Tutorial 7 — Solutions

Here is a quick summary and main results from the lectures in Week 7. Please watch the lecture recordings to gain a better understanding of this material.

Weekly summary and definitions and results for this tutorial

a) A **graph** $\mathcal{G}$ consists of a set $V_\mathcal{G}$ of vertices and a *multiset* $E_\mathcal{G}$ of edges, which are pairs of vertices.

b) Two graphs $\mathcal{G}$ and $\mathcal{H}$ are **isomorphic** if there is a bijection $f : V_\mathcal{G} \longrightarrow V_\mathcal{H}$ such that $\{u, v\}$ is an edge of $\mathcal{G}$ if and only if $\{f(u), f(v)\}$ is an edge of $\mathcal{H}$.

c) The **degree** $\deg v$ of a vertex $v \in V_\mathcal{G}$ is the number of edges in $\mathcal{G}$ that start or end at v.

d) Proposition (Degree-Sum Formula) If G is a graph then $\sum_{v\in V_\mathcal{G}} \deg v = 2\#E_\mathcal{G}$.

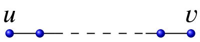
e) A graph $\mathcal{H}$ is a **subgraph** of $\mathcal{G}$ if $V_\mathcal{H} \subseteq V_\mathcal{G}$ and $E_\mathcal{H} \subseteq E_\mathcal{G}$.

f) A graph is **planar** if it can be embedded (drawn) in $\mathbb{R}^2$ without edge crossings.

g) Every graph can be embedded in $\mathbb{R}^3$ without edge crossings.

h) A **subdivision** of a graph $\mathcal{G}$ is any graph that it obtained by replacing any number of edges with .

i) A **path** between two vertices $u, v \in V_\mathcal{G}$ is a sequence of consecutive edges joining u to v:



The path is a **circuit** if it starts and ends at the same vertex (so $u = v$).

j) A circuit is **contractible** if it is possible to reduce it to a path of length 0 by repeatedly removing pairs of *consecutive* repeated edges.

k) A graph is **connected** G if any two vertices $u, v \in V_\mathcal{G}$ can be joined by a path.

l) A **tree** is a connected graph in which every circuit is contractible.

m) If $\mathcal{G}$ is a graph then a **spanning tree** $\mathcal{T}$ for $\mathcal{G}$ is a subgraph of $\mathcal{G}$ such that $\mathcal{T}$ is a tree and $V_\mathcal{T} = V_\mathcal{G}$.

n) Theorem Every connected graph has a spanning tree.

o) The **Euler characteristic** of a graph $\mathcal{G}$ is $\chi(\mathcal{G}) = \#V_\mathcal{G} - \#E_\mathcal{G}$.

p) Theorem If $\mathcal{G}$ is a connected graph then $\chi(G) \leqslant 1$ with equality if and only if G is a tree.

q) Definition Let $\mathcal{G}$ be a connected graph. The number of **independent cycles** in $\mathcal{G}$ is $1 - \chi(\mathcal{G})$.

r) A **Eulerian circuit**, or **Eulerian cycle**, is a path through in a graph $\mathcal{G}$ that goes through every edge exactly once, and every vertex at least once.

Questions to complete *before* the tutorial

1. a) Draw a graph with 4 vertices and 5 edges.
b) Check that the Degree-sum Formula holds for the graph that you drew for part (a).

2. a) Draw a graph with a non-trivial circuit. That is, give an example of a graph that is not a tree.
b) Compute the Euler characteristic of your graph in part (a).
c) Draw a graph that is a tree.
d) Compute the Euler characteristic of your tree from part (c).

3. Determine the Euler characteristic of the cycle graphs C_n, for $n = 1, 2, 3 \dots$ etc.

Solution By definition, the Euler characteristic of a graph G is

$$\chi(G) = \mathrm{V} - \mathrm{E},$$

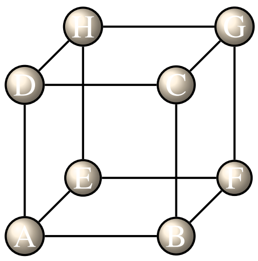
where V is the number of vertices of G and E is the number of edges. The cycle graph C_n has $V = n$ vertices and $E = n$ edges, so $\chi(C_n) = n - n = 0$.

Questions to complete *during* the tutorial

4. a) The cube graph is the graph of vertices and edges of a cube. Make sketches to show that the cube graph is planar.
b) Make sketches to show that the octahedral graph, the graph formed by the vertices and edges of the regular octahedron, is planar.

Solution

a) The diagrams below show the **standard** representation for the graph of the cube and an isomorphic planar graph for the cube that is a pair of nested squares:



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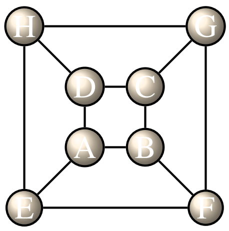
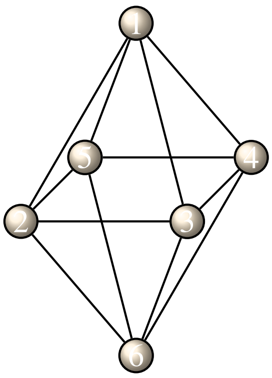


b) The vertices can be labelled 1 to 6 such that two vertices joined by an edge if and only if the sum of their labels is *not* 7.



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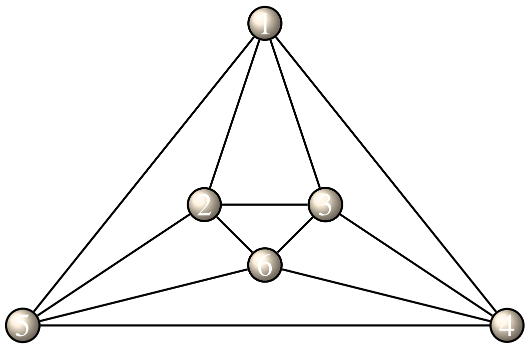


5. Show that in any graph the number of vertices of odd degree must be even.

[*Hint:* Use the Degree-sum Formula.]

Solution The parity of a sum of integers is determined by the parity of the number of odd terms. If the number of odd terms is odd the sum is odd, and if the number of odd terms is even the sum is even. As shown in class, the sum of the degrees of all the vertices of a graph is even, equal to twice the number of edges. Hence the number of the vertices of odd degree must be even.

6. a) Show that if the connected components of a graph $\mathcal{G}$ are $\mathcal{G}_1, \dots, \mathcal{G}_n$ then

$$\chi(\mathcal{G}) = \chi(\mathcal{G}_1) + \cdots + \chi(\mathcal{G}_n).$$

b) A *forest* is a graph such that all of its connected components are trees. If $\mathcal{F}$ is a forest show that $\chi(\mathcal{F})$ is equal to the number of trees in the forest.

c) Give an example of a graph $\mathcal{G}$ that is *not* a tree and $\chi(\mathcal{G}) = 1$.

Solution

a) For $1 \leqslant i \leqslant n$ let V_i be the vertex set of $\mathcal{G}_i$ and let E_i be its set of edges. Then $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_n$ (a disjoint union) and $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_n$ are the vertex set and edge set, respectively, of $\mathcal{G}$. Therefore,

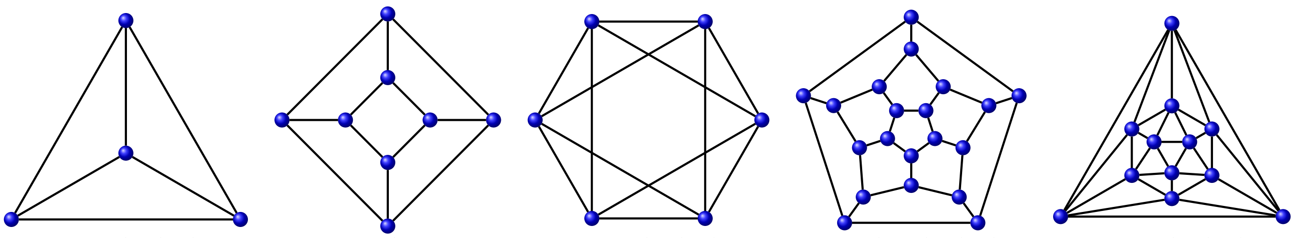
$$\begin{aligned}\chi(\mathcal{G}) = |V| - |E| &= \big(|V_1| + \cdots + |V_n|\big) - \big(|E_1| + \cdots + |E_n|\big)\\ &= \big(|V_1| - |E_1|\big) + \big(|V_2| - |E_2|\big) + \cdots + \big(|V_n| - |E_n|\big)\\ &= \chi(\mathcal{G}_1) + \chi(\mathcal{G}_2) + \cdots + \chi(\mathcal{G}_n),\end{aligned}$$

as required.

b) This follows from part (a) because, by lectures, all tree graphs have Euler Characteristic 1.

c) By a theorem from lectures, if $\mathcal{G}$ is connected then $\chi(\mathcal{G}) \leqslant 1$ with equality if and only if $\mathcal{G}$ is a tree. Therefore, $\mathcal{G}$ cannot be a connected graph. Let $\mathcal{G}$ be a graph with two connected components $\mathcal{G}_1$ and $\mathcal{G}_2$, where $\mathcal{G}_1$ is a tree and where $\mathcal{G}_2$ is a cycle graph. Then $\mathcal{G}$ is definitely not a tree and $\chi(\mathcal{G}_1) = 1$ and $\chi(\mathcal{G}_2) = 0$. Therefore, using part (a), $\chi(\mathcal{G}) = \chi(\mathcal{G}_1) + \chi(\mathcal{G}_2) = 1$ as we wanted.

7. The following graphs are projections of the five Platonic solids onto the plane:



Tetrahedral *Cubic* *Octahedral* *Dodecahedral* *Icosahedral*

a) Determine the Euler characteristic of each of these graphs.

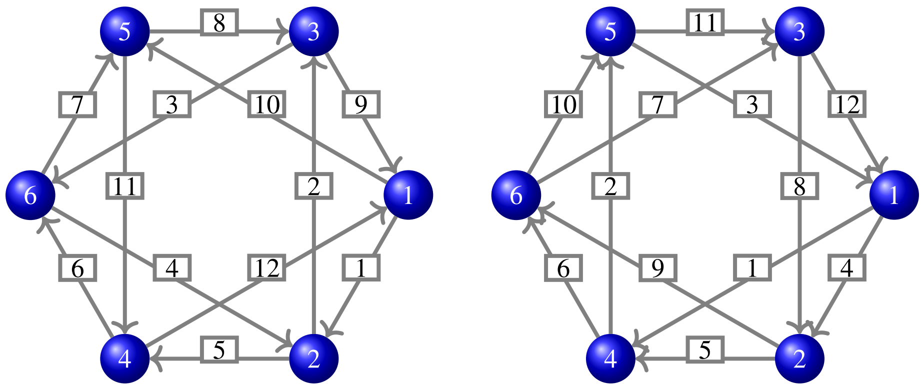
b) A *Eulerian circuit* in a graph is a circuit that goes through every edge exactly once, and every vertex at least once. Find a Eulerian cycle for the octahedral graph.

Solution

a) The Euler characteristic of a graph $\mathcal{G}$ is $\chi(\mathcal{G}) = V - E$, where V is the number of vertices in $\mathcal{G}$ and E is the number of edges. The following table lists the Euler characteristic for the Platonic solids.

Graph $\mathcal{G}$	Vertices V	Edges E	Euler $\chi(\mathcal{G})$
Tetrahedral	4	6	−2
Cubic	8	12	−4
Octahedral	6	12	−6
Dodecahedral	20	30	−10
Icosahedral	12	30	−18

b) The octahedron has many different Eulerian circuits. Label the vertices of the octahedral graph by the numbers $1, 2, 3, 4, 5, 6$ such that two vertices are joined *unless* their sum is 7. Two Eulerian circuits are:

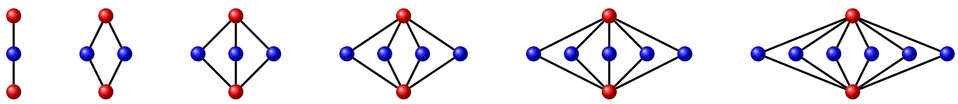


The edge labellings on these graphs correspond to the following two Eulerian circuits:

$$1 \to 2 \to 3 \to 6 \to 2 \to 4 \to 6 \to 5 \to 3 \to 1 \to 5 \to 4 \to 1$$
$$1 \to 4 \to 5 \to 1 \to 2 \to 4 \to 6 \to 3 \to 2 \to 6 \to 5 \to 3 \to 1.$$

8. The complete bipartite graph $\mathcal{K}_{m,n}$, for $m, n \geq 1$, has vertex set $V = M \sqcup N$ (disjoint union), where $m = |M|$, $n = |N|$, and with edge set $\{(x, y) \mid x \in M \text{ and } y \in N\}$. That is, $\mathcal{K}_{m,n}$ has mn edges that connect every element of M with every element of N. Show that all the bipartite graphs $\mathcal{K}_{2,n}$ are planar

Solution To see that $\mathcal{K}_{2,n}$ is planar draw its graph by placing the vertices in N on a straight line and then putting one of the vertices in M above this line and one below it. Drawing in all of the edges now shows that $\mathcal{K}_{2,n}$ is planar. For example, drawing the vertices of M in red and the vertices of N in blue, the graphs for $\mathcal{K}_{2,n}$, for $1 \leqslant n \leqslant 6$, are:



Questions to complete *after* the tutorial

9. A **spanning tree** in a graph G is any subgraph of T that is a tree and has the same vertex set as G. Let t_n be the number of distinct spanning trees in the complete graph K_n.

a) Find t_2, t_3 and t_4.
b) (Harder.) What is t_5?

Solution

a) It should be emphasised that we are counting *different* subgraphs and not subgraphs up to *isomorphism*. The graph K_2 has two vertices and one edge, so it is a tree and $t_2 = 1$. Similarly, leaving out one edge gives a spanning tree for K_3, so $t_3 = 3$. For K_4 the task is slightly harder. The graph K_4 is:

$$K_4 = 0/1, 0/2, 0/3, 1/2, 1/3, 2/3$$

So K_4 has 4 vertices and $6 = \binom{4}{2}$ edges. On the other hand, any spanning tree will have 3 edges so there are *at most* $\binom{6}{3} = 20$ possible spanning trees, as this is the number of subgraphs of K_3 with exactly 3 edges.

Thinking about it, however, all but the following three subgraphs of K_4 with 3 edges are connected:

$$1/2, 2/3, 3/1 \qquad 0/2, 2/3, 3/0 \qquad 0/1, 1/3, 3/0 \qquad 0/1, 1/2, 2/0$$

None of the other subgraphs of K_4 with 3 edges can have any non-contractible cycles. Therefore, there are $16 = 20 - 4$ spanning trees of K_4.

If you are really keen, here is the complete list of the 16 spanning trees:

0/1,0/2,0/3 0/1,0/2,1/3 0/1,0/2,2/3 0/1,0/3,1/2 0/1,0/3,2/3 0/1,1/2,1/3

0/1,1/2,2/3 0/1,1/3,2/3 0/2,0/3,1/2 0/2,0/3,1/3 0/2,1/2,1/3

0/2,1/2,2/3 0/2,1/3,2/3 0/3,1/2,1/3 0/3,1/2,2/3 0/3,1/3,2/3

b) To compute t_5, the number of spanning trees in K_5 we have to be even more clever. K_5 is a graph with 5 vertices and $10 = \binom{5}{2}$ edges. A spanning tree in K_5 will have 4 edges so there are at most $\binom{10}{4} = 210$ possible spanning trees. We need to avoid all of the subgraphs with four vertices that are either not connected or contain a cycle, or both! The disconnected graphs must have two connected components. If there is a single isolated vertex then there are $\binom{4}{2} = 6$ possible edges between them, so there are $5 \times \binom{6}{4} = 5 \times 15 = 75$ such graphs. If the connected components have 2 and 3 vertices then there is a unique way to place the 4 edges, so there are $\binom{5}{3} = 10$ such graphs. Hence, of the subgraphs of K_5 with 4 edges, $210 - 75 - 10 = 125$ of them are connected. At first sight, we still need to exclude the subgraphs that contain a cycle, however, if our graph contains C_3 or C_4 then it cannot be connected so we are already done. Hence, $t_5 = 125$.

Comparing the answers in parts (a) and (b) reveals the following:

n	Graph	# Vertices	# Edges	t_n
1	K_1	1	$0 = \binom{1}{2}$	1
2	K_2	2	$2 = \binom{2}{2}$	1
3	K_3	3	$3 = \binom{3}{2}$	$3 = 3^1$
4	K_4	4	$6 = \binom{4}{2}$	$16 = 4^2$
5	K_5	5	$10 = \binom{5}{2}$	$125 = 5^3$

Based on the values of t_3, t_4 and t_5, a brave person might guess that if $n \geqslant 2$ then $t_n = n^{n-2}$. They would be right!