Tutorial 8

Weekly summary and definitions and results for this tutorial

a) A **Eulerian circuit**, or **Eulerian cycle**, in a connected graph $\mathcal{G}$ is a circuit in $\mathcal{G}$ that goes through every edge exactly once.

b) Theorem A graph has a Eulerian circuit if and only if it is connected and every vertex has even degree.

c) Suppose that $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^n$. Then X and Y are **homeomorphic** if there exist continuous maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$, where id_X and id_Y are the identity maps on X and Y. We write $X \cong Y$.

d) If $a < b$ and $c < d$ are real numbers then $(a, b) \cong (c, d) \cong \mathbb{R}$, $[a, b) \cong [c, d) \cong (c, d]$ and $[a, b] \cong [c, d]$. Any two (filled in) polygons are homeomorphic: a (filled in) triangle is homeomorphic to a (filled in) square, pentagon, hexagon, … and all of these are homeomorphic to the disc $\mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 1\}$.

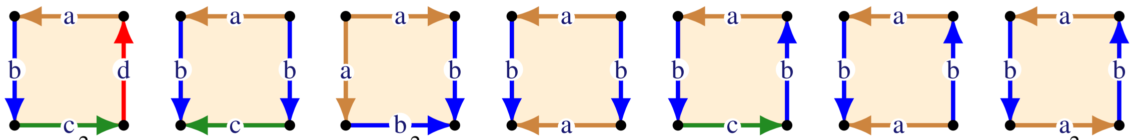
e) *Informal definition of a surface*: Let $n \geqslant 0$. A **surface** is a subset of $\mathbb{R}^n$ such that, *locally*, X looks like the graph of a function $z = f(x, y)$.

f) Examples of surfaces: planes, spheres, cubes, the tori (or donuts), coffee cups, double torus, Möbius strips, Klein bottles, projective planes, cylinders, annuli, …

g) An **identification space** for a surface T is a collection of surfaces $X_1, \dots, X_r$ together with a continuous map $f : \bigcup_r X_r \longrightarrow T$. So f *identifies* $x \in X_i$ and $y \in X_j$ whenever $f(x) = f(y)$.

h) *Formal definition of the surfaces considered in this course:* A surface with a **polygonal decomposition** is the identification space given by a (finite) collection of polygons with at most two polygons being identified along any edge.

i) Polygonal decompositions, or identification spaces, for some important surfaces:



$\mathbb{D}^2$ Disc — $\mathbb{A}$ Annulus — S^2 Sphere — $\mathbb{T}$ Torus — $\mathbb{M}$ Möbius strip — $\mathbb{K}$ Klein bottle — $\mathbb{P}^2$ Projective plane

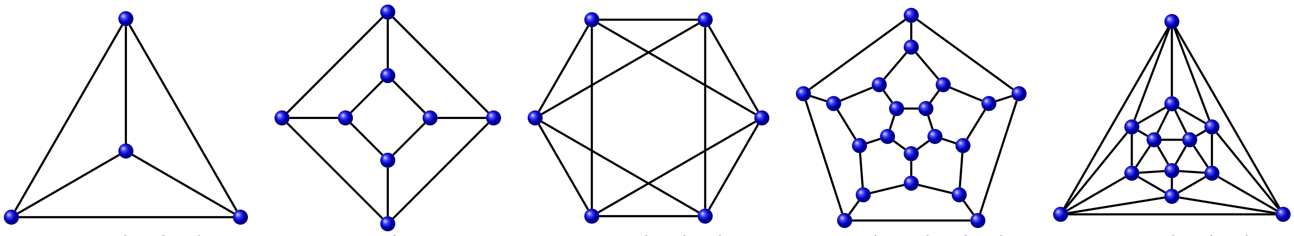
j) The **Euler characteristic** of a surface S with polygonal decomposition (V, E, F) is:

$$\chi(S) = |V| - |E| + |F|.$$

k) The **boundary** of a surface S with a polygonal decomposition is the union of the free, or unpaired, edges. The boundary ∂S is a disjoint union circuits, which are the **boundary circles** of S.

Questions to complete *before* the tutorial

1. In the tutorial of week 7, we talked about the projections of the five Platonic solids onto the plane:



Tetrahedral *Cubic* *Octahedral* *Dodecahedral* *Icosahedral*

We found a Eulerian cycle for the octahedral graph last week. Determine, for each of the graphs of the platonic solids, whether they have a Eulerian cycle.

2. Recall that K_n is the complete graph on n vertices, for $n \geqslant 1$.

a) Determine the Euler characteristic of K_n.
b) For which values of n is K_n a tree?
c) For which values of n does K_n have a Eulerian circuit?

3. Recall that the complete bipartite graph $K_{m,n}$, for $m, n \geq 1$, has vertex set $V = M \sqcup N$ (disjoint union), where $m = |M|$, $n = |N|$, and with edge set $\{(x, y) \mid x \in M \text{ and } y \in N\}$. That is, $K_{m,n}$ has mn edges that connect every element of M with every element of N.

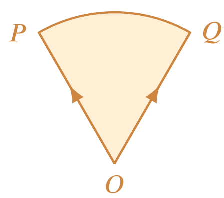
a) Find a formula for $\chi\left(K_{m,n}\right)$.
b) For which values of m and n does $K_{m,n}$ have a Eulerian circuit?

Questions to complete *during* the tutorial

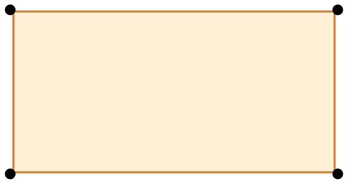
4. a) Show that a connected graph with $n \geqslant 3$ and every vertex of degree 2 is isomorphic to the cyclic graph C_n.
b) Deduce a graph has every vertex of degree two if and only if it is a disjoint union of cycle graphs.
c) Show that any connected graph in which the vertex degrees are 1 or 2 is a path graph or a cycle graph.
d) Deduce that a graph has every vertex of degree one or two if and only if it is a disjoint union of cycles graphs and path graphs. In this case show the Euler characteristic of the graph counts the number of path graph components.

5. Check that you understand the polygonal decompositions of the surfaces $\mathbb{D}^2$, $\mathbb{A}$, S^2, $\mathbb{T}$, $\mathbb{M}$, $\mathbb{K}$ and $\mathbb{P}^2$ given in the lecture summary at the start of the tutorial.

6. Let W be a sector of the disc between two distinct radii OP and OQ. What surface do we get when we identify OP with OQ?

$W =$ 

7. Label the vertices of a rectangle A, B, C, D as we move anticlockwise around the sides.



What surfaces do we get when we identify:

a) AB with AD?
b) AB with AD and CB with CD?

Questions to complete *after* the tutorial

8. In lectures it was explained in an intuitive way why the annulus $\mathbb{A}$ and the cylinder are homeomorphic. Up to homeomorphism, the annulus is the set

$$A = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, \tfrac{1}{2} \leqslant x^2 + y^2 \leqslant 1 \right\} \subseteq \mathbb{R}^2$$

and up to homeomorphism the cylinder is the set

$$C = \left\{ (x, y, z) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 = 1 \text{ and } \tfrac{1}{2} \leqslant z \leqslant 1 \right\} \subseteq \mathbb{R}^3.$$

In particular, the annulus A embeds in $\mathbb{R}^2$ whereas the cylinder C embeds in $\mathbb{R}^3$.

a) Draw the sets A and C and verify that they are the annulus and the cylinder, respectively.
b) Show that $A \cong C$ by constructing explicit continuous maps $f : A \longrightarrow C$ and $g : C \longrightarrow A$ such that $f \circ g = \mathrm{id}_C$ and $g \circ f = \mathrm{id}_A$.