Tutorial 8 — Solutions

Weekly summary and definitions and results for this tutorial

a) A **Eulerian circuit**, or **Eulerian cycle**, in a connected graph $\mathcal{G}$ is a circuit in $\mathcal{G}$ that goes through every edge exactly once.

b) Theorem A graph has a Eulerian circuit if and only if it is connected and every vertex has even degree.

c) Suppose that $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^n$. Then X and Y are **homeomorphic** if there exist continuous maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$, where id_X and id_Y are the identity maps on X and Y. We write $X \cong Y$.

d) If $a < b$ and $c < d$ are real numbers then $(a, b) \cong (c, d) \cong \mathbb{R}$, $[a, b) \cong [c, d) \cong (c, d]$ and $[a, b] \cong [c, d]$. Any two (filled in) polygons are homeomorphic: a (filled in) triangle is homeomorphic to a (filled in) square, pentagon, hexagon, … and all of these are homeomorphic to the disc $\mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 1\}$.

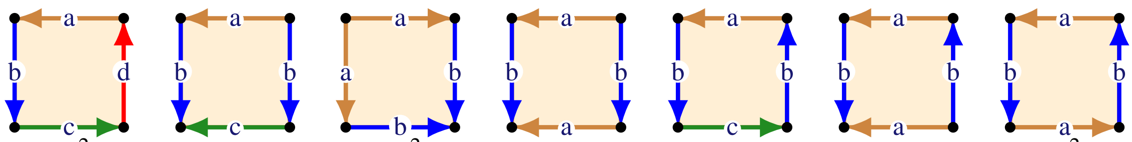
e) *Informal definition of a surface*: Let $n \geqslant 0$. A **surface** is a subset of $\mathbb{R}^n$ such that, *locally*, X looks like the graph of a function $z = f(x, y)$.

f) Examples of surfaces: planes, spheres, cubes, the tori (or donuts), coffee cups, double torus, Möbius strips, Klein bottles, projective planes, cylinders, annuli, …

g) An **identification space** for a surface T is a collection of surfaces $X_1, \dots, X_r$ together with a continuous map $f : \bigcup_r X_r \longrightarrow T$. So f *identifies* $x \in X_i$ and $y \in X_j$ whenever $f(x) = f(y)$.

h) *Formal definition of the surfaces considered in this course:* A surface with a **polygonal decomposition** is the identification space given by a (finite) collection of polygons with at most two polygons being identified along any edge.

i) Polygonal decompositions, or identification spaces, for some important surfaces:



$\mathbb{D}^2$	$\mathbb{A}$	S^2	$\mathbb{T}$	$\mathbb{M}$	$\mathbb{K}$	$\mathbb{P}^2$
Disc	Annulus	Sphere	Torus	Möbius strip	Klein bottle	Projective plane

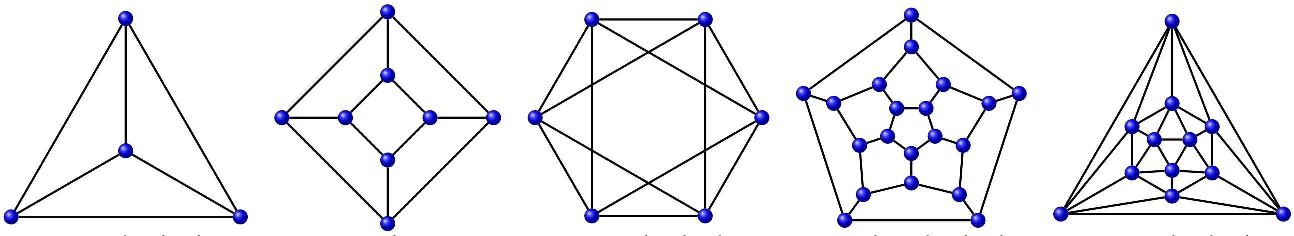
j) The **Euler characteristic** of a surface S with polygonal decomposition (V, E, F) is:

$$\chi(S) = |V| - |E| + |F|.$$

k) The **boundary** of a surface S with a polygonal decomposition is the union of the free, or unpaired, edges. The boundary ∂S is a disjoint union circuits, which are the **boundary circles** of S.

Questions to complete *before* the tutorial

1. In the tutorial of week 7, we talked about the projections of the five Platonic solids onto the plane:



We found a Eulerian cycle for the octahedral graph last week. Determine, for each of the graphs of the platonic solids, whether they have a Eulerian cycle.

Solution

Graph $\mathcal{G}$	Vertex degreee
Tetrahedral	3
Cubic	3
Octahedral	4
Dodecahedral	3
Icosahedral	5

By Euler's Criterion, a graph G has a Eulerian circuit if and only if it is connected and every vertex has even degree. Therefore, for the graphs of the Platonic solids, only the octahedral graph has a Eulerian circuit.

2. Recall that K_n is the complete graph on n vertices, for $n \geqslant 1$.

a) Determine the Euler characteristic of K_n.
b) For which values of n is K_n a tree?
c) For which values of n does K_n have a Eulerian circuit?

Solution

a) The graph K_n has n vertices and $\binom{n}{2} = \frac{1}{2}n(n-1)$ edges each of degree $n-1$. Therefore, the Euler characteristic of K_n is $\chi(K_n) = n - \frac{1}{2}n(n-1) = \frac{1}{2}n(3-n)$.
b) The graph K_n is connected, so it is a tree if and only if $\chi(K_n) = \frac{1}{2}n(3-n) = 1$. Hence K_n is a tree if and only if $n = 1$ or $n = 2$.
Reality check: draw a picture of K_n for some values of n to convince yourself this is true.
c) As K_n is connected, it is Eulerian if and only if the degree of each vertex is even. That is, K_n is Eulerian if and only if n is odd.

3. Recall that the complete bipartite graph $K_{m,n}$, for $m, n \geq 1$, has vertex set $V = M \sqcup N$ (disjoint union), where $m = |M|$, $n = |N|$, and with edge set $\{(x, y) \mid x \in M \text{ and } y \in N\}$. That is, $K_{m,n}$ has mn edges that connect every element of M with every element of N.

a) Find a formula for $\chi\left(K_{m,n}\right)$.

b) For which values of m and n does $K_{m,n}$ have a Eulerian circuit?

Solution

a) The complete bipartite graph $K_{m,n}$ has $V = m + n$ vertices and $E = mn$ edges. Therefore, its Euler characteristic is $\chi\left(K_{m,n}\right) = m + n - mn$.
b) The graph $K_{m,n}$ is connected and all of the vertices of $K_{m,n}$ in M have degree n and all of the vertices in N have degree m. Therefore, by Euler's criteria, the graph $K_{m,n}$ has an Eulerian circuit if and only if both m and n are even.

Questions to complete *during* the tutorial

4. a) Show that a connected graph with $n \geqslant 3$ and every vertex of degree 2 is isomorphic to the cyclic graph C_n.
b) Deduce a graph has every vertex of degree two if and only if it is a disjoint union of cycle graphs.
c) Show that any connected graph in which the vertex degrees are 1 or 2 is a path graph or a cycle graph.
d) Deduce that a graph has every vertex of degree one or two if and only if it is a disjoint union of cycles graphs and path graphs. In this case show the Euler characteristic of the graph counts the number of path graph components.

Solution

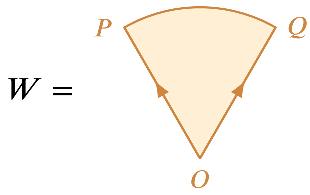
a) We repeat the argument from the lecture notes for proving Euler's criteria. If we start a walk from any vertex we can always continue the walk along the edge other than we have just previously used until eventually we must arrive back at the original vertex to form a circuit. Because vertices all have degree 2 this circuit must be a cycle. By connectivity this cycle must contain all vertices, because, since every vertex has degree 2, no walk which starts on the cycle can ever leave it.

A second way to prove this is to argue by induction on the number of vertices, imitating the proof that we used to classify graphs with Eulerian circuits.
b) A disjoint union of cycle graphs has every vertex of degree 2. The converse follows from (a), because the connected components of such a graph are connected graphs in which every vertex has degree 2.
c) If every vertex has degree 2 then the graph is a cycle graph by (a). Suppose the graph has a vertex of degree 1. Then we start a walk from this vertex, and every time we arrive at a vertex of degree 2 we can continue the walk along the previously unvisited edge. We cannot arrive back at our original vertex of degree 1. Hence we must arrive at different vertex of degree 1. At this stage we have a path graph. By connectivity this path graph must contain all the vertices, because, since every vertex has degree 1 or 2, no walk in the graph which starts on the path can ever leave it.
d) A disjoint union of cycle graphs and path graphs has every vertex of degree 1 or 2. The converse follows from (c), because the connected components of such a graph are connected graphs in which every vertex has degree 1 or 2.

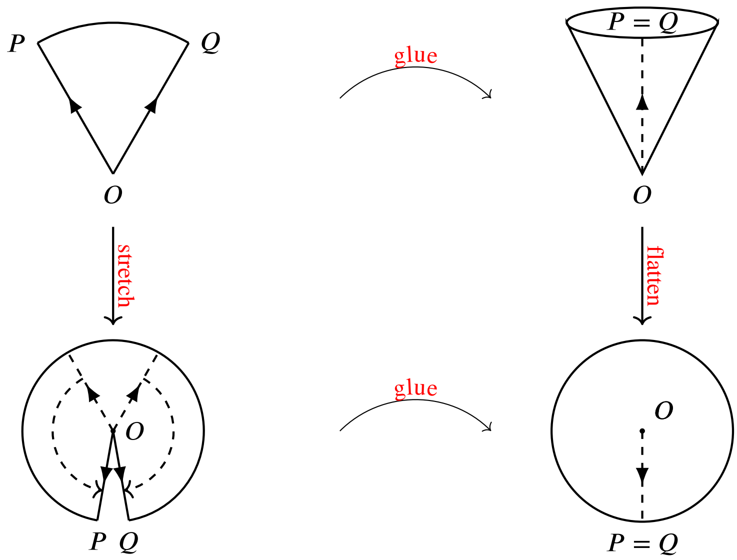
Recall that the Euler characteristic of a graph is the sum of the Euler characteristics of its connected components. Since cycle graphs have Euler Characteristic 0 and path graphs have Euler characteristic 1, the Euler characteristic of a disjoint union of cycle and path graphs is the number of path graphs in the union.

5. Check that you understand the polygonal decompositions of the surfaces $\mathbb{D}^2$, $\mathbb{A}$, S^2, $\mathbb{T}$, $\mathbb{M}$, $\mathbb{K}$ and $\mathbb{P}^2$ given in the lecture summary at the start of the tutorial.

6. Let W be a sector of the disc between two distinct radii OP and OQ. What surface do we get when we identify OP with OQ?



Solution When we identify the two line segments OP and OQ on an arc then we naturally obtain a cone. The following diagram shows two ways to deform a cone into a disc so, topologically, W is a disc.



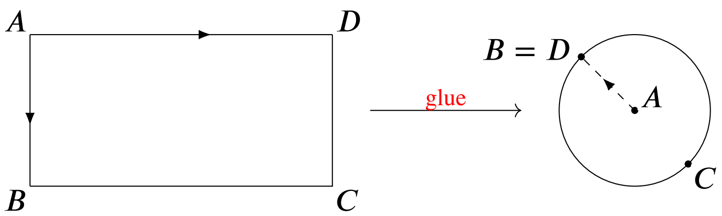
7. Label the vertices of a rectangle A, B, C, D as we move anticlockwise around the sides.

What surfaces do we get when we identify:

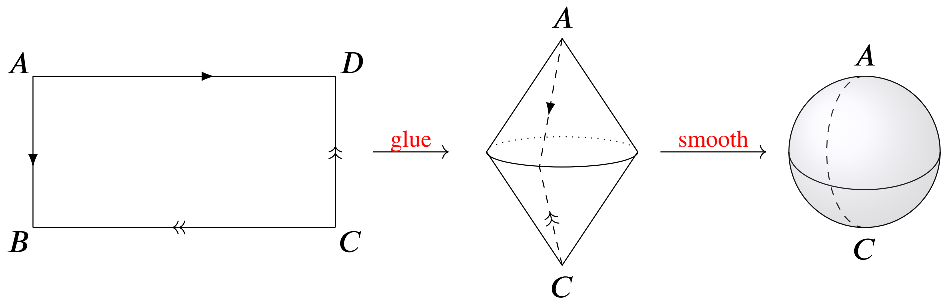
a) AB with AD?
b) AB with AD and CB with CD?

Solution

a) The disc $\mathbb{D}^2$:



b) The sphere S^2:



The equation of the sphere corresponds to the diagonal BD in the identification diagram on the left-hand side.

Questions to complete *after* the tutorial

8. In lectures it was explained in an intuitive way why the annulus $\mathbb{A}$ and the cylinder are homeomorphic. Up to homeomorphism, the annulus is the set

$$A = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, \tfrac{1}{2} \leqslant x^2 + y^2 \leqslant 1 \right\} \subseteq \mathbb{R}^2$$

and up to homeomorphism the cylinder is the set

$$C = \left\{ (x, y, z) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 = 1 \text{ and } \tfrac{1}{2} \leqslant z \leqslant 1 \right\} \subseteq \mathbb{R}^3.$$

In particular, the annulus A embeds in $\mathbb{R}^2$ whereas the cylinder C embeds in $\mathbb{R}^3$.

a) Draw the sets A and C and verify that they are the annulus and the cylinder, respectively.
b) Show that $A \cong C$ by constructing explicit continuous maps $f : A \longrightarrow C$ and $g : C \longrightarrow A$ such that $f \circ g = \mathrm{id}_C$ and $g \circ f = \mathrm{id}_A$.

Solution

a)
b) We need to define two continuous maps $f : A \longrightarrow C$ and $g : C \longrightarrow A$ such that $f \circ g = \mathrm{id}_A$ and $g \circ f = \mathrm{id}_C$. Looking at the definitions of A and C, define

$$f(x, y) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, x^2 + y^2 \right) \quad \text{and} \quad g(x, y, z) = \left(x\sqrt{z}, y\sqrt{z} \right).$$

Then $(f \circ g)(x, y, z) = (x, y, z)$ and $g \circ f)(x, y) = (x, y)$, so $f \circ g = \mathrm{id}_C$ and $g \circ f = \mathrm{id}_A$. The maps f and g are continuous because they are continuous in each component. Hence, $A \cong C$.