Topology – week 11

Math3061

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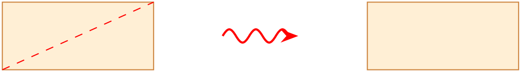
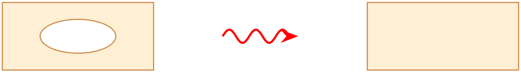
Map coloring assumptions

A map on a surface S is a polygonal subdivision such that:

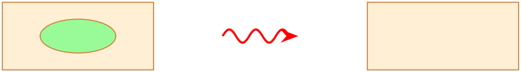
- All vertices have degree at least 3
- No region (i.e. face or polygon) has a border with itself



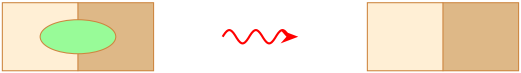
- No region contains a hole



- No region is completely surrounded by another



- No internal region has only two borders (i.e. edges)



The last three assumptions are purely for convenience because, in each case, we can color these maps using the same number of colors

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Moreover,

- $\partial_V \geq 3$ since vertices have degree at least 3
- $\partial_F \leq |F| - 1$ because no region borders itself
- If M is a map on a closed surface S, then we proved that
$$\partial_F \leq 6\left(1 - \frac{\chi(S)}{|F|}\right)$$

Maps on surfaces with $\chi(S) \leq 0$

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Let M be a map on a closed surface S with $\chi(S) \leq 0$. Then

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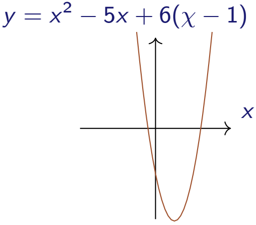

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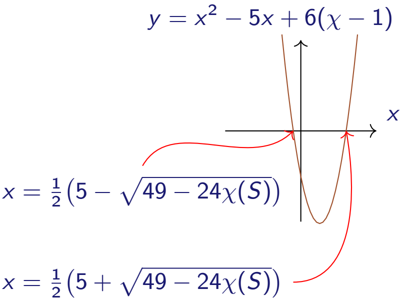

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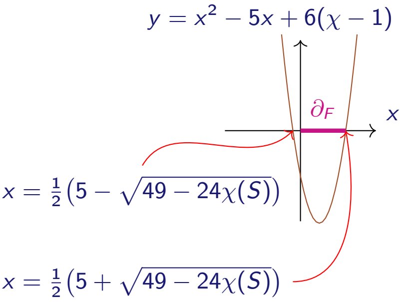

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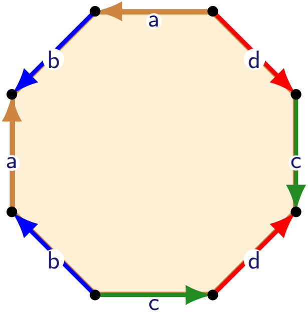

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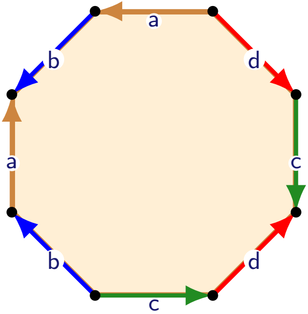

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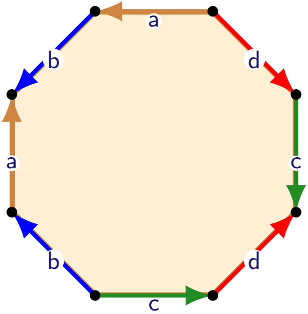

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This is not a contradiction because we are assuming that no region has a border with itself, which is never true for a polygonal decomposition that has only one face

Heawood's theorem

Theorem

Suppose that S is a closed surface. Then

$$C(S) \leq \begin{cases} 6, & \text{if } S = S^2 \text{ or } S = \mathbb{P}^2, \\ \frac{7+\sqrt{49-24\chi(S)}}{2}, & \text{otherwise} \end{cases}$$

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Since $\partial_F < c$ there is at least one face f with $\deg(f) < c$

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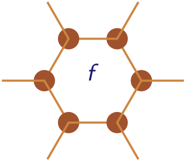

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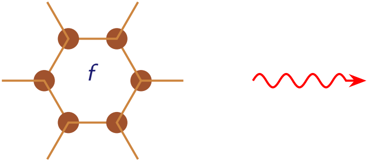

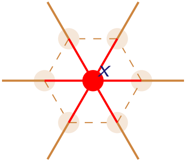

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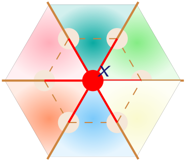

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2. It is easy to see that $C(S^2) \geq 4$ but it is really hard to show that $C(S^2) = 4$: the first proofs of the Four color theorem used complicated reductions and then exceedingly long brute force computer calculations

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Remarks

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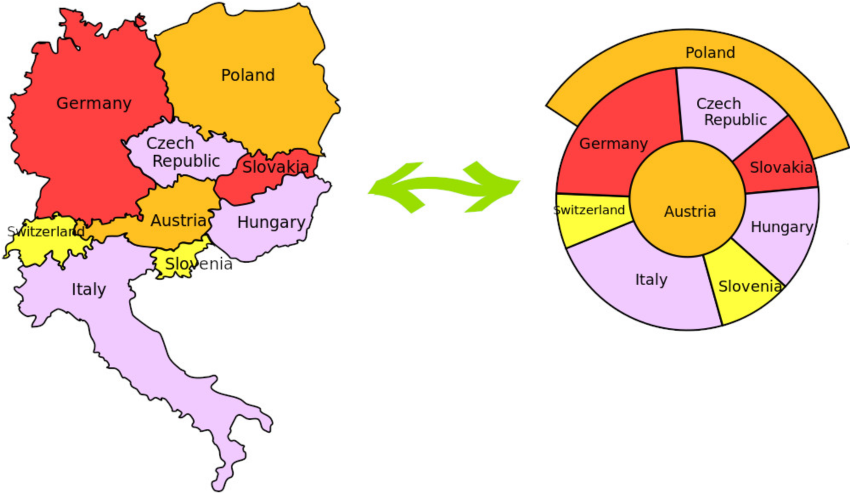
2. It is easy to see that $C(S^2) \geq 4$ but it is really hard to show that $C(S^2) = 4$: the first proofs of the Four color theorem used complicated reductions and then exceedingly long brute force computer calculations
3. If $S = S^2$ then $\chi(S^2) = 2$ so $\frac{7+\sqrt{49-24\chi(S)}}{2} = 4$!?

Why is $C(S^2) \geq 4$ easy to see? Well:



Coloring the torus

Heawood's estimate for the torus is $C(\mathbb{T}) \leq \frac{7+\sqrt{49-24\chi(\mathbb{T})}}{2} \leq 7$

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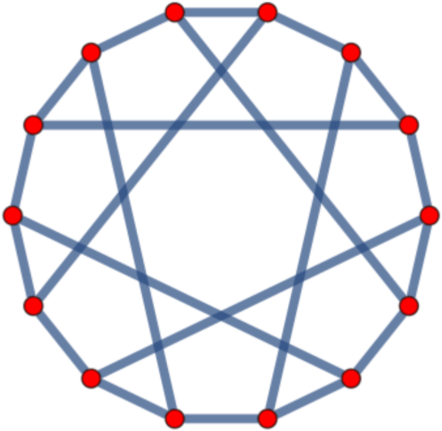
Hence, $C(\mathbb{T}) = 7$ (see the tutorials)



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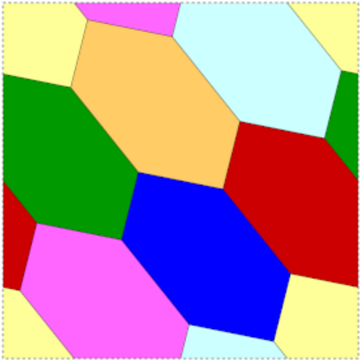
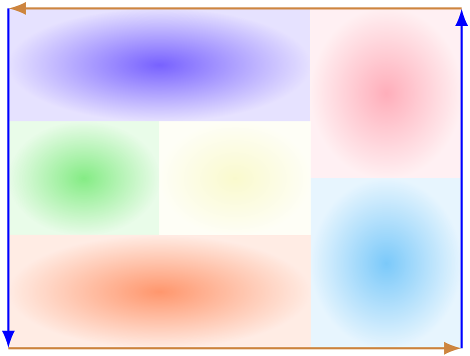


Coloring the projective plane

Heawood's estimate for the projective plane $\mathbb{P}^2$ is

$$C(\mathbb{P}^2) \leq \frac{7+\sqrt{49-24\chi(\mathbb{P}^2)}}{2} \leq 6$$

Here is a map on $\mathbb{P}^2$ that requires 6 colors:



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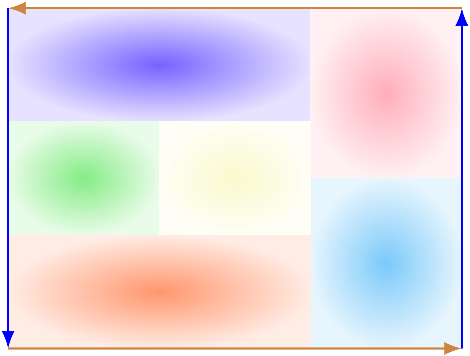

Hence, $C(\mathbb{P}^2) = 6$

Coloring the Klein bottle

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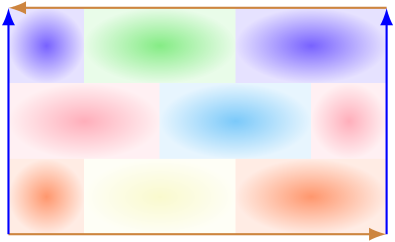
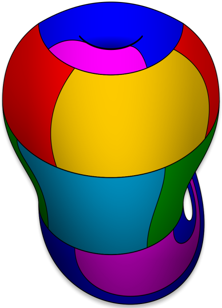
In fact, Franklin (1930) proved that $C(\mathbb{K}) = 6$





Using these maps you can show that $C(\mathbb{K}) \geq 6$

The four color theorem

Theorem

Every map on $\mathbb{D}^2$ can be colored using four colors.
That is, $C(\mathbb{D}^2) = C(\mathbb{R}^2) = C(S^2) = 4$

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There were several incorrect proofs published before Appel and Haken proved this result. One of the incorrect proofs was due to Kempe and 11 years later Heawood found a counterexample to their proof. In doing this, Heawood gave their upper bound for the chromatic number $C(S)$ of any closed surface and he gave a conjecture for coloring surfaces and graphs, which was finally proved in 1968 by Ringel and Young.

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Every map on $\mathbb{D}^2$ can be colored with five colors

By stereographic projection, it is enough to show that $C(S^2) \leq 5$

Let $M = (V, E, F)$ be a map on S^2. We argue by induction on $|F|$

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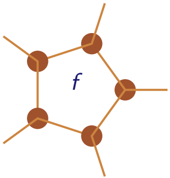

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As in the proof of Heawood's theorem, the idea is now to modify the 5-coloring on N to give a 5-coloring on M.

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By induction the new map N is 5-colorable

As in the proof of Heawood's theorem, the idea is now to modify the 5-coloring on N to give a 5-coloring on M. This time the proof is more complicated and there are several cases to consider

Case 1: $\deg(f) < 5$

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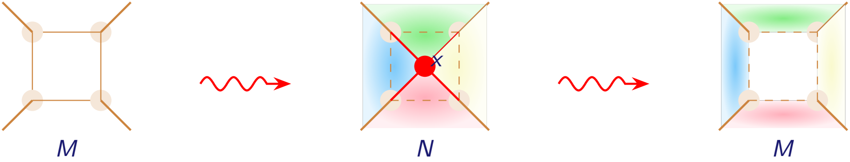

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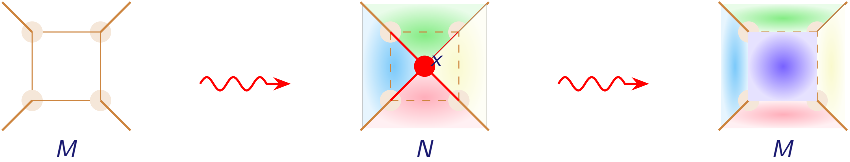

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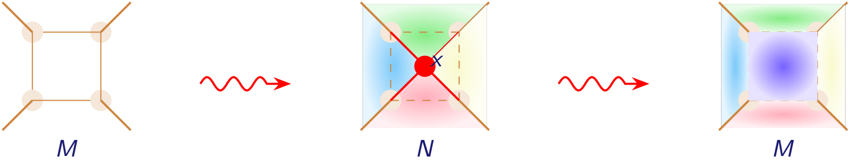

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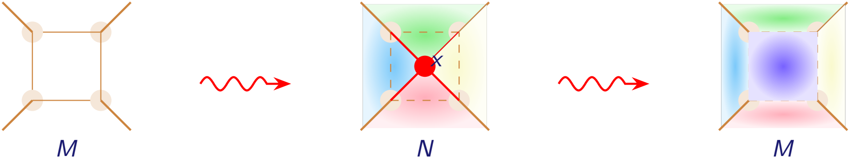

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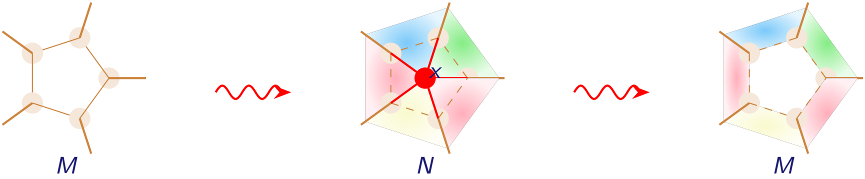

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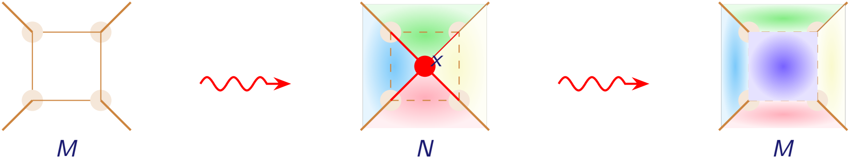

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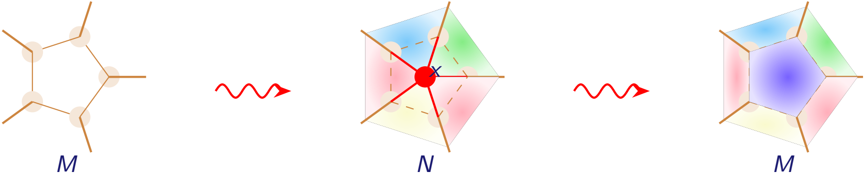
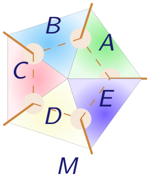


As we have used at most 4 colors in N around x, it follows that M is 5-colorable

Case 3: $\deg(f) = 5$ and all of the colors in N around x are different



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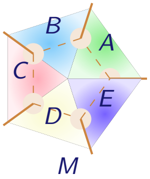

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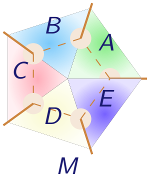

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Consider the polygonal decomposition P contained in N that has these five faces together with all of the regions in N that have the same colors as the faces A and C

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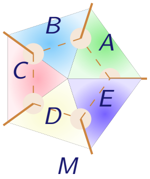

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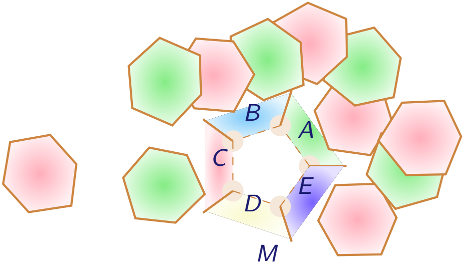
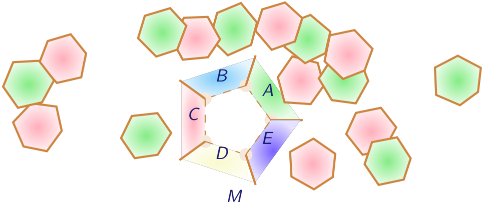


Case 3a: The regions A and C are not connected in P



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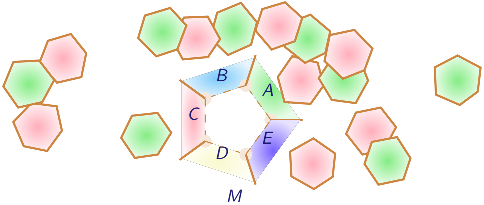
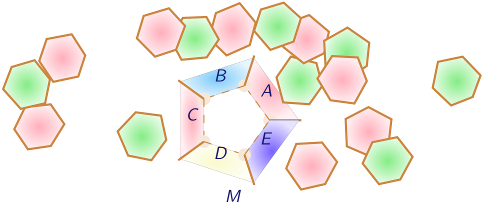


$\Longrightarrow$ Swapping the colors A and C in the connected component of P that contains A gives a new map N' with a valid coloring



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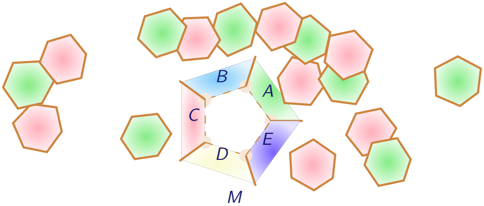

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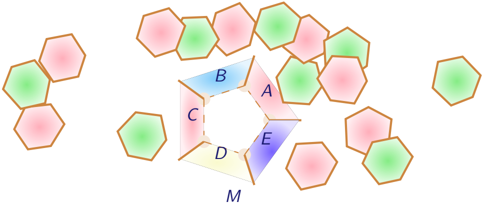

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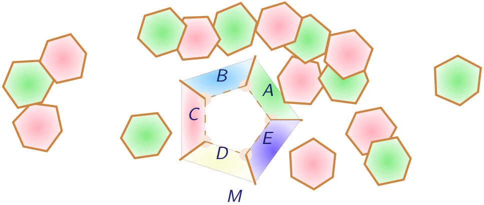

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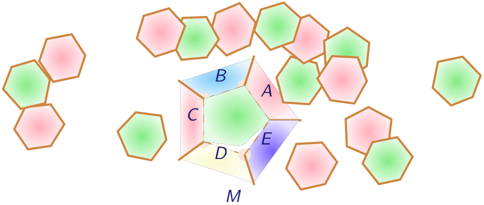

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$\Longrightarrow$ The map M is 5-colorable

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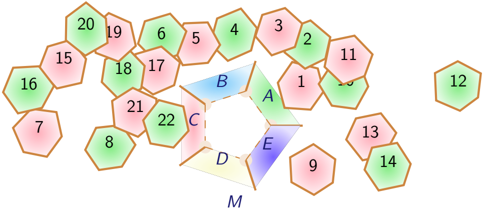

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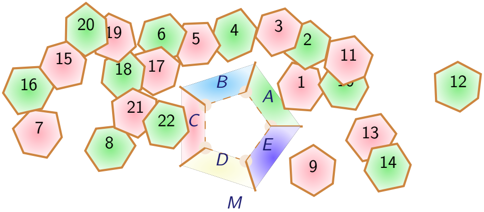

$\Longrightarrow$ As A and C are connected, B and E cannot be connected!

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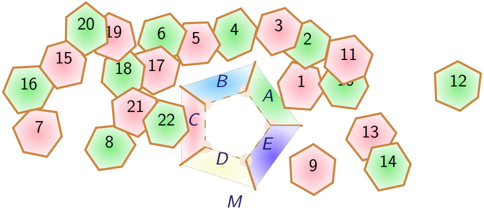

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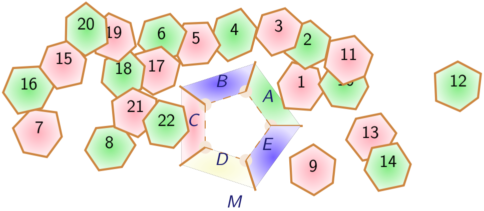

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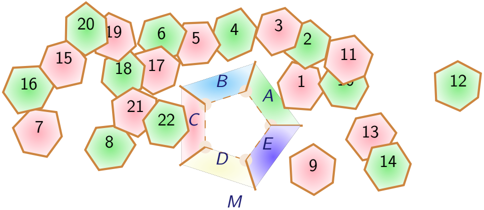

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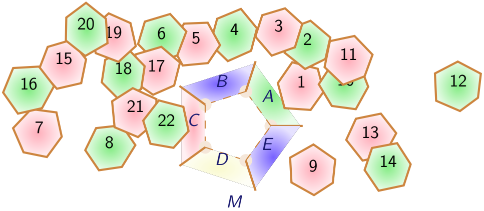

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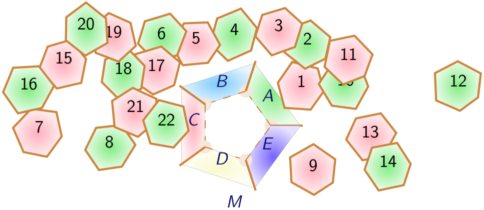

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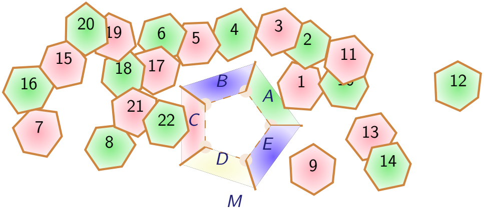

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This completes the proof of the Five color Theorem

Knots

Intuitive definition A knot is a piece of string with the ends tied together

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A knot is the image of an injective continuous map from S^1 into $\mathbb{R}^3$, where $S^1 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$ is the unit circle in $\mathbb{R}^2$

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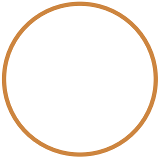
Equivalently, a knot is a closed path in $\mathbb{R}^3$ that has no self-intersections

Examples



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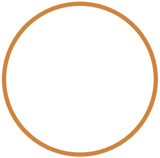

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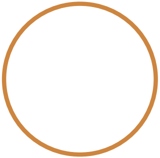

Unknot

Trefoil

Reverse trefoil

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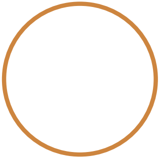

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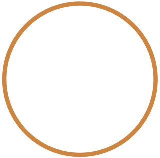
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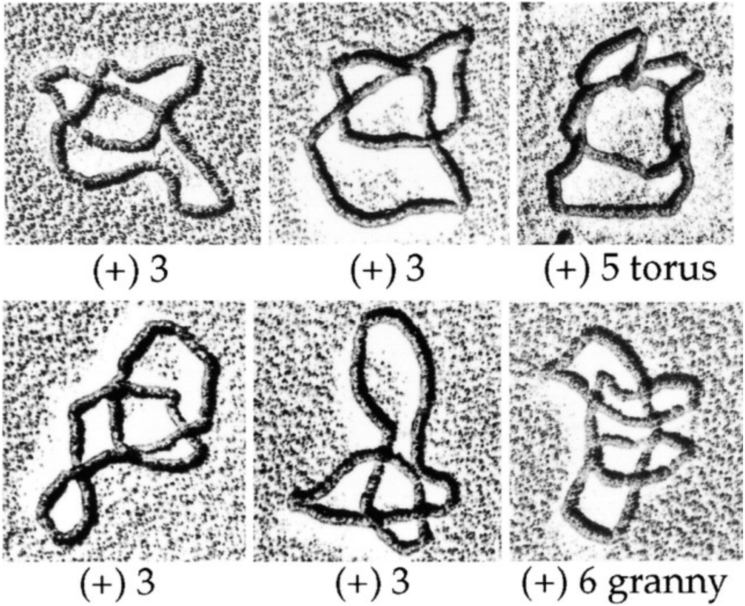
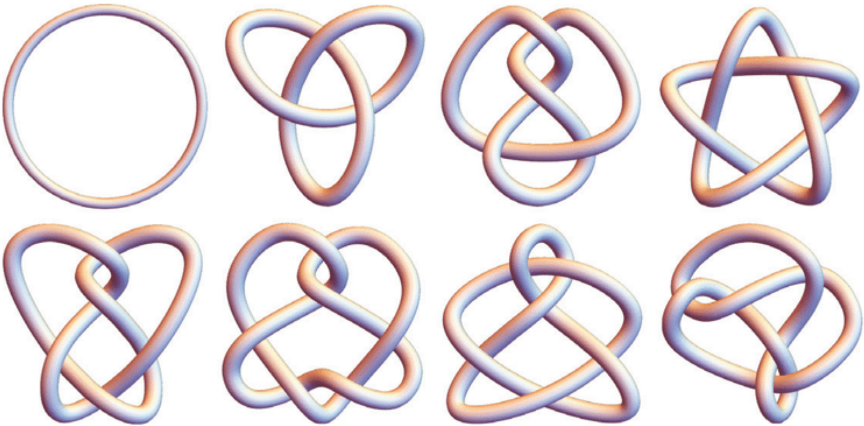
Knot theory is a beautiful mathematical subject with applications in mathematics, computer science, computer chip design, biology, ...

A picture of life



Another picture of life

More knots



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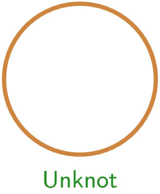

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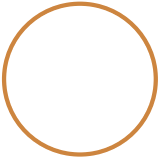

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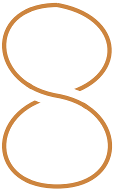

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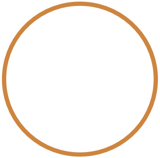

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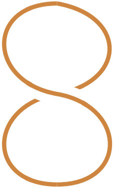

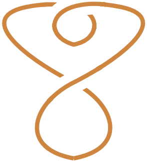

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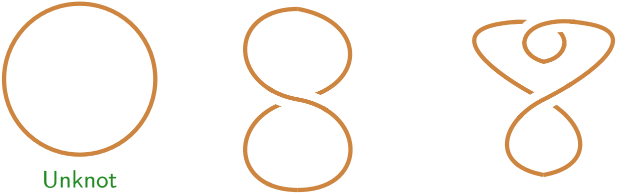

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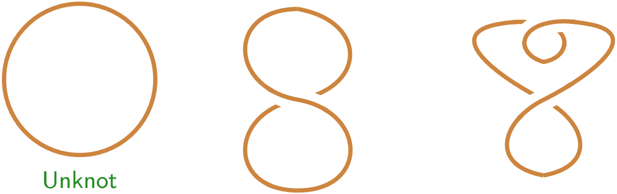

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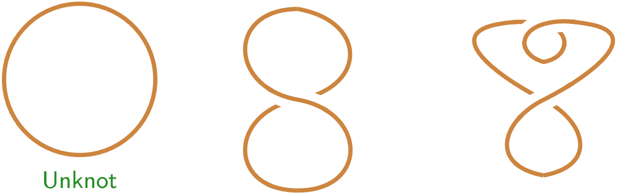
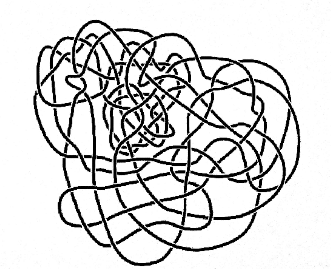


Another unknot

It is difficult to tell if a knot is the unknot



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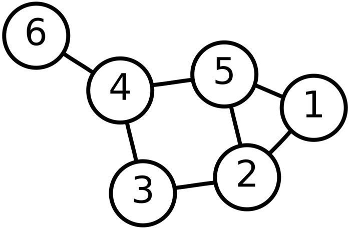
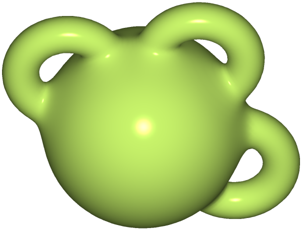
A knot K is trivial if it is equivalent to the unknot otherwise it is non-trivial

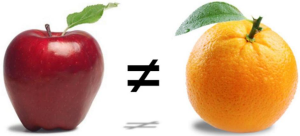
Different notions of “equal”

Objects	Graphs	Surfaces	Knots
Equivalence	Isomorphism of graphs	Homeomorphism	Equivalence of knots

In other words, graphs, surfaces and knots should never be directly compared – they are different beasts







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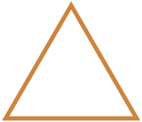

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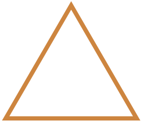

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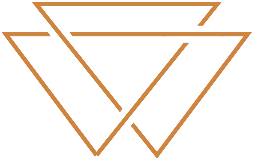

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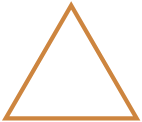

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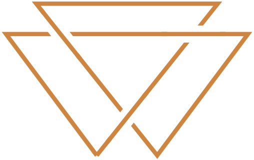

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Figure eight

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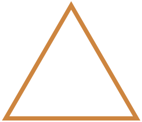

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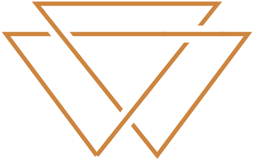

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Figure eight

Remark Two polygonal knots K and L are equivalent if they have a common subdivision

Only polygonal knots

From now on all knots are polygonal knots and we drop the adjective polygonal

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This is not a huge restriction: anything you can draw is polygonal. Any "finite thing" is a polygonal knot, but "limits" are not so we ignore them

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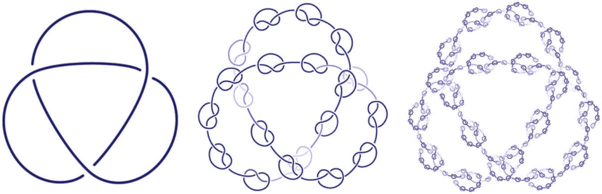

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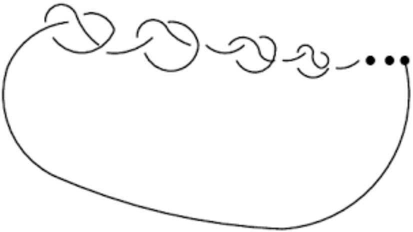

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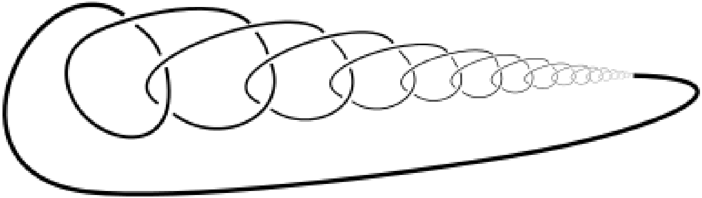

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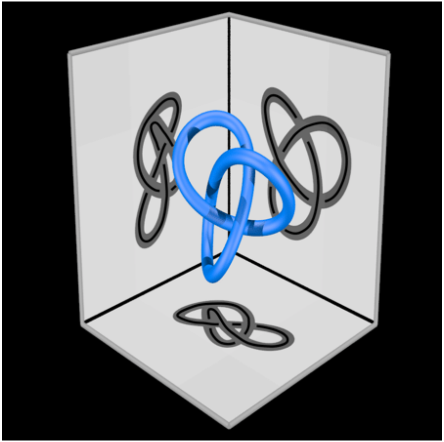
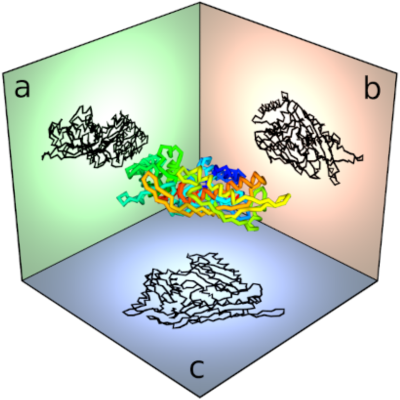
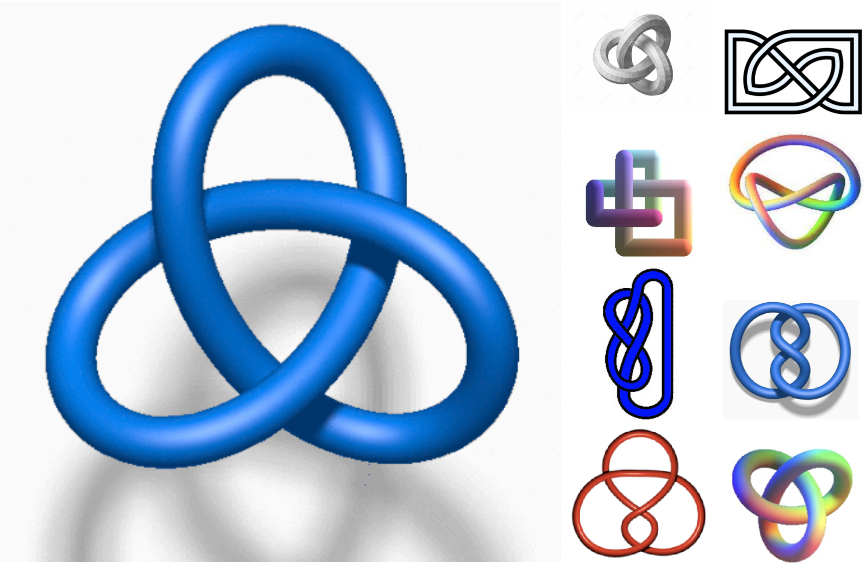
$\Longrightarrow$ Knot projections can be misleading so we have to check that our constructions are independent of the choice of knot projection

Projections = shadows





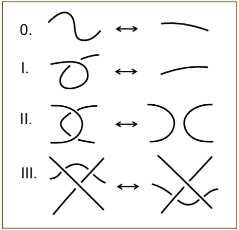
The trefoil knot times nine



Reidemeister's theorem

Theorem

Two knot diagrams represent the same knot if and only if they are related by a (finite) sequence of moves of the following three types



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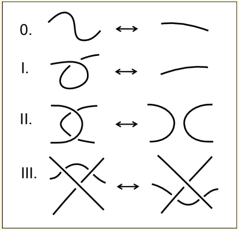

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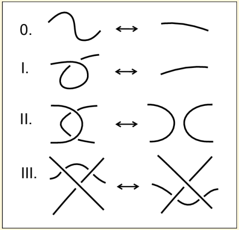

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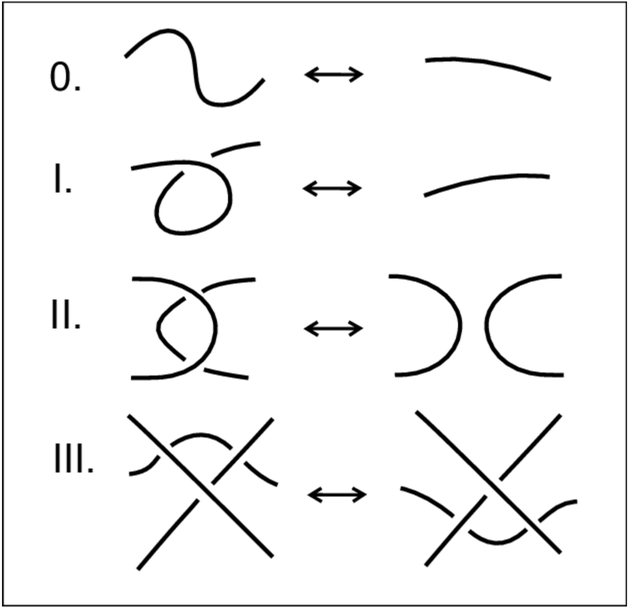
The point: Reidemeister's theorem reduces topology to combinatorics of diagrams

The Reidemeister moves on one slide



The knotty trefoil

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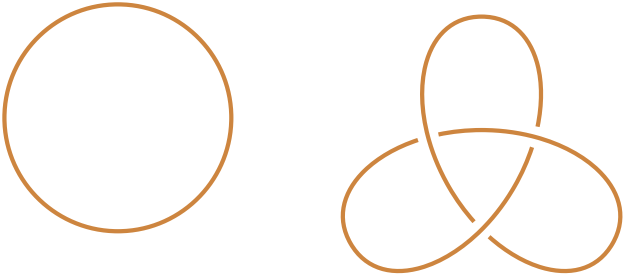

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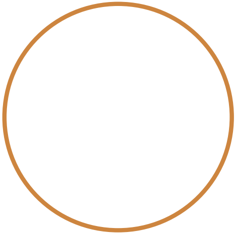

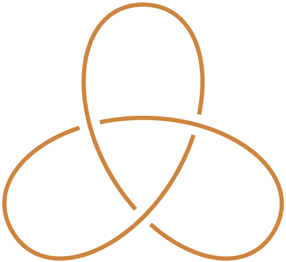

It seems clear that these two knots are different but, so far, we have not seen an easy way to distinguish between them

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A coloring of a knot (projection) is the assignment of colors to the different segments,or connected components, so that at each crossing all segments have either the same color or they all have different colors and at least two colors are used

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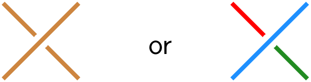

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Let $C_3(K)$ be the number of different colorings of K using 3 colors

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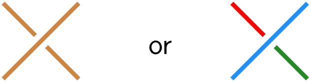

$\Longrightarrow$ If a knot (projection) is 3-colorable then it has a coloring that uses exactly 3 colors

Let $C_3(K)$ be the number of different colorings of K using 3 colors

Remark

- A knot can always be colored using a single color, so $C_3(K) \geq 3$ for all knots K

Knot colorings

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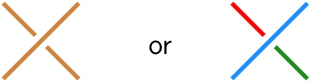

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- A knot can always be colored using a single color, so $C_3(K) \geq 3$ for all knots K
- As soon as more than one color is used we must use all three colors, so K is 3 colorable if and only if $C_3(K) > 3$

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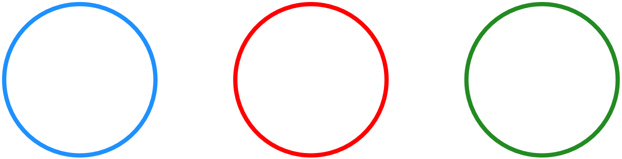

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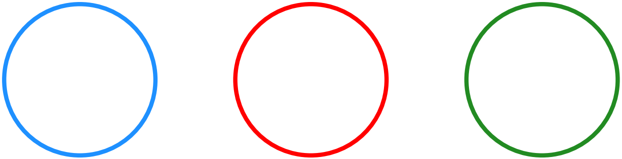

Which of the following are knots are 3-colorable?

coloring the trefoil knot

Question What is $C_3(T)$ if T is the trefoil knot?

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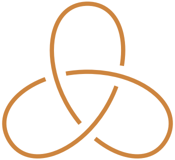

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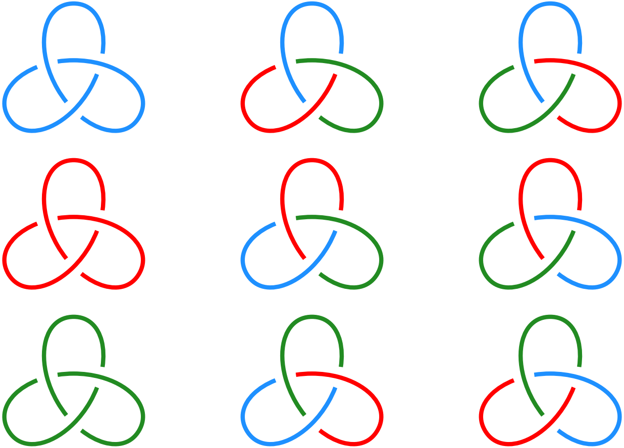

Three colorability

Theorem

The integer $C_3(K)$ is a knot invariant
That is, $C_3(K)$ depends only on K, up to ambient isotopy, and it is independent of the choice of knot projection

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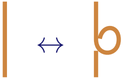
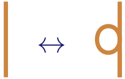
- Twisting  and 

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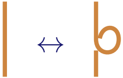
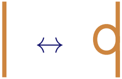
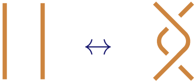
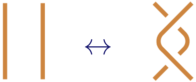
- Twisting  and 
- Looping  and 

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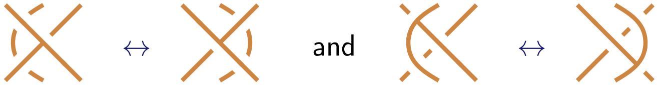

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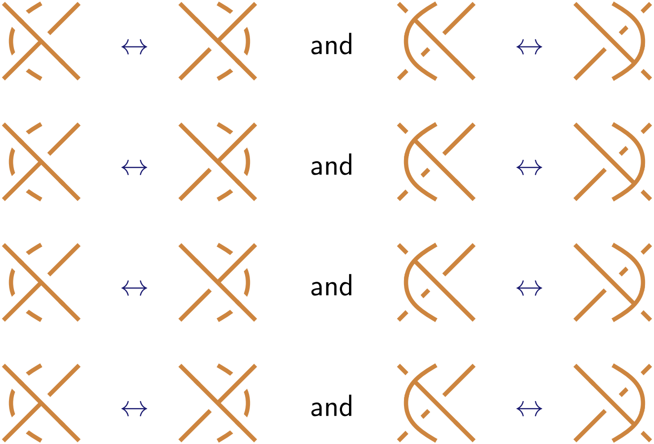

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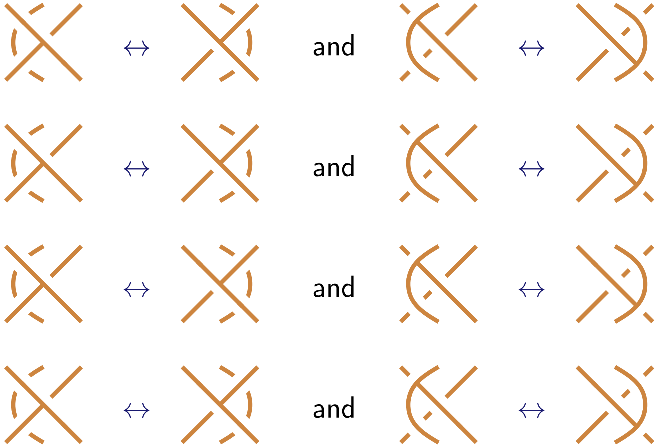

Key point For each Reidemeister move there is a unique way to complete any coloring given the existing colors of the segments going in and out