Topology – week 9

Math3061

Daniel Tubbenhauer, University of Sydney

Classifying surfaces using invariants

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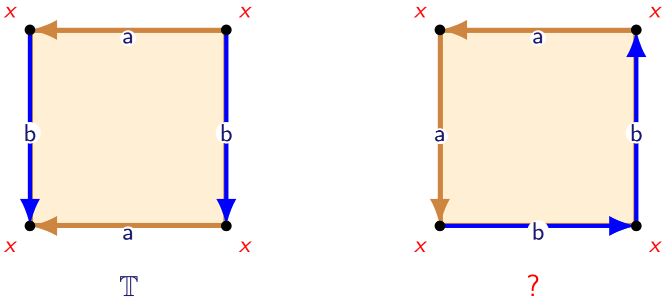

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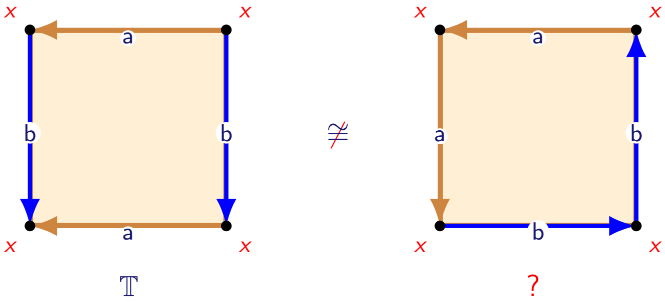

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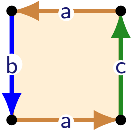
$\mathbb{M} =$ 

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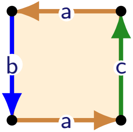
$\mathbb{M} =$ 

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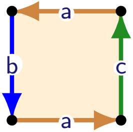
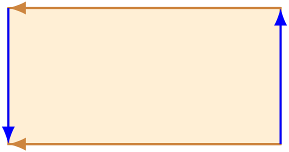
$\mathbb{M} =$ 

- Are S^2, $\mathbb{A}$, $\mathbb{D}^2$, $\mathbb{T}$, $\mathbb{P}^2$, $\mathbb{K}$, ... orientable or non-orientable?
- Can a surface be orientable and non-orientable for different polygonal decompositions? (That would be bad!)

The Klein bottle $\mathbb{K}$

$\mathbb{K} \quad =$ 

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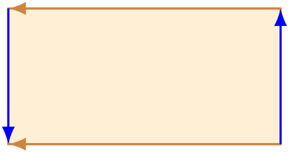
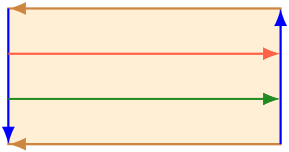
$\mathbb{K}$ =  = 

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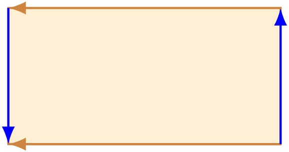
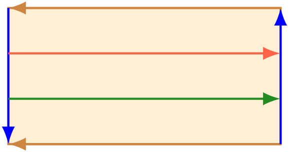
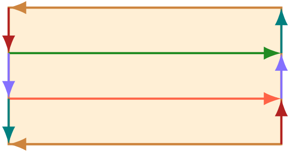
$\mathbb{K}$ $=$  $=$ 

$=$ 

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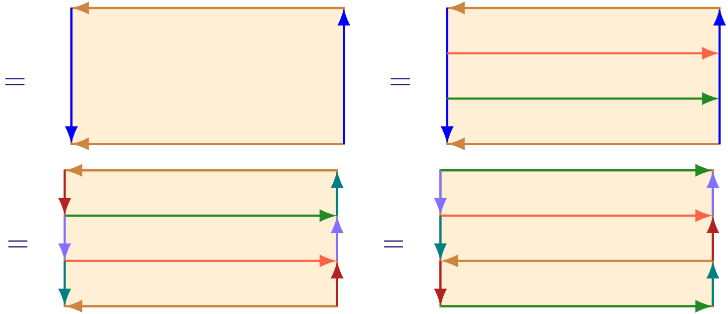

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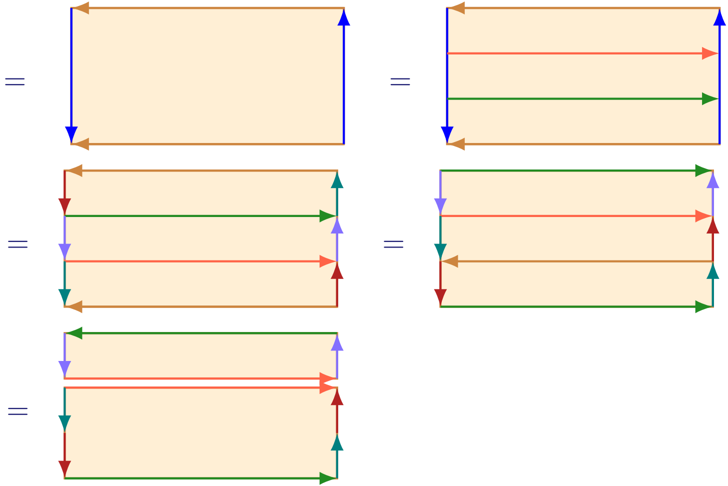

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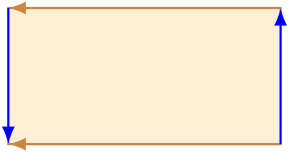

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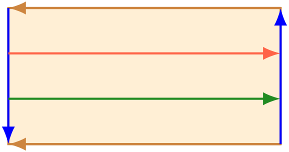

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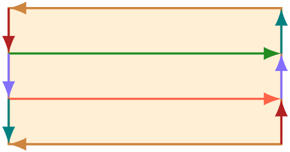

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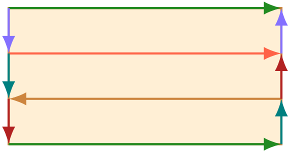

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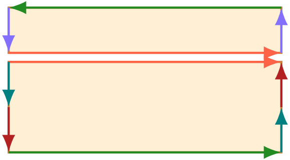

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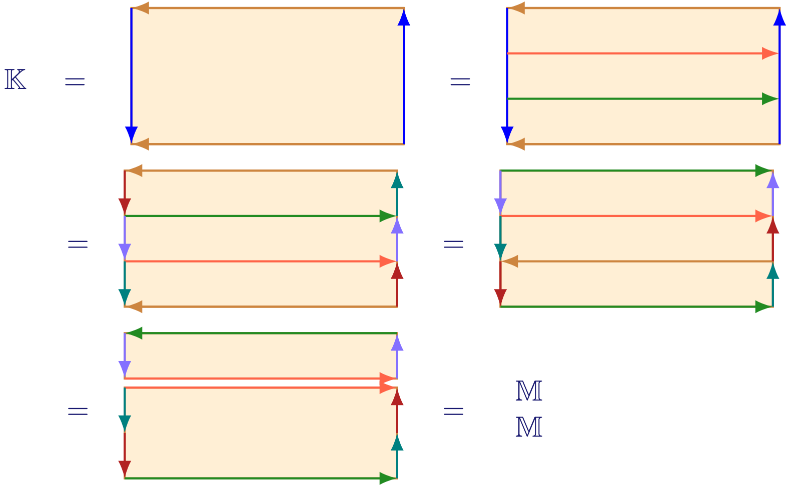

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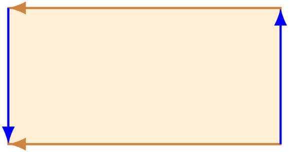
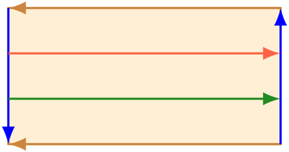
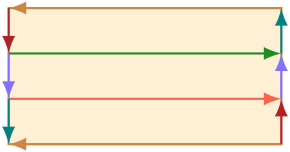
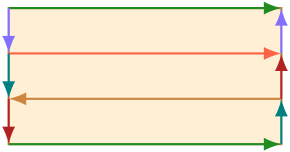
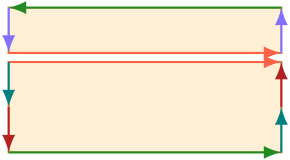
$\mathbb{K}$ =  = 

=  = 

=  = $\mathbb{M}$ $\mathbb{M}$

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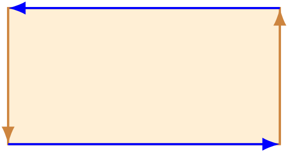
. . . although it might be more accurate to say that the Klein bottle is a Möbius strip without boundary

The projective plane $\mathbb{P}^2$

$\mathbb{P}^2 \quad =$ 

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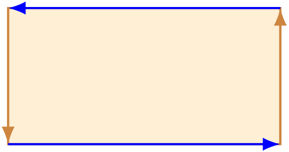
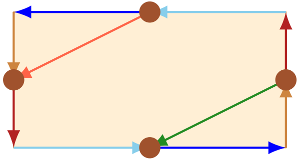
$\mathbb{P}^2 \quad = $  $=$ 

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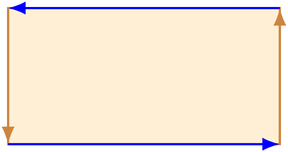

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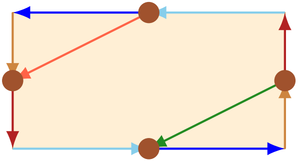

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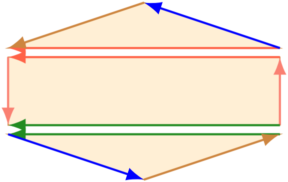

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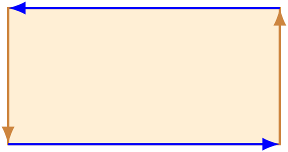

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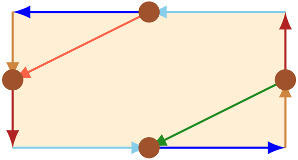

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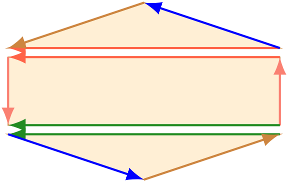

=

$\mathbb{D}^2$

$\mathbb{M}$

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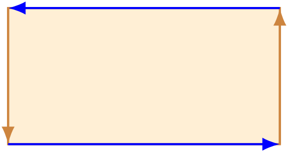

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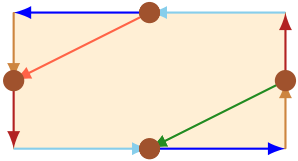

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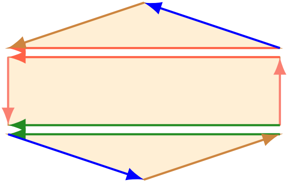

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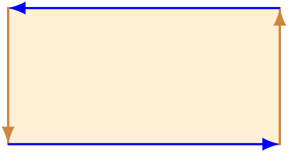
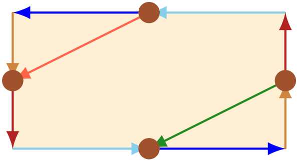
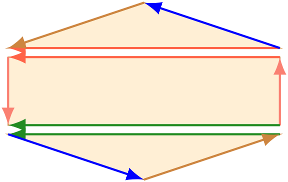
$\mathbb{P}^2 \quad =$  $=$ 

$=$  $=$ $\begin{matrix}\mathbb{D}^2 \\ \mathbb{M} \\ \mathbb{D}^2\end{matrix}$

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... or maybe $\mathbb{P}^2$ and not $\mathbb{K}$
is a Möbius strip without boundary?

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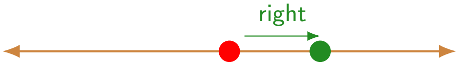

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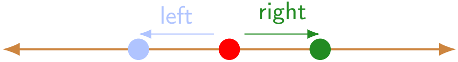

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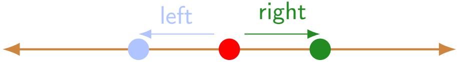

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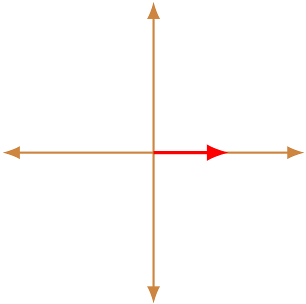

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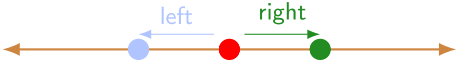

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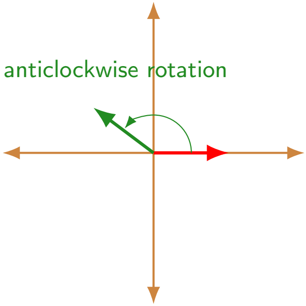

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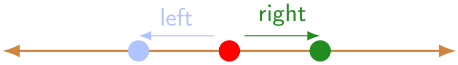

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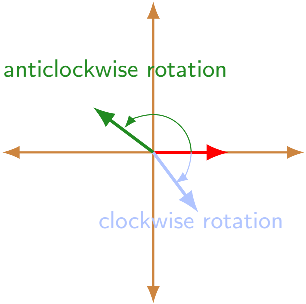

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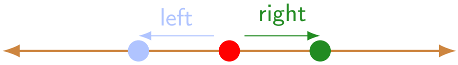

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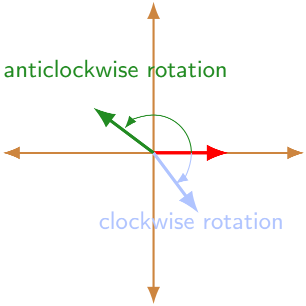

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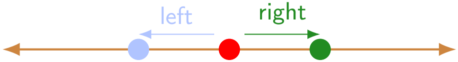

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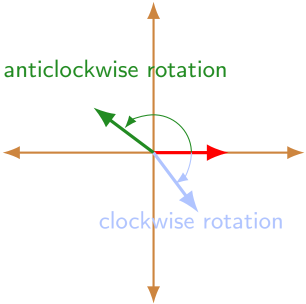

- Three dimensions $\mathbb{R}^3$???
- Higher dimensions $\mathbb{R}^n$, for $n \geq 3$???

Direction in higher dimensions

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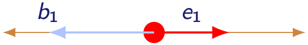

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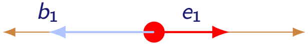
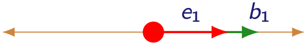


$\operatorname{sign}(B) = +1$



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$$\det(B) = \det\begin{pmatrix} \vdots & \vdots & & \vdots \\ b_1 & b_2 & \ldots & b_n \\ \vdots & \vdots & & \vdots \end{pmatrix} \rightsquigarrow \operatorname{sign}(B) = \pm 1$$

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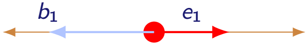

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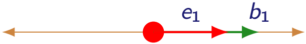

- Two dimensions $\mathbb{R}^2$

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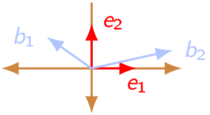

Direction in higher dimensions

To generalise direction, choose an ordered basis $B = \{b_1, b_2, \ldots, b_n\}$ of $\mathbb{R}^n$

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Direction on the Möbius strip

Pick a point $m \in \mathbb{M}$ on the Möbius strip and an ordered basis $B = \{b_1, b_2, b_3\}$ positioned at m with $b_3 = b_1 \times b_2$ pointing outwards

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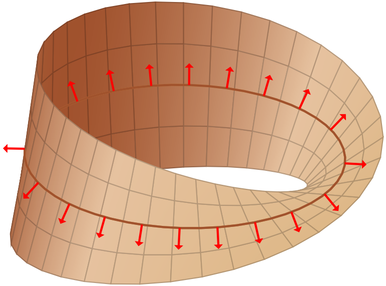
Now imagine m, and the coordinate axes moving, continuously around the Möbius strip so that the xyz-coordinate axes around $\mathbb{M}$



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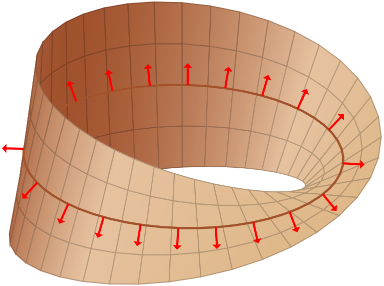

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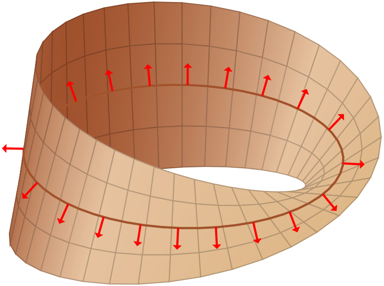

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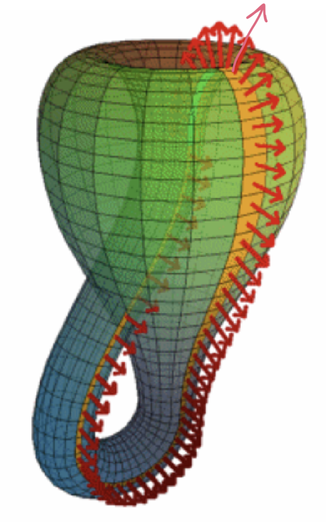
The vector b_3 is always normal to the surface of the Möbius strip. The direction of b_3 can change from pointing outside to inside because the Möbius strip is a surface with a boundary that only has one side

Direction on the Klein bottle $\mathbb{K}$



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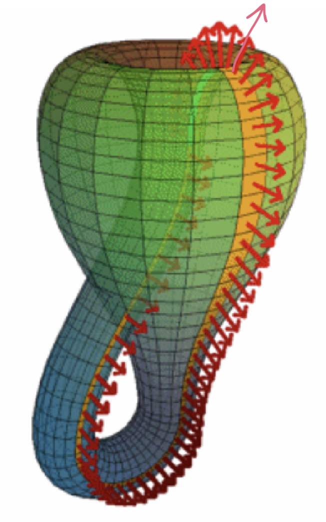

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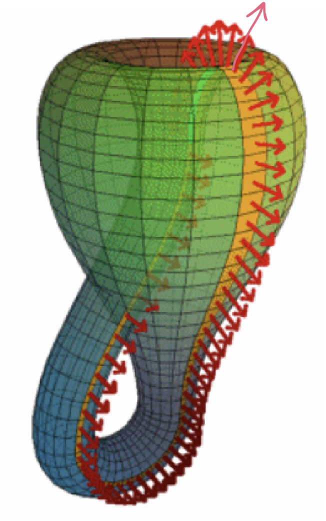

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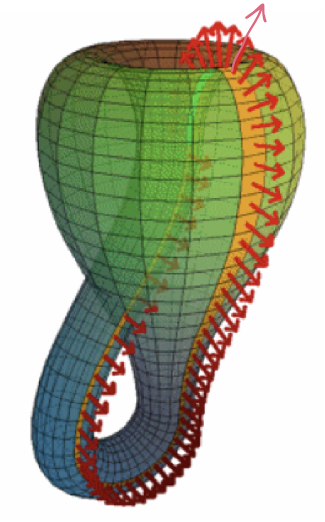

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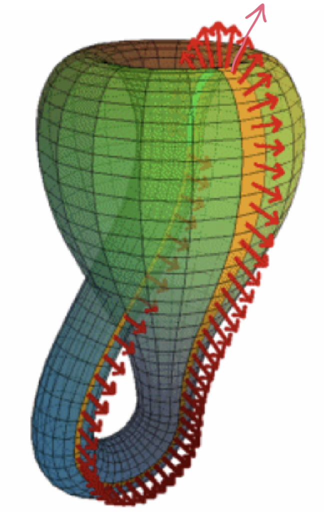

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Warning: this is a drawing of $\mathbb{K}$ in $\mathbb{R}^3$ but it is not the actual Klein bottle! Similarly, the pictures of the sphere S^2 in $\mathbb{R}^3$ are not really the sphere!

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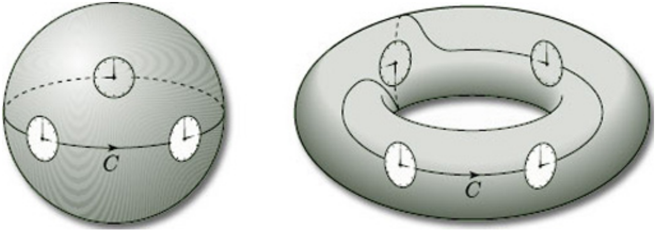

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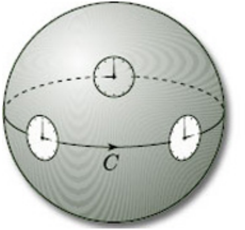

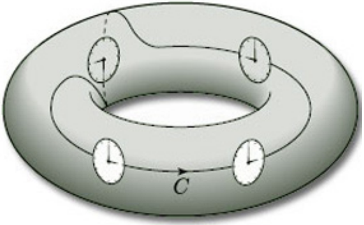

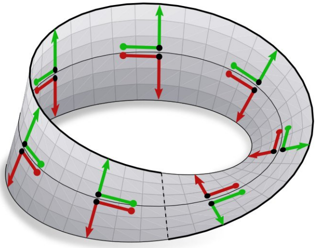

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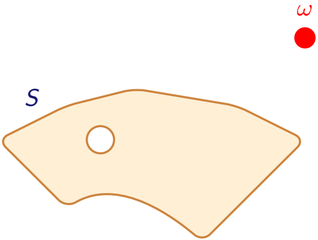

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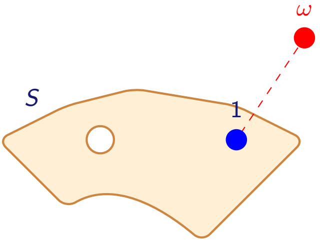

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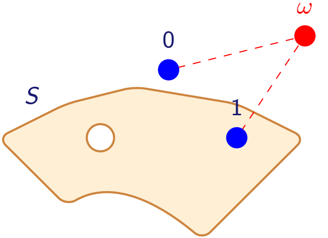

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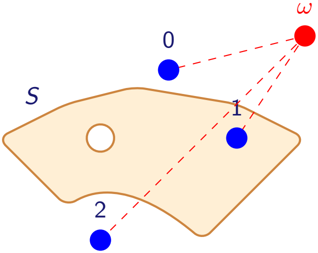

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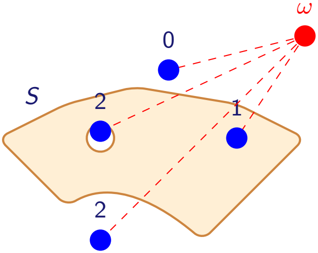

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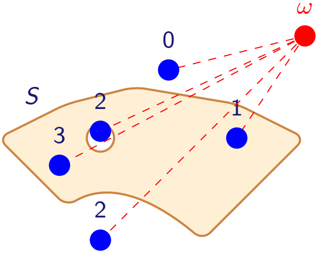

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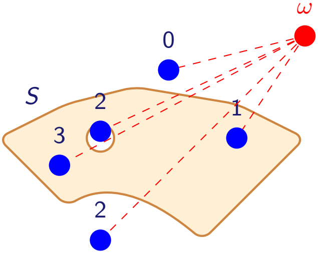

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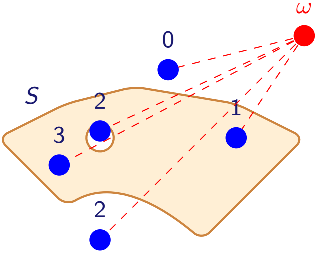

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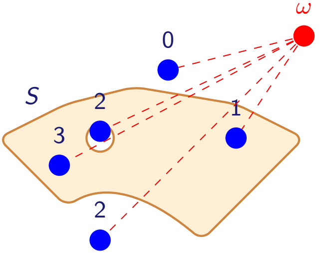

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Notice that since S is a closed surface it does not have boundary, so the "circle" in the picture, which contains a point x with $s(x) = 2$, should be interpreted as a tube through the surface

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Corollary

Let S be a non-orientable closed surface. Then S does not embed in $\mathbb{R}^3$.

You can't fill a liquid into the Klein bottle

Strictly speaking the liquid is neither in- nor outside

Jordan curve theorem

This argument used to prove theorem can be made rigorous for surfaces with finite polygonal decompositions but for "general surfaces" it is difficult to prove that $\mathbb{R}^3 = S \cup V_{\text{in}} \cup V_{\text{out}}$.

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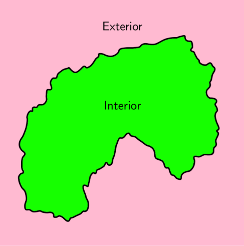

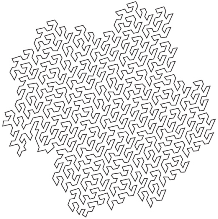

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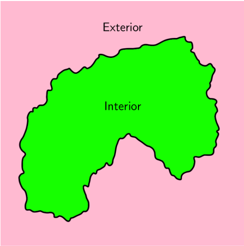
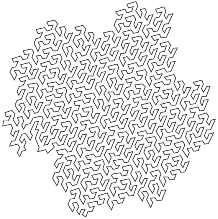
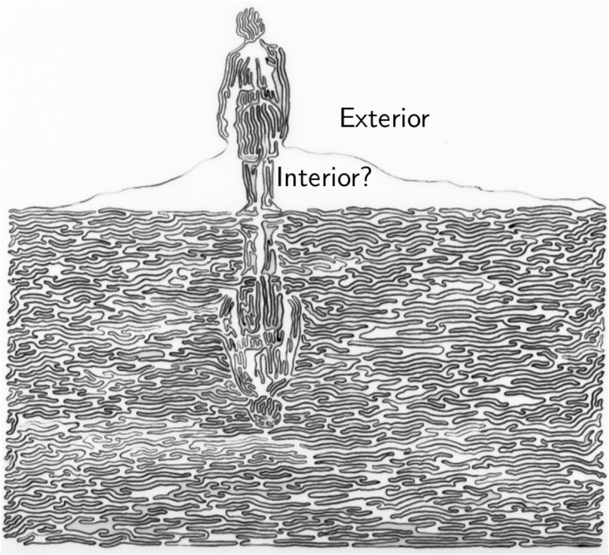




The left is easy, but can you tell for the right what is "in" or "out"?

Jordan curve theorem - 2

The main meat is that one needs to deal with "crazy" curves:



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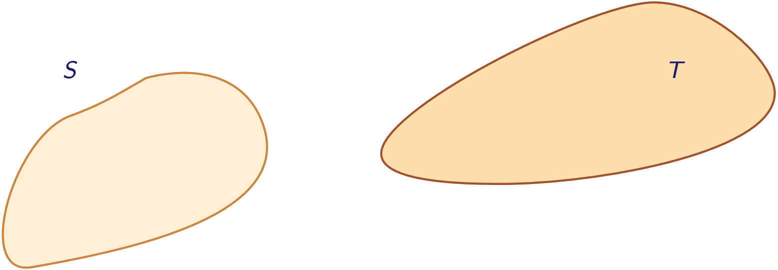

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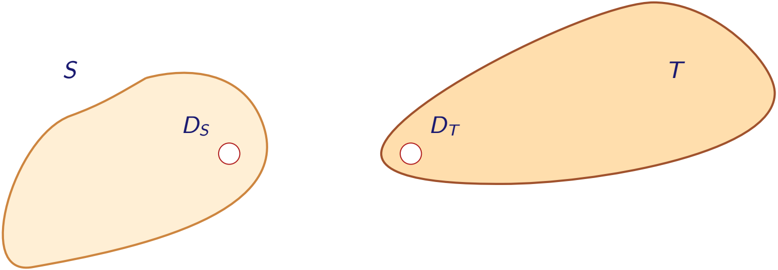

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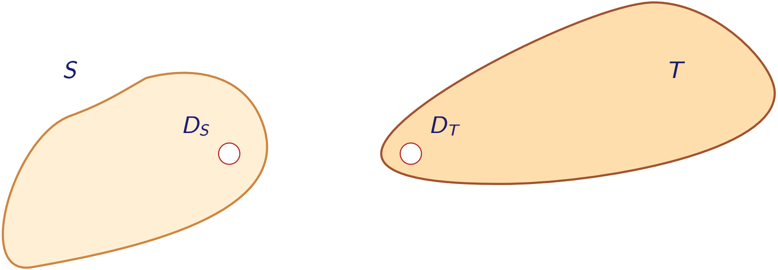

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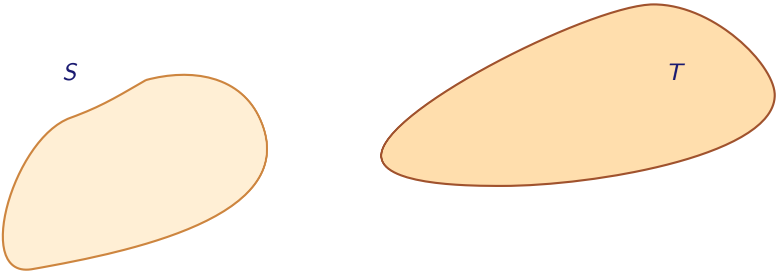

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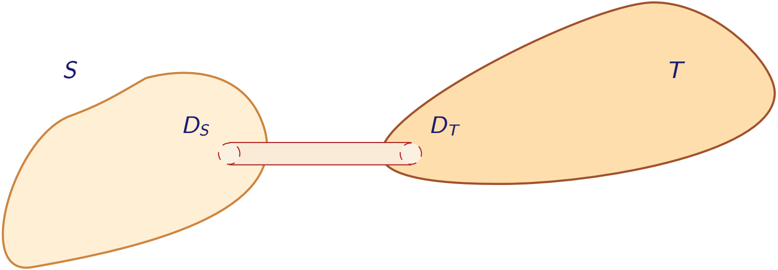

Identifying D_S and D_T is the same as connecting them with a cylinder

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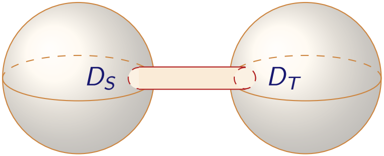

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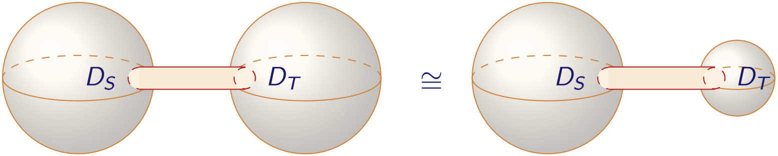

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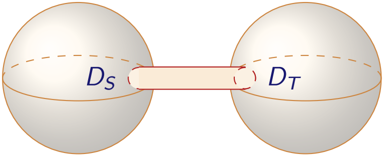

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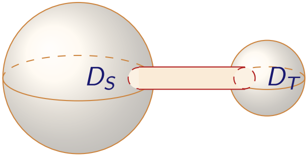

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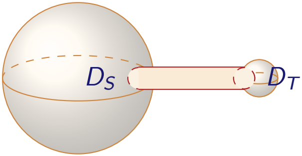

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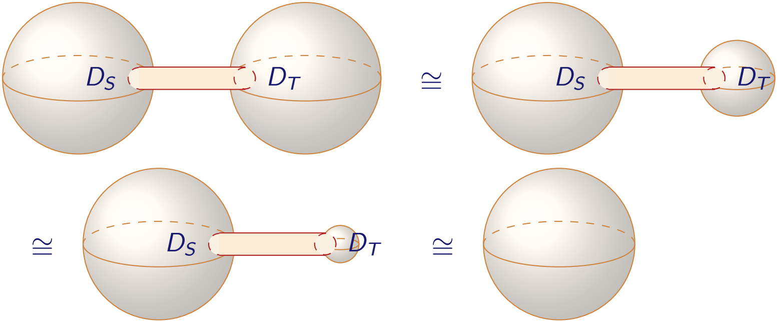

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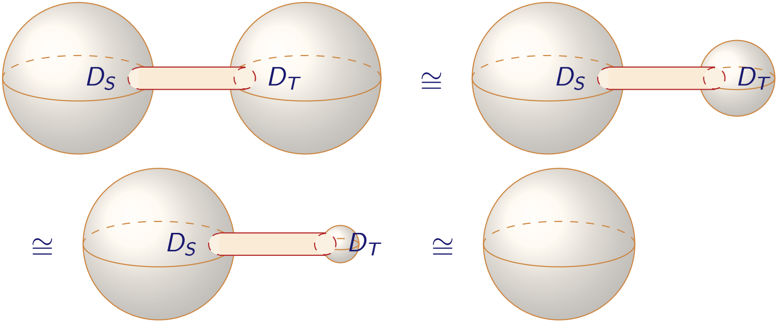

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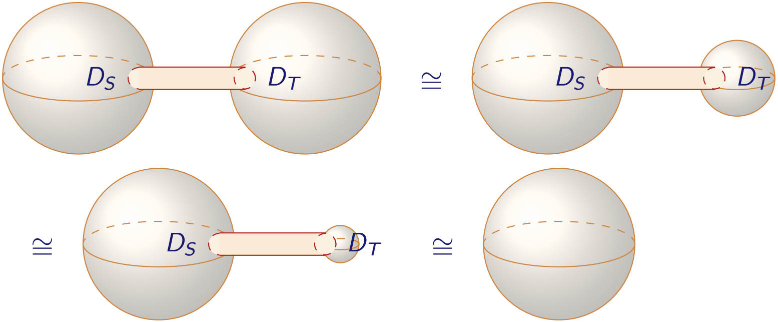

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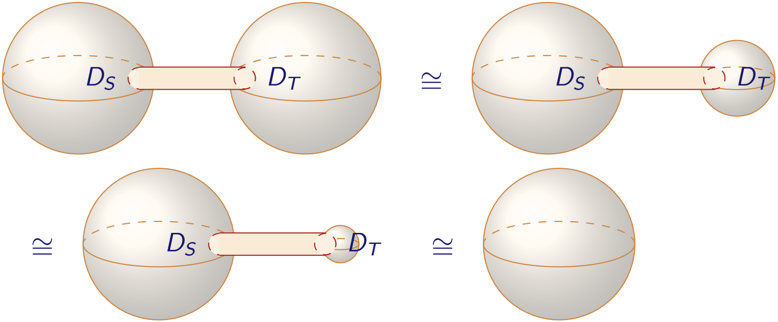

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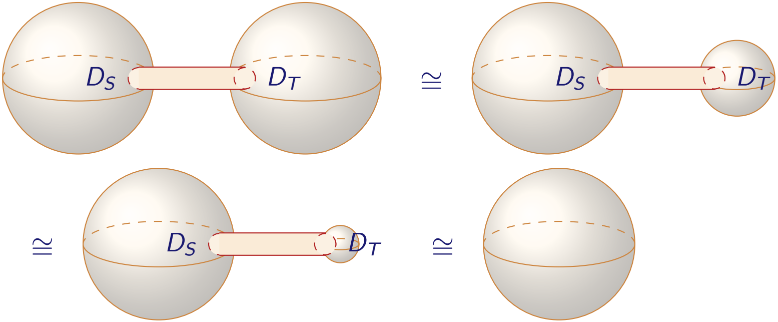

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So S^2 is the unit under the operation $\#$

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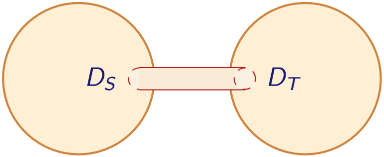

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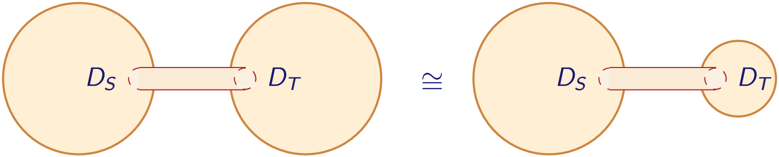

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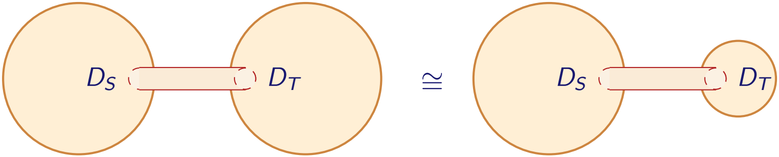

This is not the same as collapsing a sphere, which closes up the hole, because the disk has a boundary!

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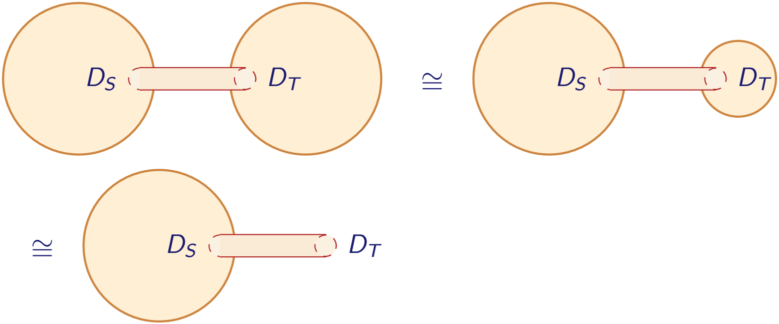

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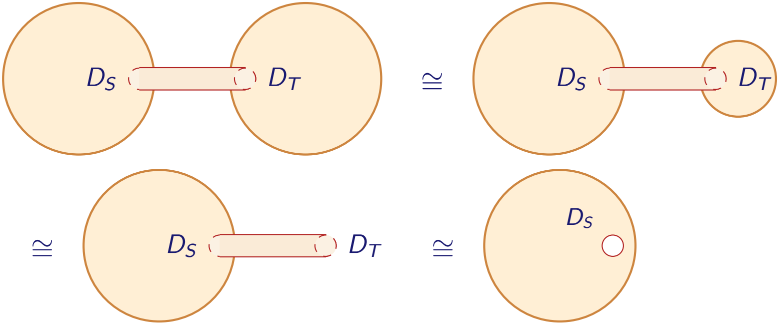

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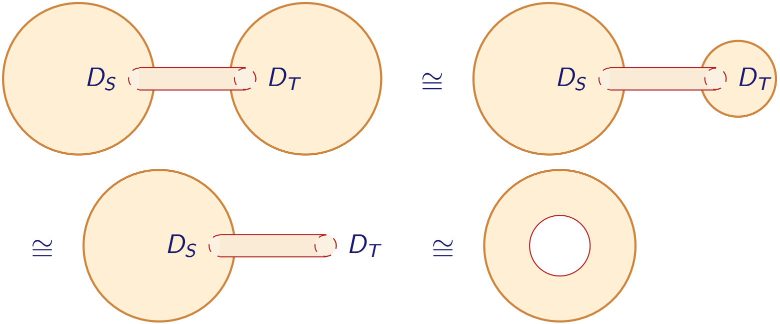

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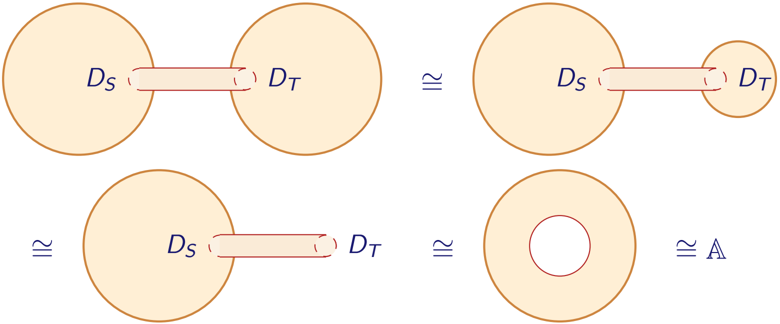

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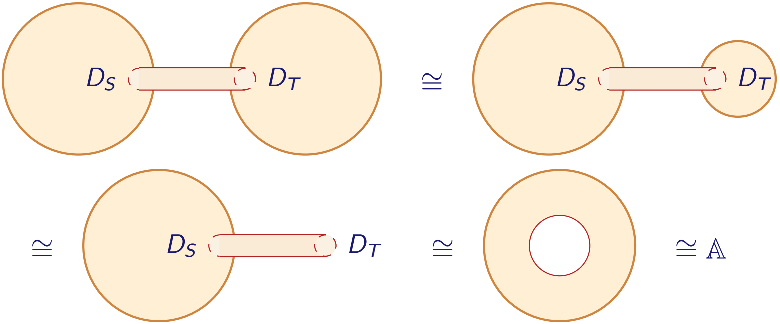

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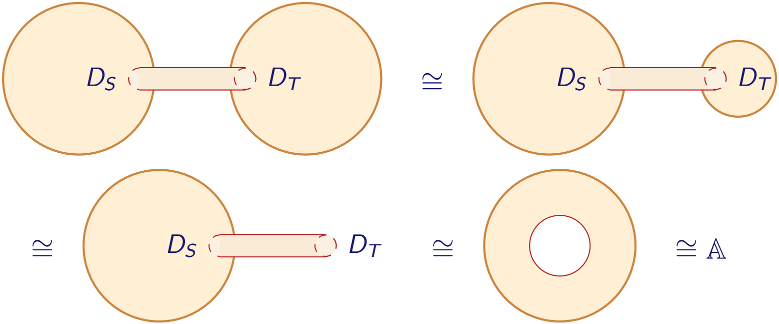

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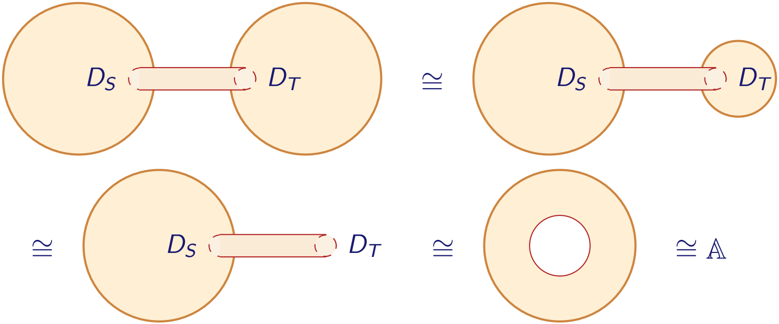

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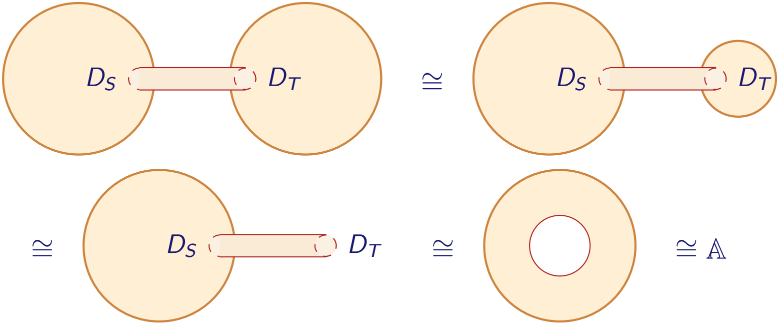

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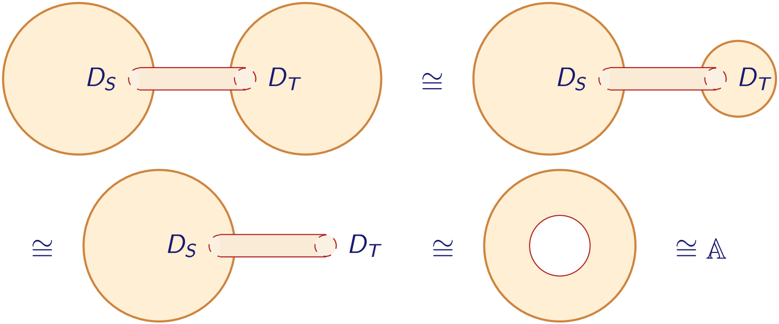

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Connected sums with tori

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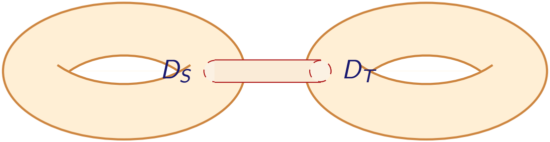

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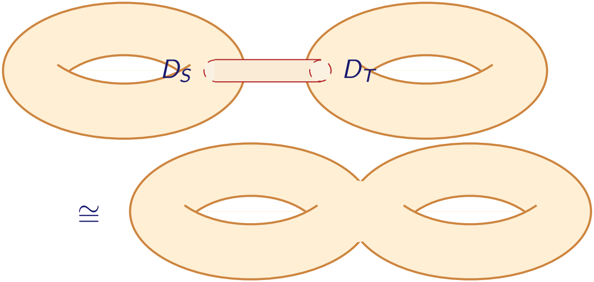

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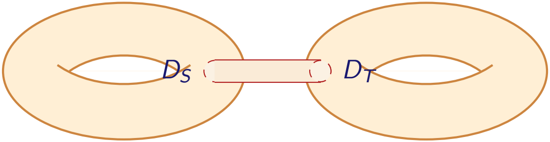

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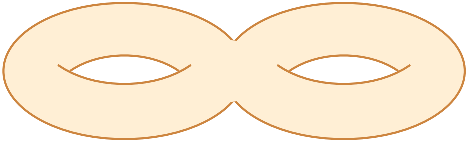

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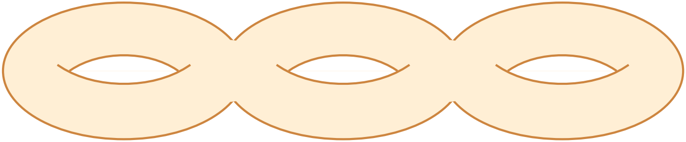

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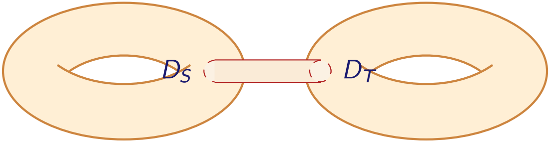

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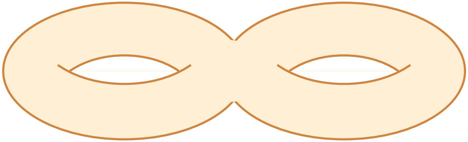

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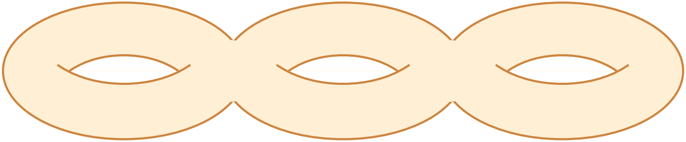
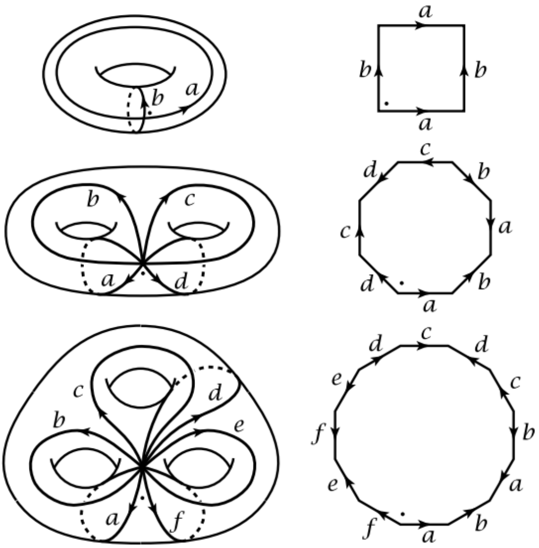


. . . and, more generally, t-tori $\#^t \mathbb{T}$

We already know t-tori



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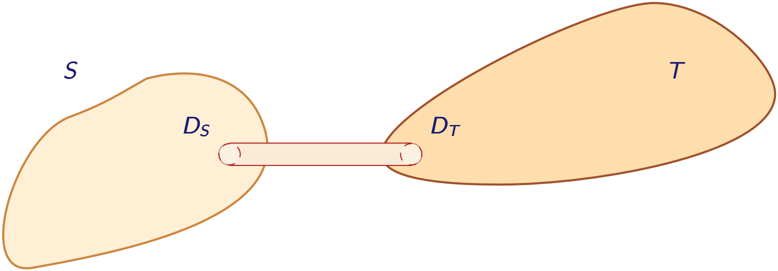

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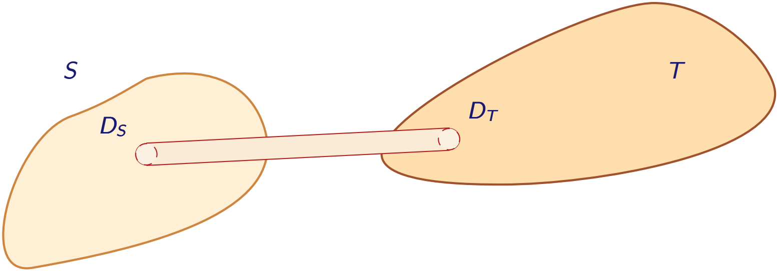

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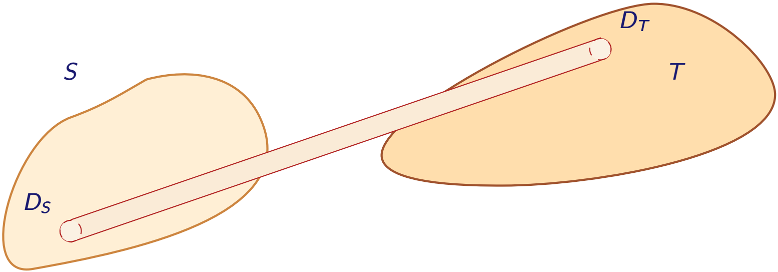

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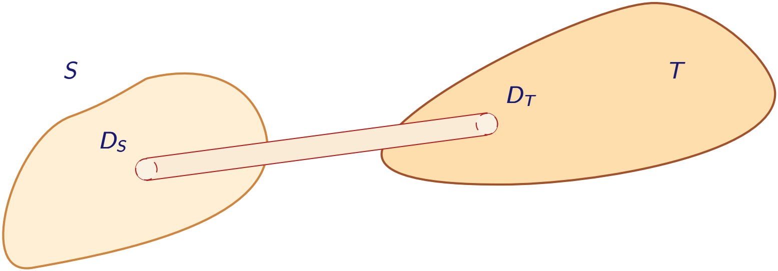

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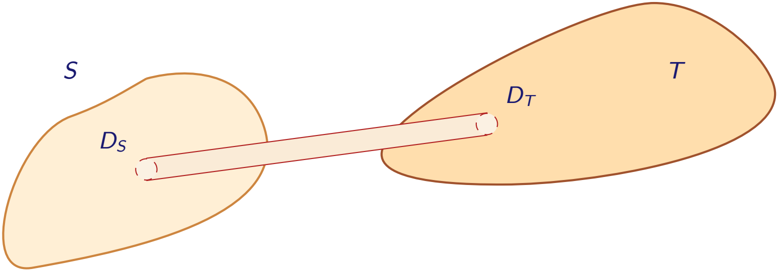

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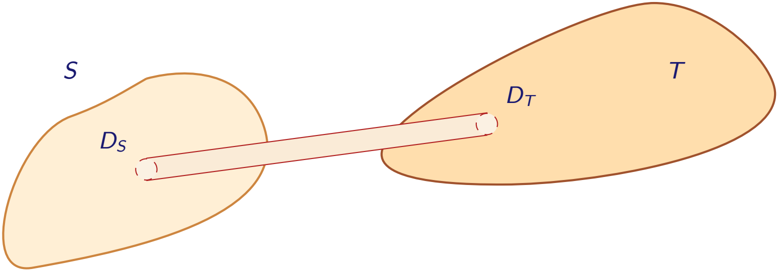

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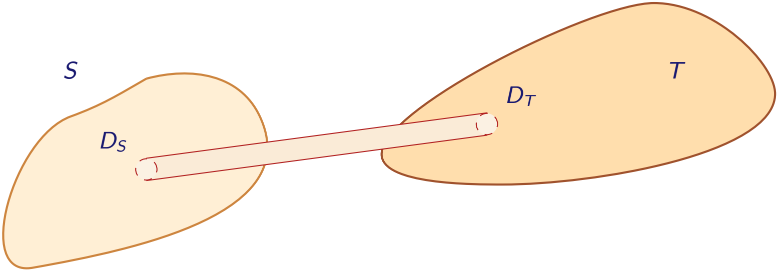

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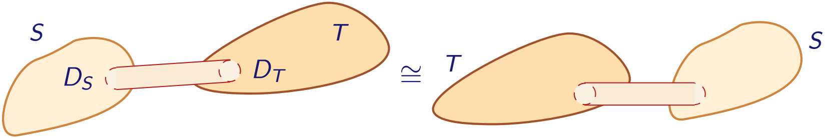

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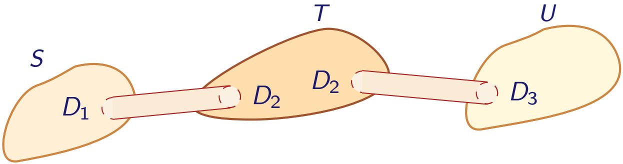

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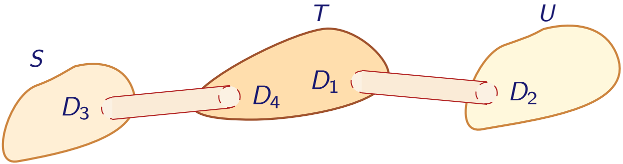
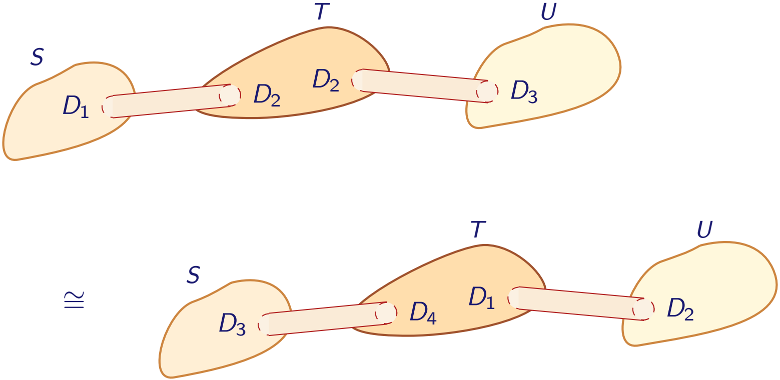


In these diagrams, D_1 and D_2 are cut first and then D_3 and D_4

Associativity of connected sums...

- $S \# (T \# U) \cong (S \# T) \# U$



In these diagrams, D_1 and D_2 are cut first and then D_3 and D_4

$\Longrightarrow$ $\#$ is a "surface addition or multiplication"

Connected sums of Euler characteristic

Theorem

Let S and T be surfaces with polygonal decompositions. Then

$$\chi(S \# T) = \chi(S) + \chi(T) - 2$$

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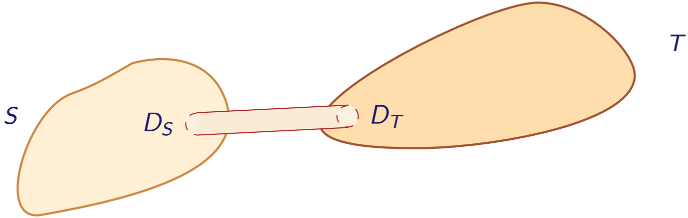

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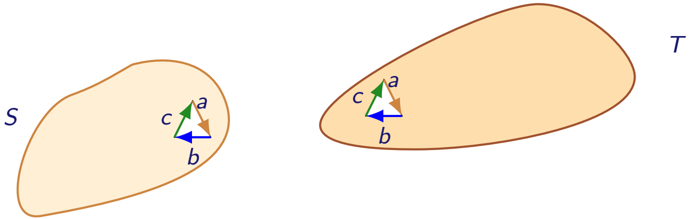

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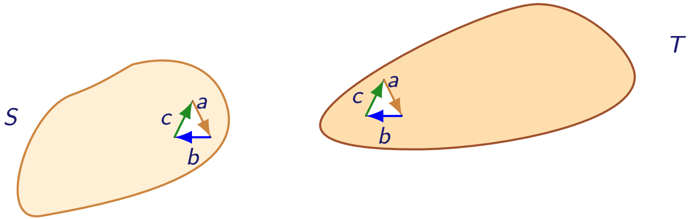

$$\implies \quad \chi(S \# T) = \big(\chi(S) - (3 - 3 + 1)\big) + \big(\chi(T) - (3 - 3 + 1)\big)$$

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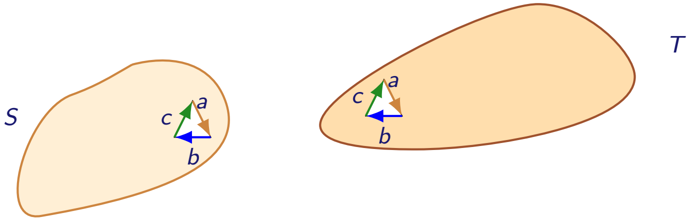


$$\implies \quad \chi(S \# T) = \big(\chi(S) - (3 - 3 + 1)\big) + \big(\chi(T) - (3 - 3 + 1)\big)$$

Moral The -2 comes from cutting out two disks

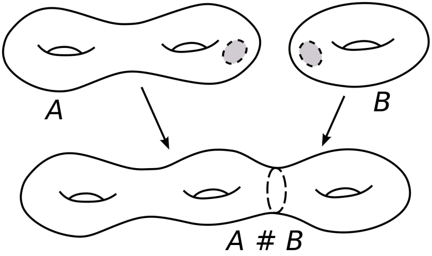
Examples

- If S is any surface then $S \cong S \# S^2$

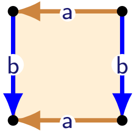
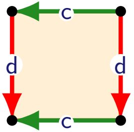
$$\implies \quad \chi(S) = \chi(S) + \underbrace{\chi(S^2)}_{=2} - 2 = \chi(S)$$

- $\mathbb{A} \cong \mathbb{D}^2 \# \mathbb{D}^2 \quad \implies \quad \chi(\mathbb{A}) = \chi(\mathbb{D}^2) + \chi(\mathbb{D}^2) - 2 = 1 + 1 - 2 = 0$

- $\chi(\mathbb{T} \# \mathbb{T} \# \mathbb{T}) = (\chi(\mathbb{T}) + \chi(\mathbb{T}) - 2) + \chi(\mathbb{T}) - 2 = -4$



Connected sums and polygonal decompositions

$\mathbb{T} \# \mathbb{T} \quad =$  $\#$ 

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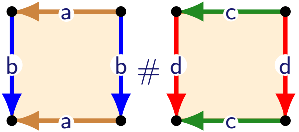

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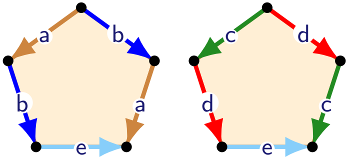

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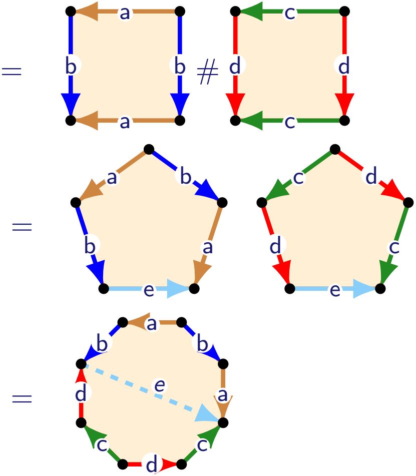

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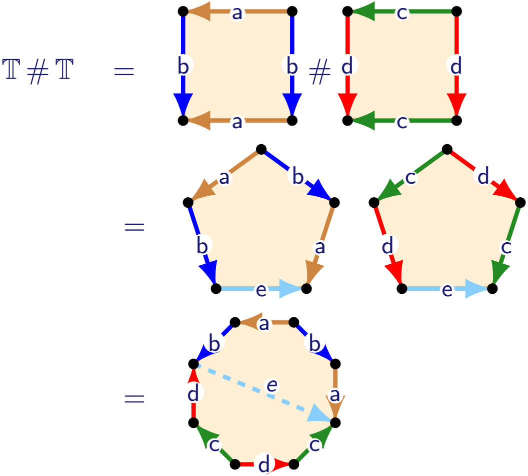
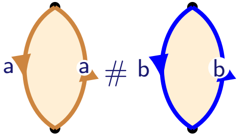


$\Longrightarrow$ For surfaces without a boundary you can cut the disks anywhere!

Connected sums with projective planes

- What is $\mathbb{P}^2 \# \mathbb{P}^2$?

$\mathbb{P}^2 \# \mathbb{P}^2 \quad \cong$ 

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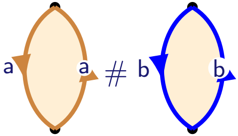
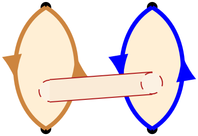
$\mathbb{P}^2 \# \mathbb{P}^2 \quad \cong$  $\cong$ 

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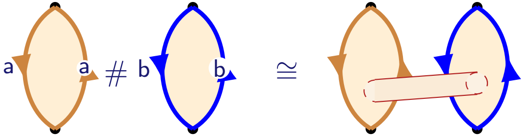

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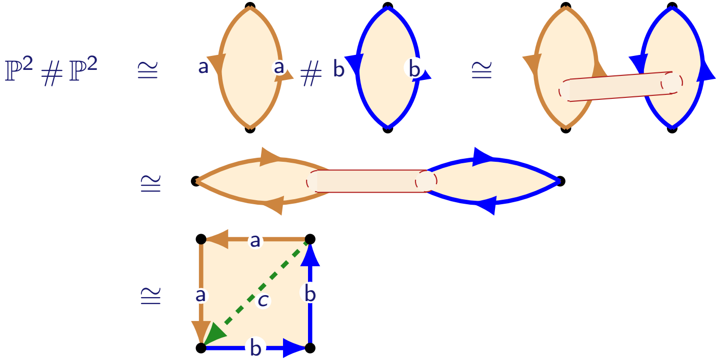

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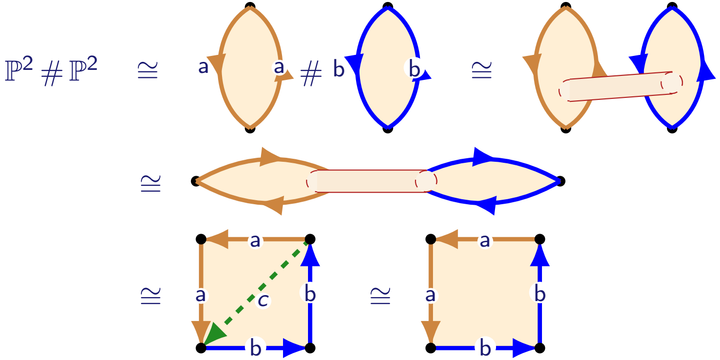

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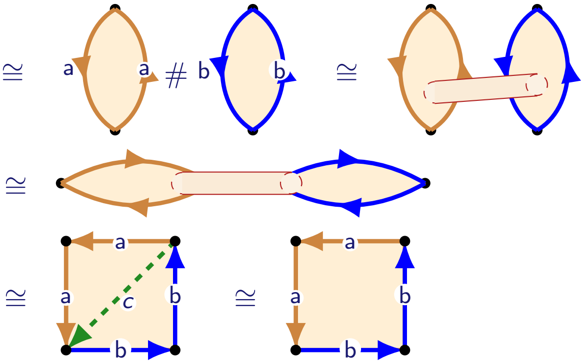
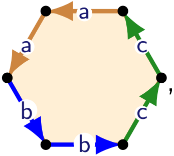


Similarly, $\#^3\mathbb{P}^2 \cong$



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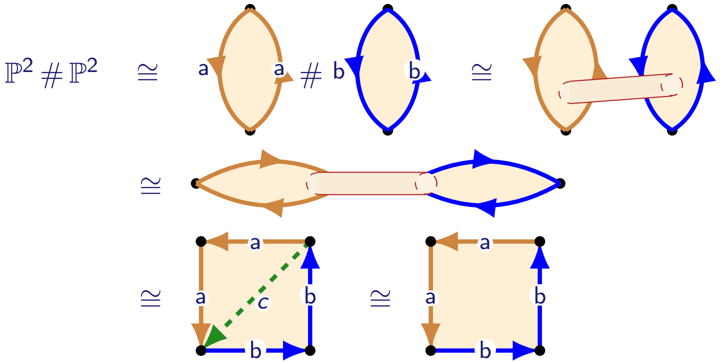
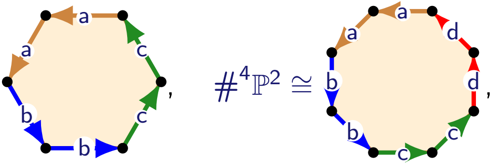
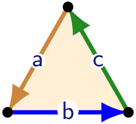
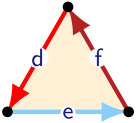


Similarly, $\#^3\mathbb{P}^2 \cong$, $\#^4\mathbb{P}^2 \cong$, ...



Connected sums and polygonal decompositions...

$\mathbb{D}^2 \# \mathbb{D}^2 \quad \cong$ 

$\#$ 

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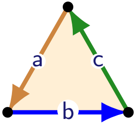

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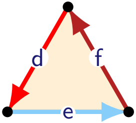

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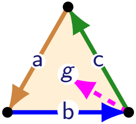

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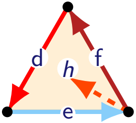

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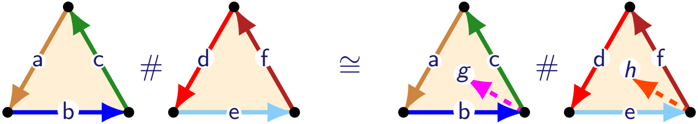

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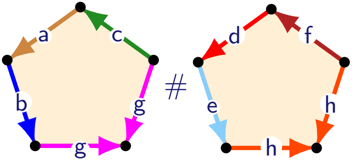

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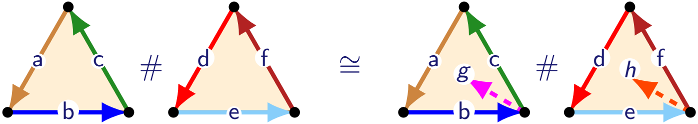

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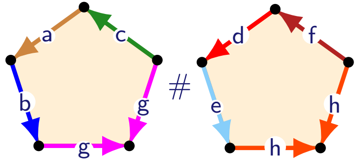

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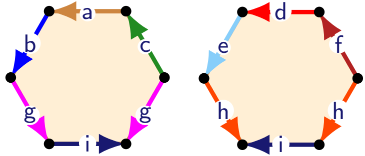

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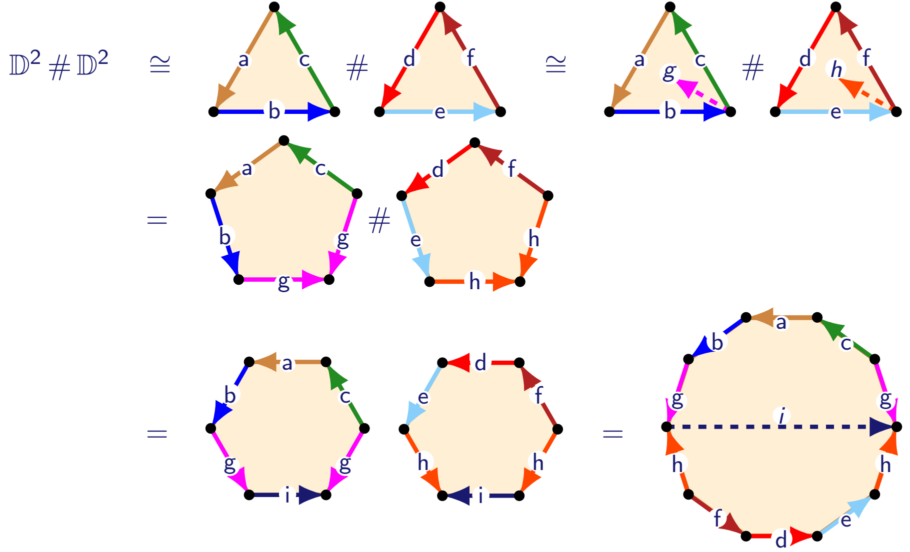

Connected sums and polygonal decompositions...

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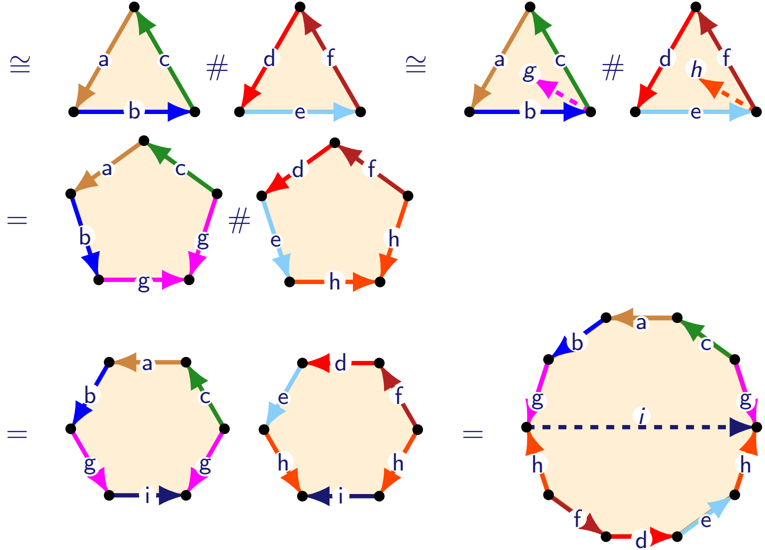

$\Longrightarrow$ For surfaces with a boundary, you can cut into the interior, if necessary, to form the connected sum

Surgery

We have already seen that it is possible to change one polygonal decomposition into another using surgery

There are two basic operations:

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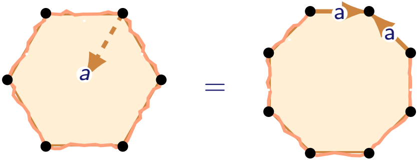
- Adding and removing edges:



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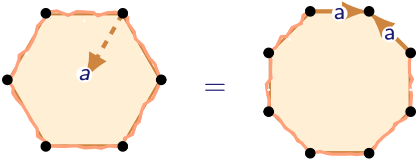
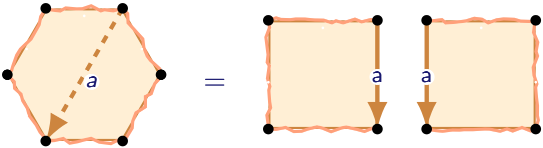


- Cutting and gluing



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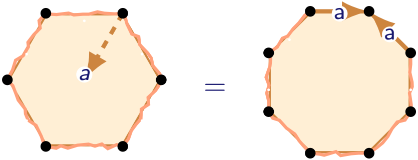

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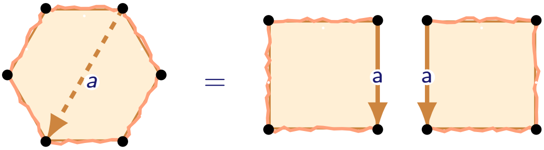

Perhaps surprisingly, these two operations and subdivision are all that we need

Surgery on the Möbius strip

Lemma

$\mathbb{M} \cong \mathbb{D}^2 \# \mathbb{P}^2$ (= a *punctured projective plane*)

Proof

Surgery on the Möbius strip

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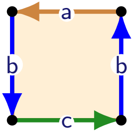
$\mathbb{M} =$ 

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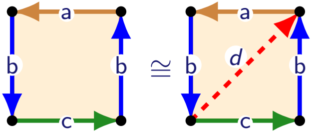

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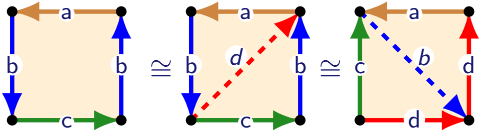
$\mathbb{M} =$ 

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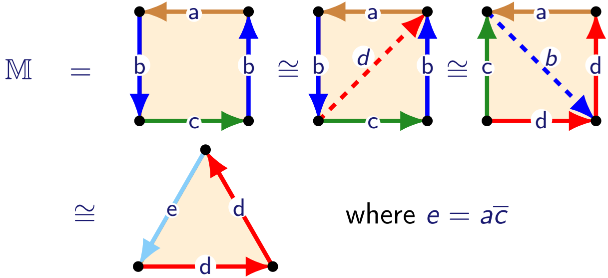

where $e = a\bar{c}$

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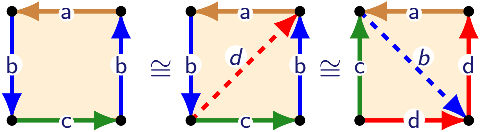
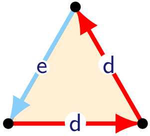


$\cong$



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$\cong$ $\mathbb{D}^2 \# \mathbb{P}^2$

Surgery on the Möbius strip

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Proof

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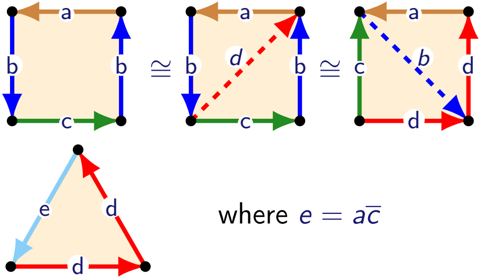

$\cong$ where $e = a\bar{c}$

$\cong \mathbb{D}^2 \# \mathbb{P}^2$

$\Longrightarrow$ A Möbius strip is a punctured projective plane

Surgery on the Möbius strip

Lemma

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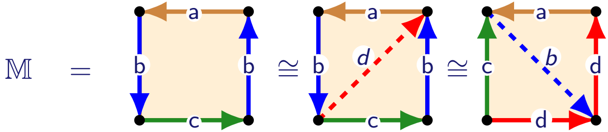
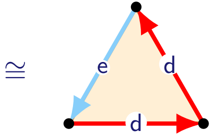
Proof





where $e = a\bar{c}$

$\cong \quad \mathbb{D}^2 \# \mathbb{P}^2$

$\Longrightarrow$ A Möbius strip is a punctured projective plane

$\Longrightarrow$ Every non-orientable surface contains the projective plane

Surgery on the Klein bottle

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$$\mathbb{K} \cong \mathbb{P}^2 \# \mathbb{P}^2 \cong \#^2 \mathbb{P}^2$$

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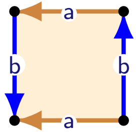
$\mathbb{K} \quad =$ 

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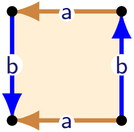
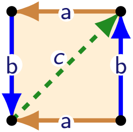
$\mathbb{K} =$  $=$ 

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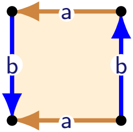
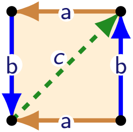
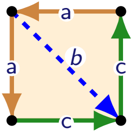
$\mathbb{K} =$  $=$  $=$ 

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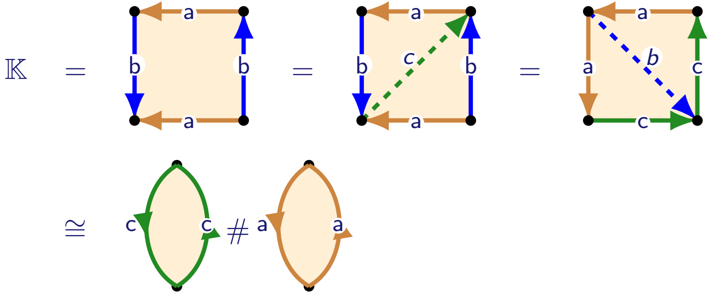

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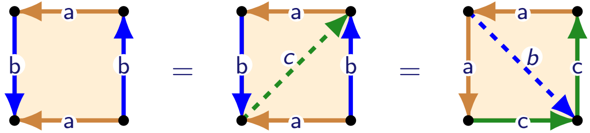
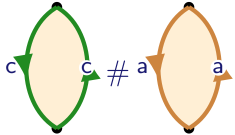
Proof

$\mathbb{K} =$



$\cong$



$\cong \mathbb{P}^2 \# \mathbb{P}^2$

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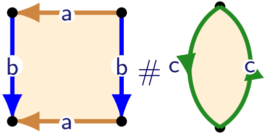
$\mathbb{T} \# \mathbb{P}^2 \quad =$ 

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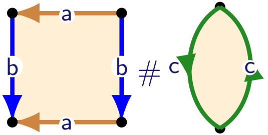
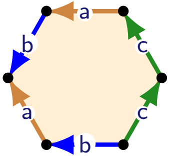
$\mathbb{T} \# \mathbb{P}^2 =$  $\cong$ 

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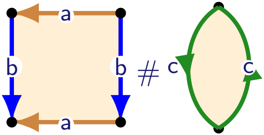

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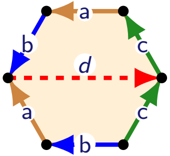

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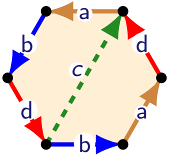

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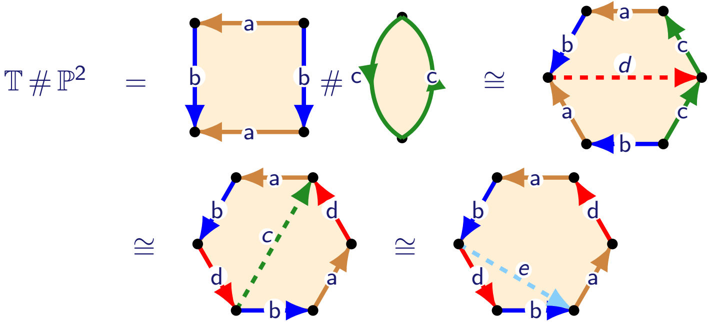

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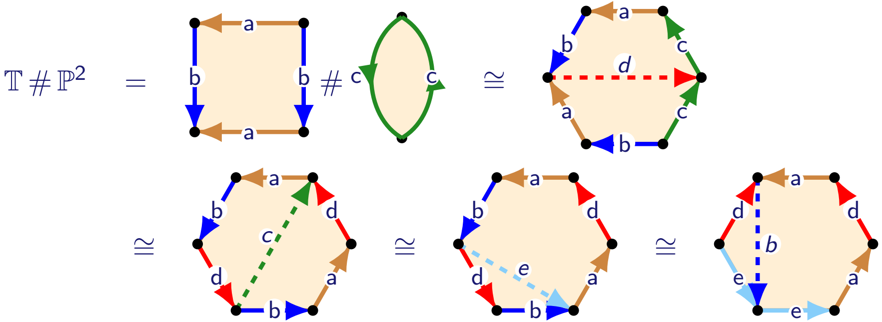

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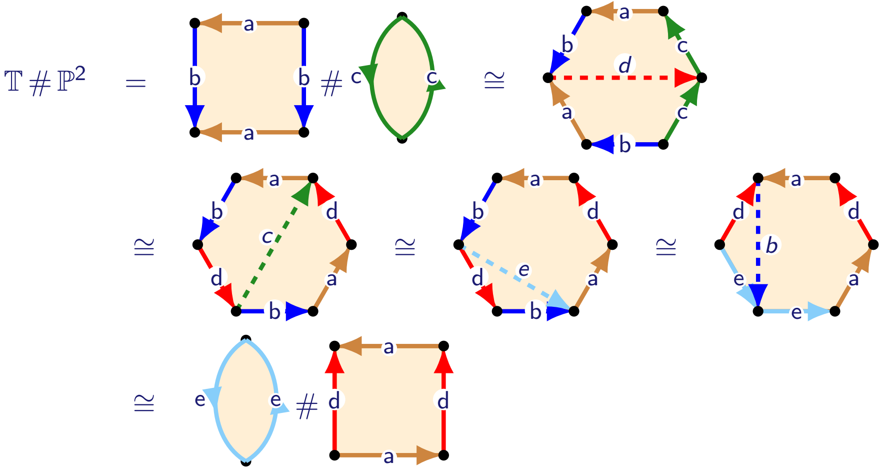

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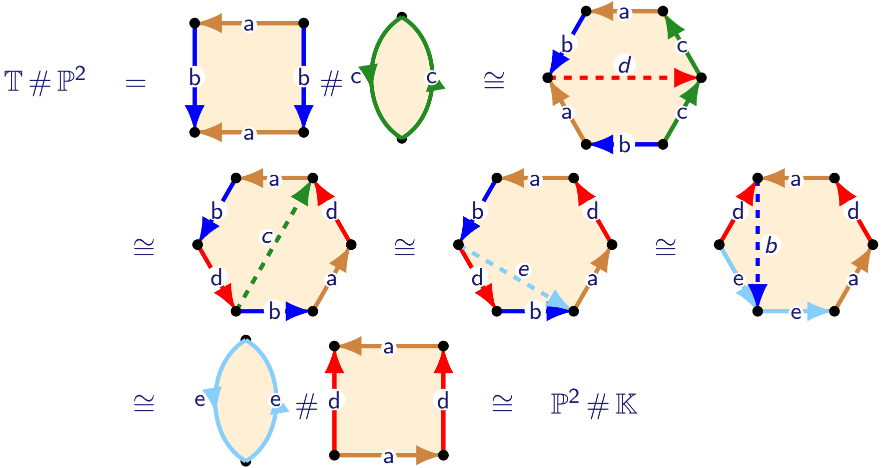

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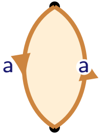
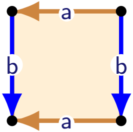
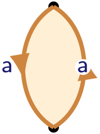
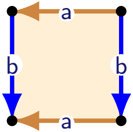
Why? $\mathbb{T}$ embeds in $\mathbb{R}^3$ but $\mathbb{K}$ does not!

Oriented and unoriented edges

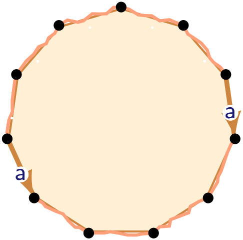
Compare: $\mathbb{P}^2 =$ 

and $\mathbb{T} =$ 

Compare: $\mathbb{P}^2 =$ 

and $\mathbb{T} =$ 

Paired edges on a polygon are oriented if they point in opposite directions and unoriented if they point in the same direction



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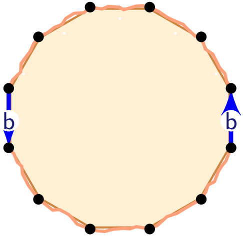

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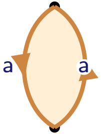
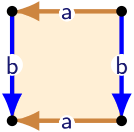
Compare: $\mathbb{P}^2 =$ 

and $\mathbb{T} =$ 

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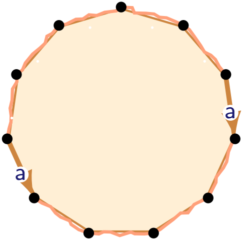

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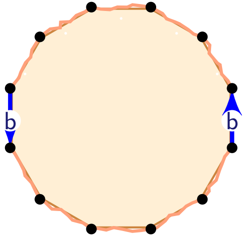

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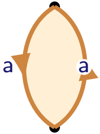
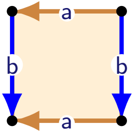
Compare: $\mathbb{P}^2 =$ 

and $\mathbb{T} =$ 

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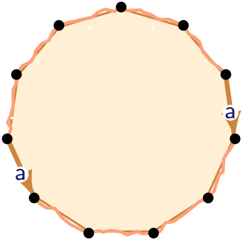

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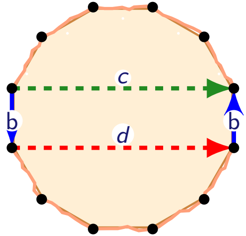

$\mathbb{M}$

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Classification of connected surfaces

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Let S be a connected surface. Then there exist non-negative integers d, p and t such that

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2. *the boundary of S is the disjoint union of d circles*
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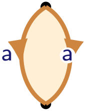
$S =$ 

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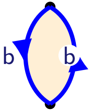
$S =$ 

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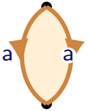
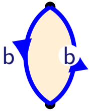
$S =$ 

$\cong S^2$ or $S =$ 

$\cong \mathbb{P}^2$

$\Longrightarrow$ The theorem is true in this case

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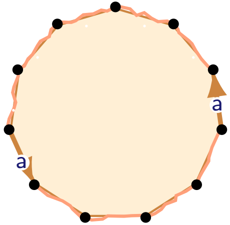

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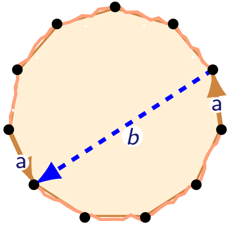

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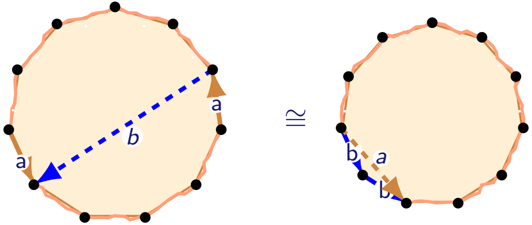

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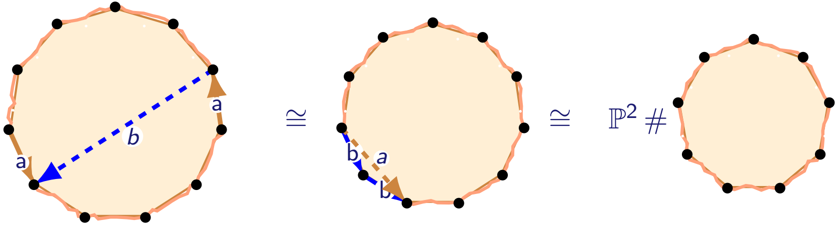

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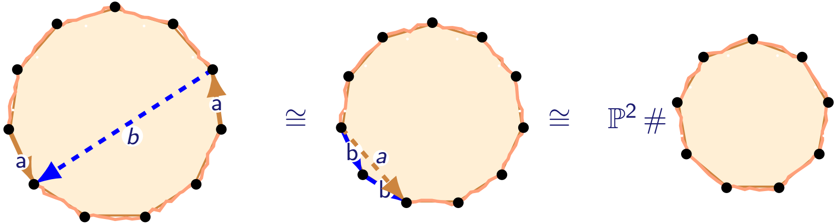

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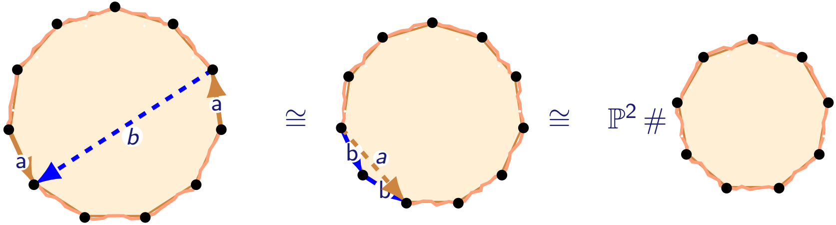

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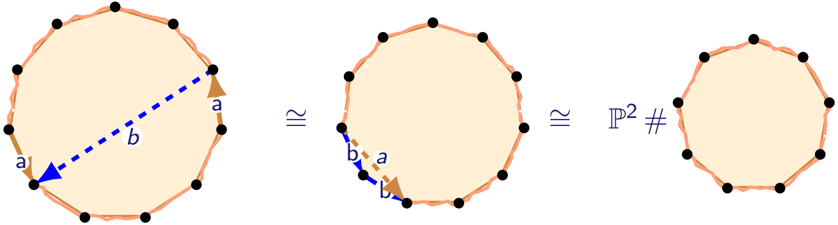

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$\Longrightarrow \quad S \cong S^2 \# \#^d \mathbb{D}^2 \# \#^{p+1} \mathbb{P}^2 \# \#^t \mathbb{T}$ as required

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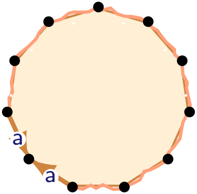
If S has adjacent oriented edges then

$S \cong$ 

Proof of the classification theorem...

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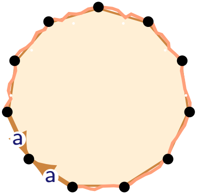
$S \cong$ 

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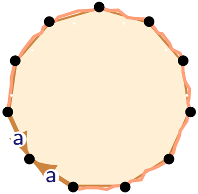
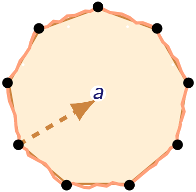
$S \cong$  $\cong$ 

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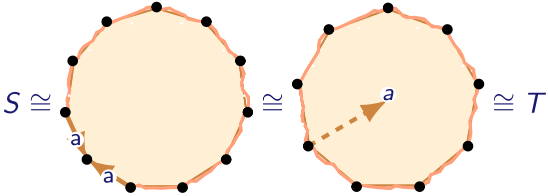

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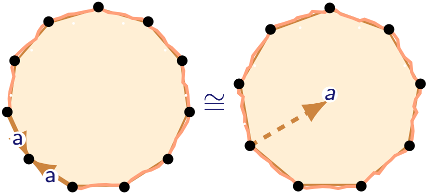

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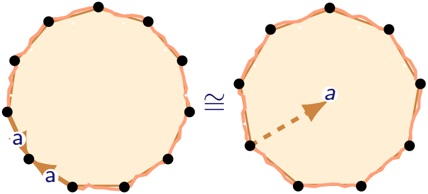


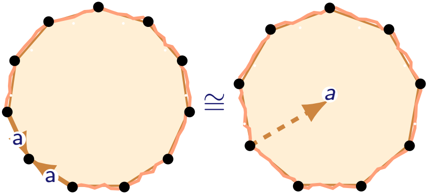
$\cong T$

$$\Longrightarrow \quad S \cong T \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T} \text{ by induction}$$

Hence, we can assume that the paired edges are not adjacent

Case II: All paired edges in S are oriented

If S has adjacent oriented edges then

$S \cong$  $\cong T$

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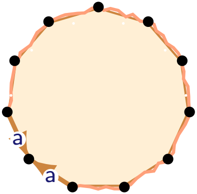
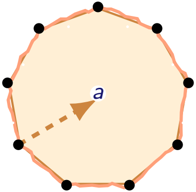
$S \cong$  $\cong$  $\cong T$

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Similarly, we can assume that S does not have any adjacent free edges as such edges can be replaced with a single free edge

Fix an (oriented) paired edge a such that the number of edges between the two copies of a is minimal

Proof of the classification theorem...

Case IIa: All edges on one side of a are free

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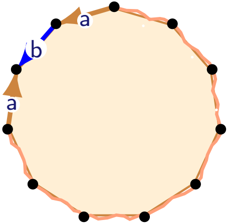
$\Longrightarrow \quad S \quad \cong$ 

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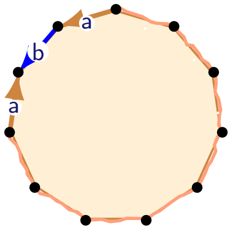
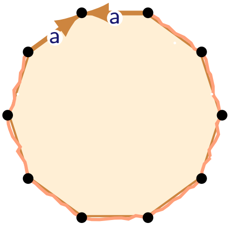
$\Longrightarrow \quad S \quad \cong$ 

$\cong \quad \mathbb{D}^2 \#$ 

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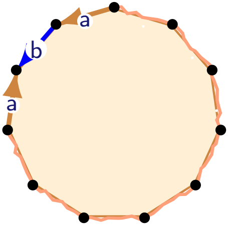

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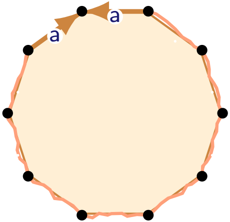

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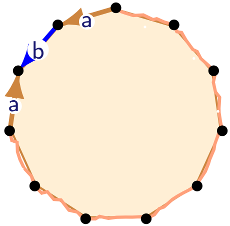

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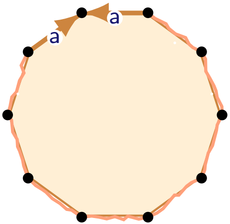

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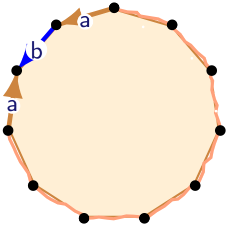

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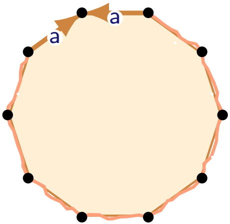

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Proof of the classification theorem...

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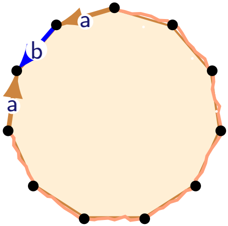

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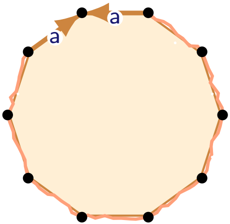

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Hence, we can assume that there are paired edges on both sides of a

Case IIb: There are paired edges on both sides of a

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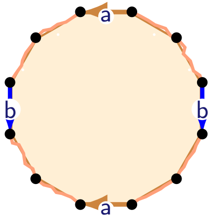
$\implies \quad S \quad \cong$ 

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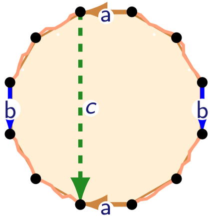

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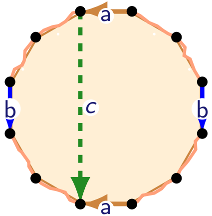
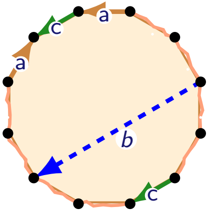
$\Longrightarrow \quad S \quad \cong$  $\cong$ 

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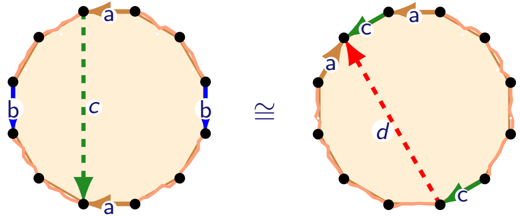
$\Longrightarrow \quad S \cong$ 

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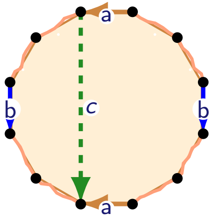

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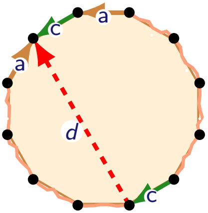

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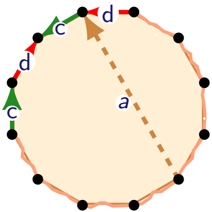

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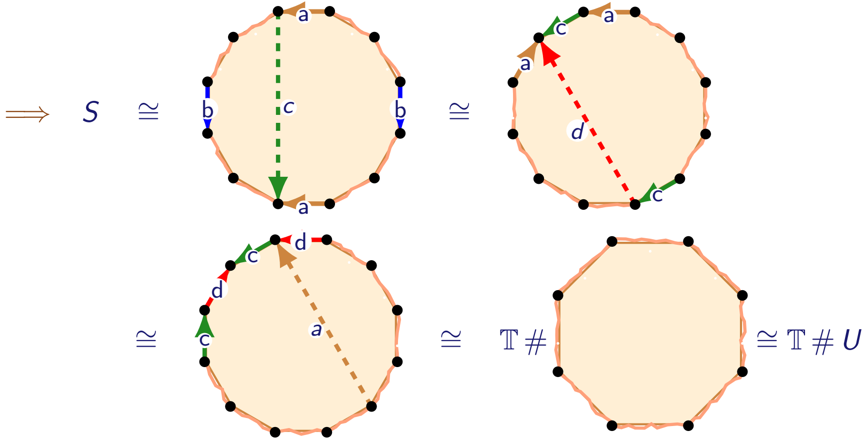

Proof of the classification theorem...

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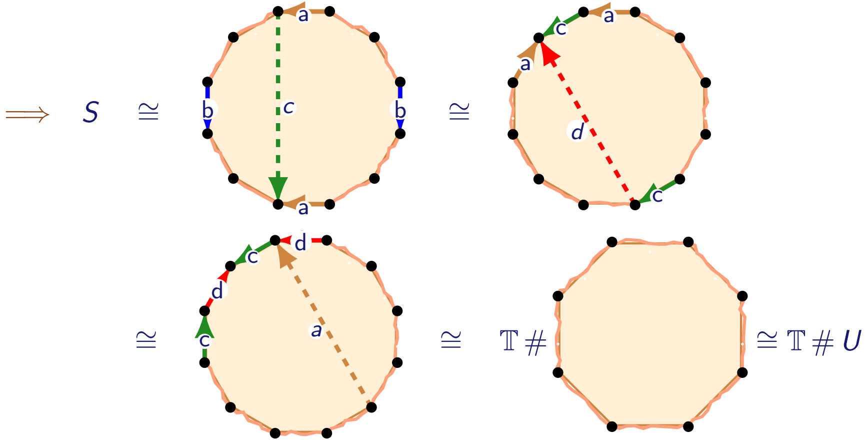

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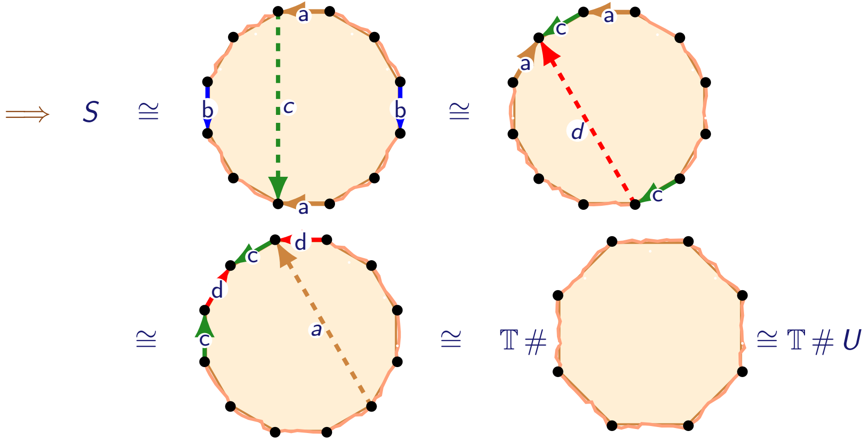

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$\implies \quad S \cong \mathbb{D}^2 \# U \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^{t+1} \mathbb{T}$

Proof of the classification theorem...

We have now proved that every surface can be written in the form

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$\Longrightarrow$ Hence, we can assume $t = 0$ if $p \neq 0$

That is, we can assume $pt = 0$ — equivalently, $p = 0$ or $t = 0$

Proof of the classification theorem...

It remains to prove if $S \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$ with $tp = 0$ then S is uniquely determined up to homeomorphism by (d, p, t)

Let $T = S^2 \# \#^e \mathbb{D}^2 \# \#^q \mathbb{P}^2 \# \#^s \mathbb{T}$, with $sq \neq 0$

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- $p \neq 0 \Leftrightarrow q \neq 0$ as homeomorphisms preserve orientability
- Homeomorphisms preserve Euler characteristic. By tutorial 9,
  - $\chi(S^2 \# \#^a \mathbb{D}^2 \# \#^b \mathbb{P}^2) = 2 - a - b$
  - $\chi(S^2 \# \#^a \mathbb{D}^2 \# \#^c \mathbb{T}) = 2 - a - 2c$

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All parts of the classification theorem are now proved!!

Hence, we now know all surfaces up to homeomorphism!

Orientability

Corollary

A surface S is non-orientable if and only if its polygonal decomposition contains an unoriented edge

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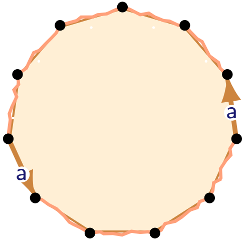

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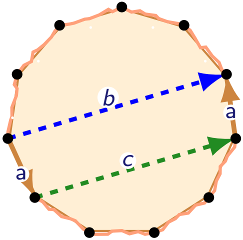

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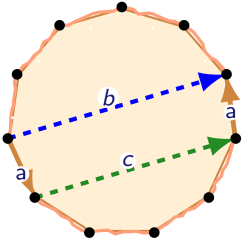

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It is now not hard to find an explicit polygonal decomposition of

$$S = S^2 \# \#^d \mathbb{D}^2 \# \#^t \mathbb{T}$$

and check that surgery cannot create unoriented edges in S

Standard forms

Theorem

Let S be a connected surface. Then there exist non-negative integers d, p and t with $pt = 0$ such that

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The standard form of a surface that is not connected has each component in standard form

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Conversely, these three characteristics of S determine (d, p, t)

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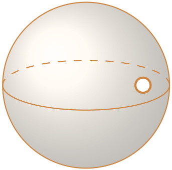
$S^2 \# \mathbb{D}^2 \quad =$ 

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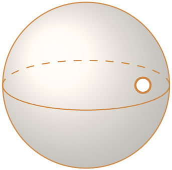

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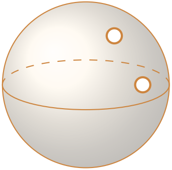

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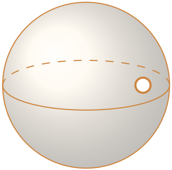
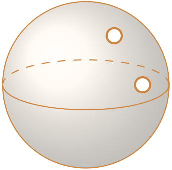
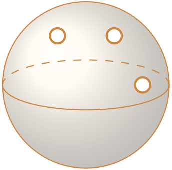
$S^2 \# \mathbb{D}^2 \quad =$ 

$S^2 \# \#^2 \mathbb{D}^2 \quad =$ 

$S^2 \# \#^3 \mathbb{D}^2 \quad =$ 

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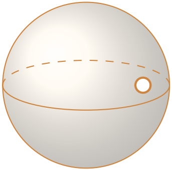

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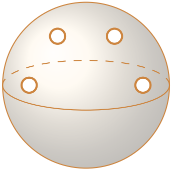

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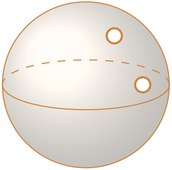

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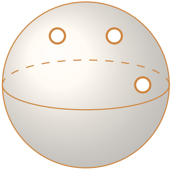

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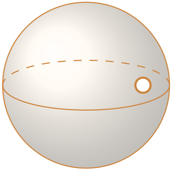

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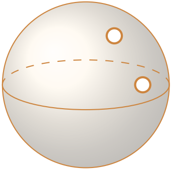

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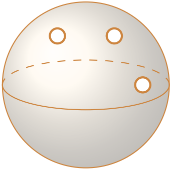

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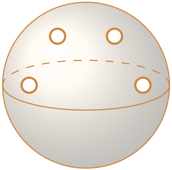

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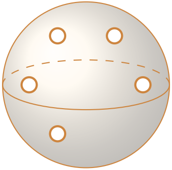

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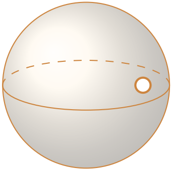

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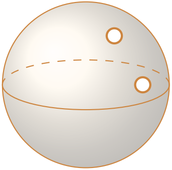

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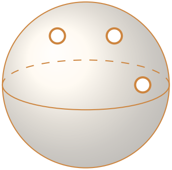

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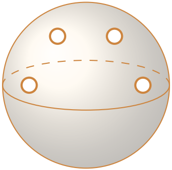

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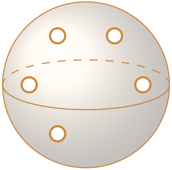

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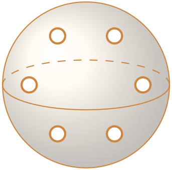

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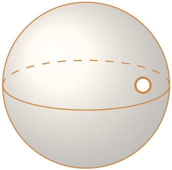

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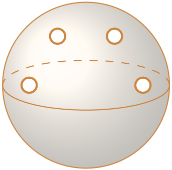

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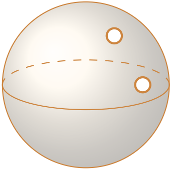

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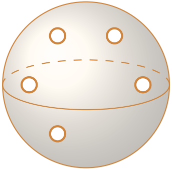

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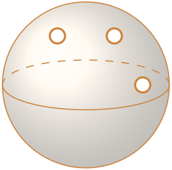

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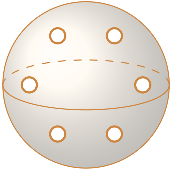

More generally, $S \# \#^d \mathbb{D}^2$ is S with d punctures

A spheres with zero and one puncture

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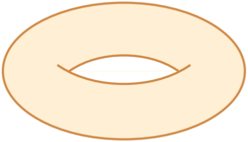
$S^2 \# \mathbb{T} \quad \cong \quad \mathbb{T} \quad \cong$ 

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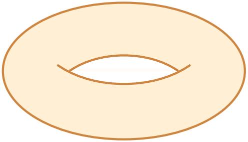
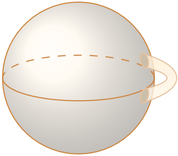
$S^2 \# \mathbb{T} \quad \cong \quad \mathbb{T} \quad \cong$  $\cong$ 

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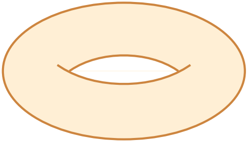
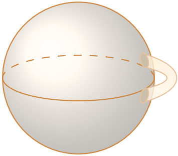
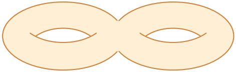
$S^2 \# \mathbb{T} \quad \cong \quad \mathbb{T} \quad \cong$  $\cong$ 

$S^2 \# \#^2 \mathbb{T} \quad \cong \quad \#^2 \mathbb{T} \quad \cong$ 

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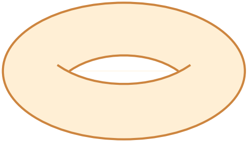
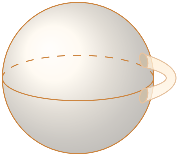
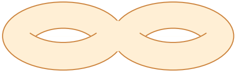
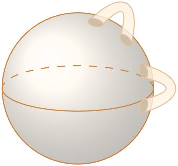
$S^2 \# \mathbb{T} \cong \mathbb{T} \cong$  $\cong$ 

$S^2 \# \#^2 \mathbb{T} \cong \#^2 \mathbb{T} \cong$  $\cong$ 

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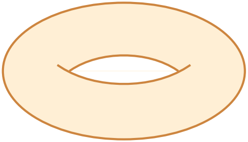
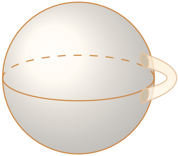
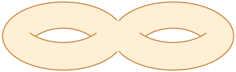
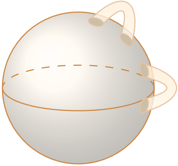
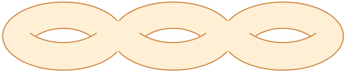
$S^2 \# \mathbb{T} \cong \mathbb{T} \cong$  $\cong$ 

$S^2 \# \#^2 \mathbb{T} \cong \#^2 \mathbb{T} \cong$  $\cong$ 

$S^2 \# \#^3 \mathbb{T} \cong$ 

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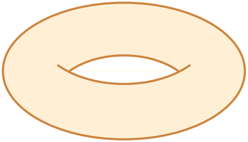
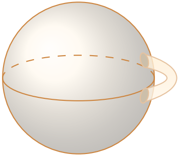
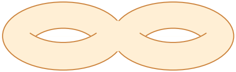
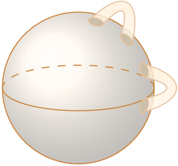
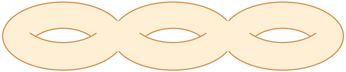
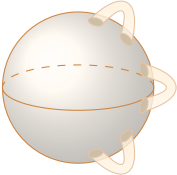
$S^2 \# \mathbb{T} \quad \cong \quad \mathbb{T} \quad \cong$  $\cong$ 

$S^2 \# \#^2 \mathbb{T} \quad \cong \quad \#^2 \mathbb{T} \quad \cong$  $\cong$ 

$S^2 \# \#^3 \mathbb{T} \quad \cong$  $\cong$ 

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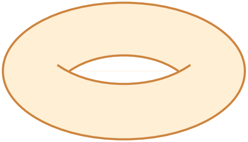
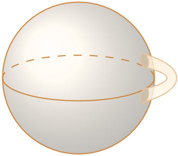
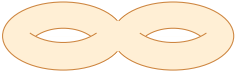
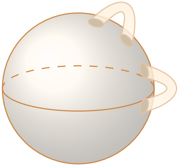
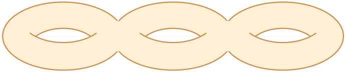
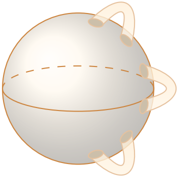
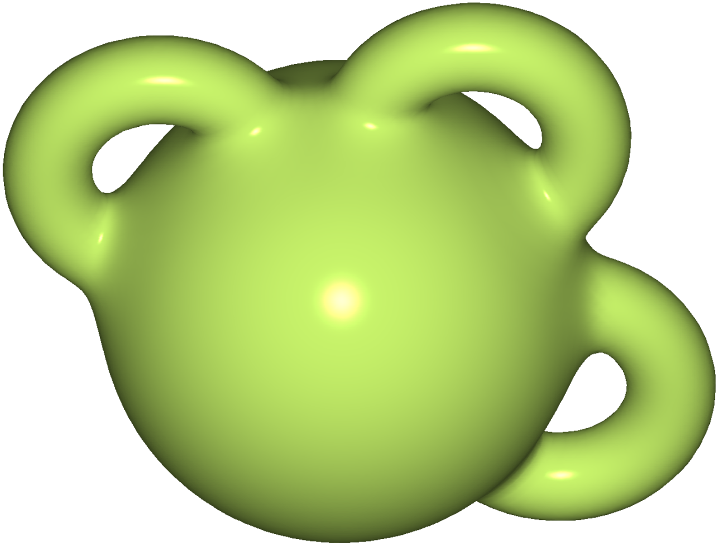
$S^2 \# \mathbb{T} \quad \cong \quad \mathbb{T} \quad \cong$  $\cong$ 

$S^2 \# \#^2 \mathbb{T} \quad \cong \quad \#^2 \mathbb{T} \quad \cong$  $\cong$ 

$S^2 \# \#^3 \mathbb{T} \quad \cong$  $\cong$ 

Continuing like this constructs a sphere with t-handles $\#^t \mathbb{T}$

Handle decomposition



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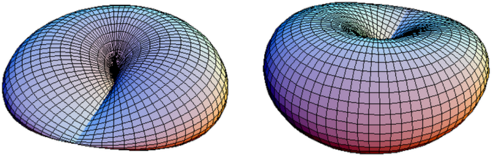

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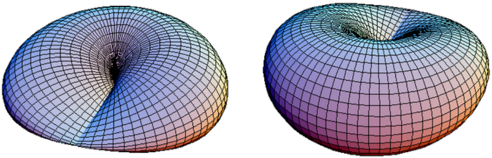

In $\mathbb{R}^3$ this surface self-intersects. We draw surfaces with cross caps as:

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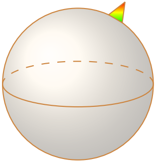

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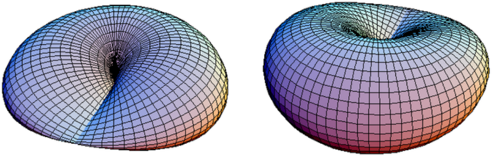

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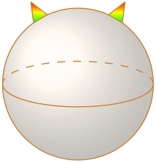
$S^2 \# \#^2 \mathbb{P}^2 \quad \cong$ 

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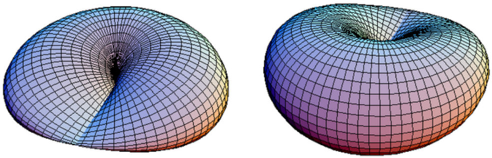

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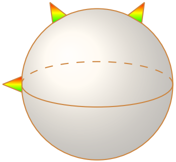

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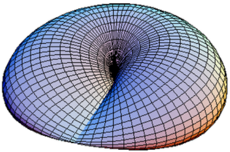

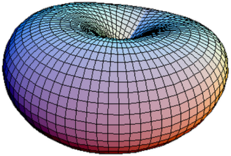

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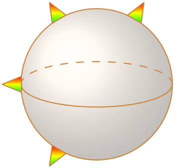
$$S^2 \# \#^4 \mathbb{P}^2 \quad \cong$$



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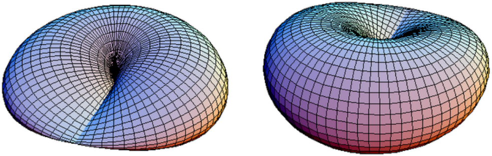

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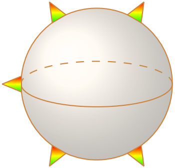
$$S^2 \# \#^5 \mathbb{P}^2 \quad \cong$$



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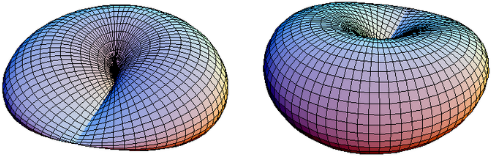

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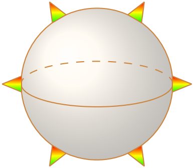
$$S^2 \# \#^6 \mathbb{P}^2 \quad \cong$$



What do standard surfaces look like?

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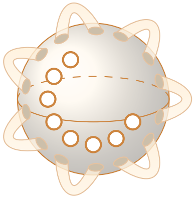
$$\#^8\mathbb{D}^2 \# \#^7\mathbb{T} \quad \cong$$



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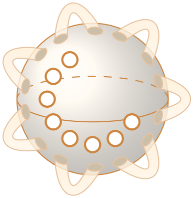
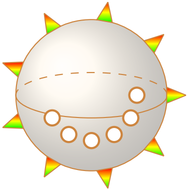


$$\#^6\mathbb{D}^2 \# \#^9\mathbb{P}^2 \quad \cong$$



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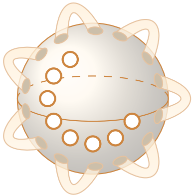

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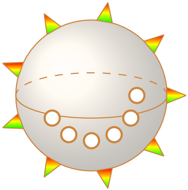
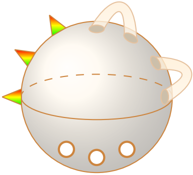


$$\#^3\mathbb{D}^2 \# \#^2\mathbb{T} \# \#^3\mathbb{P}^2 \quad \cong$$



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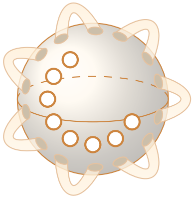

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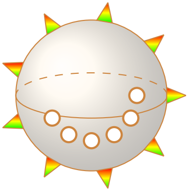

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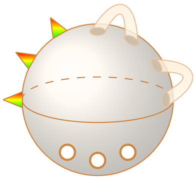
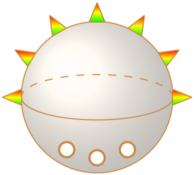


$\cong$



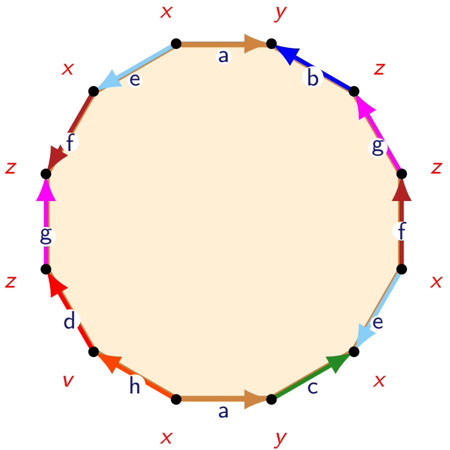
Putting a surface in standard form

Given a polygonal decomposition for a surface we can put it in standard form by:

- Find all of the vertices (identified edges implicitly identify vertices)
- Count the number d of boundary circles
- S is orientable $(p = 0)$ if all edges are oriented otherwise it is non-orientable $(t = 0)$
- Compute $\chi(S) = 2 - d - p - 2t$ to determine the missing variable, which is t if S is orientable and or p if non-orientable

Example 1

What is the surface with the below polygonal decomposition?

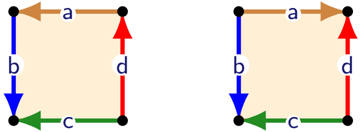


$a\,c\,\overline{e}\,f\,g\,b\,\overline{a}\,e\,f\,\overline{g}\,\overline{dh}$ (overline=opposite direction)

$\Longrightarrow$ This is $\#^1\mathbb{D}^2 \# \#^0\mathbb{T} \# \#^4\mathbb{P}^2$

Example 2

What is the standard form of the surface with polygonal decomposition?



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