S_1) = 2 - 10 - p$ and $\chi(S_2) = 2 - 10 - 2t$ and $\chi(S) = -7$ we see that $S$ cannot be $S_2$ because the Euler characteristic of $S_2$ is always even. Hence, $S = S_1$ and $p$, the number of cross caps, is given by $-11 = -8 - p$, so that $p = 3$. That is, $S = \#^{10}\mathbb{D}^2 \#\#\#^3 \mathbb{P}^2$ is a sphere with 10 punctures and 3 cross caps.

d) As the surface $S$ has 10 punctures it is either

$$S_1 = S^2 \#\#^{10}\mathbb{D}^2 \#\#\#^p \mathbb{P}^2 \quad \text{or} \quad S_2 = \#\#^{10}\mathbb{D}^2 \#\#^t \mathbb{T},$$

where either $p > 0$ or $t > 0$. As $\chi(S_1) = 2 - 10 - p$ and $\chi(S_2) = 2 - 10 - 2t$ and $\chi(S) = -8$ we see that both surfaces are possible. If $S = S_1$ then $-8 = -8 - p$, so that $p = 0$. That is, $S = S^2 \#\#^{10}\mathbb{D}^2$. If $S = S_2$ then $-8 = -8 - 2t$ and again $t = 0$ and $S = S^2 \#\#^{10}\mathbb{D}^2$. Hence, $S$ is a is a sphere with 10 punctures.

e) As the surface $S$ has 10 punctures it is either

$$S_1 = S^2 \#\#^{10}\mathbb{D}^2 \#\#\#^p \mathbb{P}^2 \quad \text{or} \quad S_2 = \#\#^{10}\mathbb{D}^2 \#\#^t \mathbb{T},$$

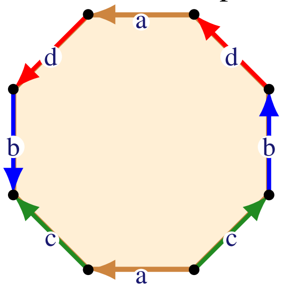
where either $p > 0$ or $t > 0$. As $\chi(S_1) = 2 - 10 - p$ and $\chi(S_2) = 2 - 10 - 2t$ and $\chi(S) = -8$ we see that both surfaces are possible. If $S = S_1$ then $-10 = -8 - p$, so that $p = 2$. That is, $S = S^2 \#\#^{10}\mathbb{D}^2 \#\#^2 \mathbb{P}^2$. If $S = S_2$ then $-10 = -8 - 2t$, so that $t = 1$ and $S = S^2 \#\#^{10}\mathbb{D}^2 \# \mathbb{T}$. Hence, $S$ is a sphere with 10 punctures and either 2 cross caps or one handle.

**6.** Let $S$ be the surface given by the word $a\, d\, b\, \bar{c}\, \bar{a}\, c\, b\, d$.

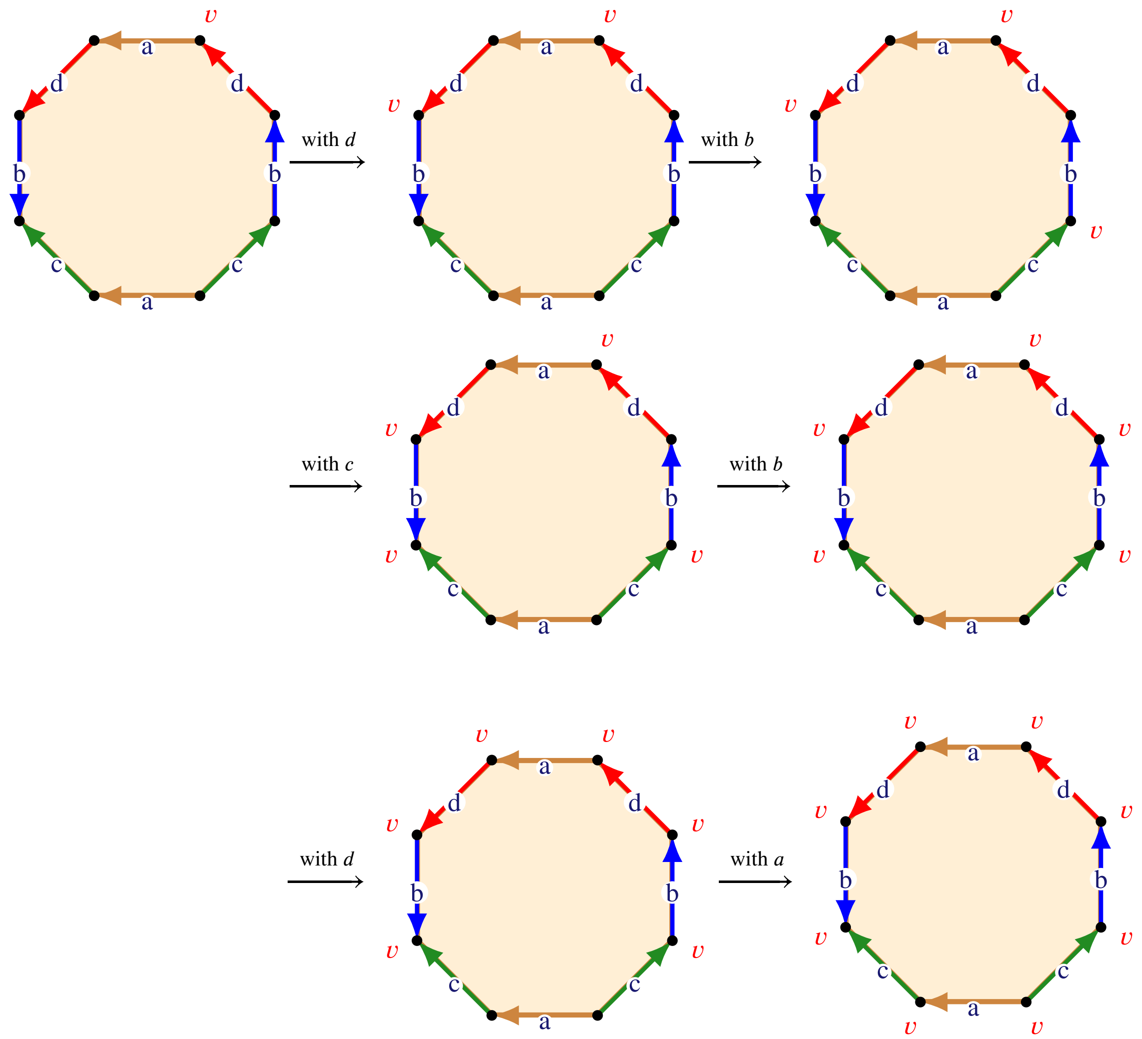
a) Draw a polygonal decomposition of $S$.
b) How many vertices are there in your polygonal decomposition of $S$?
c) How many boundary circles does $S$ have?
d) Is $S$ orientable? Explain.
e) Compute the Euler characteristic $\chi(S)$ of the surface $S$.
f) Describe $S$ as a *standard surface* – that is, as a sphere with punctures, handles and cross caps.

*Solution*

a) Reading the word anti-clockwise, the surface $S$ corresponds to the polygonal decomposition



b) Looking in turn at the three paired edges shows that:




Therefore, $v$ is the only vertex. Hence, this polygonal decomposition has 1 vertex.

c) The surface $S$ has no unpaired edges, so it has no boundary circles.

d) This surface is non-orientable because, for example, $b$ is an unoriented edge.

e) The polygonal decomposition for $S$ has 1 vertex, 4 edges and 1 face. So $\chi(S) = 1 - 4 + 1 = -2$.

f) By part (c), $S$ has no boundary circles and it is non-orientable by (d), so $S \cong \#^p\mathbb{P}^2$, for some $p \geqslant 0$. As $\chi(\#^p\mathbb{P}^2) = 2 - p$ we have $2 - p = -2$ by part (e). Therefore, $p = 4$ and $S \cong \#^4\mathbb{P}^2$. That is, $S$ is a sphere with four cross caps.