Topology – week 8

Math3061

Daniel Tubbenhauer, University of Sydney

Eulerian circuits and graphs

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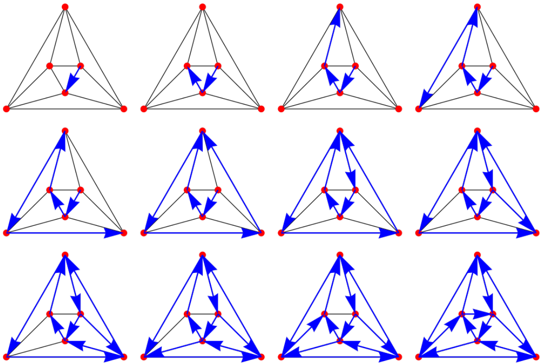
Example



Warning Eulerian graphs do not need to be connected because they may have vertices of degree 0!

Finding Eulerian circuits

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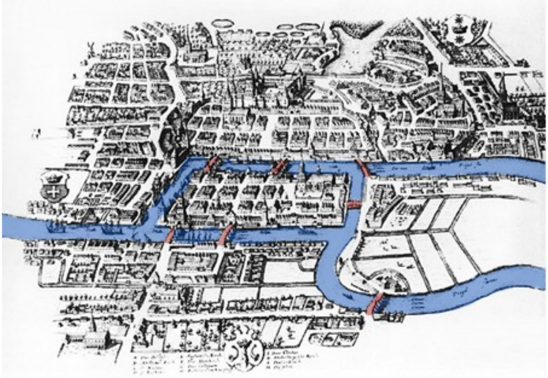

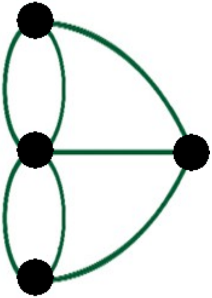

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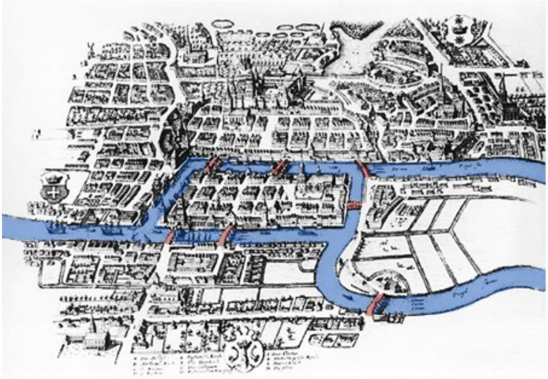

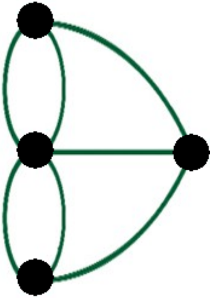

In answering this question Euler laid the foundations of graph theory

Classifying Eulerian graphs

Theorem

Let $G = (V, E)$ be a connected graph. Then G is Eulerian if and only if every vertex has even degree

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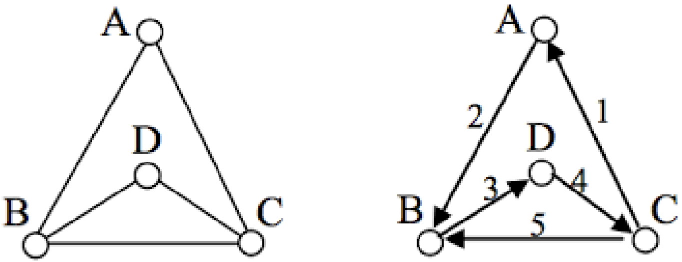
Proof

Assume that there is at least one vertex v of odd degree. Since we want to visit every edge exactly once we will eventually get stuck in v or another vertex of odd degree while trying to create an Eulerian cycle. Hence, G can not have an Eulerian cycle



Classifying Eulerian graphs

Proof continued

Conversely, if every vertex has even degree, then G is not a tree so contains some circuit C. If C is an Euler circuit we are done, and if not remove all edges of C from G. The resulting (potentially disconnected) graph G' has still even degrees for all of its vertices but fewer edges than G

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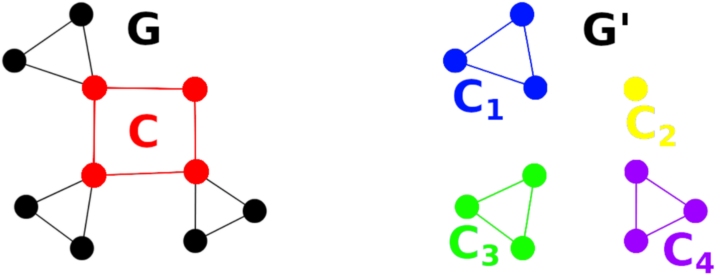
So we can argue by induction on the number of edges (the base case has no edges and is thus clear), and inductively we can assume that the connected components of G' have Euler circuits $C_1, \dots, C_n$

Proof continued

We piece C and $C_1, \ldots, C_n$ together into an Euler cycle: we walk along C and whenever we hit a vertex of C_i we take a detour over C_i



Eulerian paths

A Eulerian path is a path that is not a circuit and which passes through every edge exactly once

Corollary

Let $G = (V, E)$ be a connected graph that is not Eulerian. Then G has a Eulerian path if and only if it has exactly two vertices of odd degree

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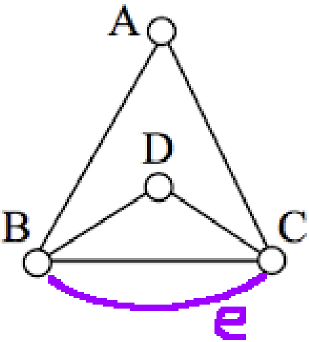
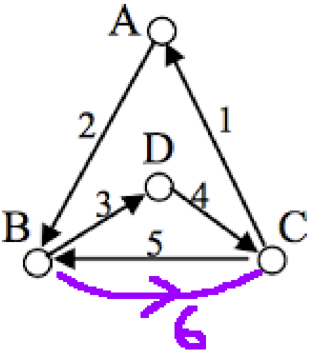
Proof

Only vertices of odd degree can be a start or an end vertex, so we need precisely two of them (all other must be of even degree by the same argument as before)

Proof continued

Conversely, if v and w are the two vertices of even degree, then we put an additional edge e between them. We get a graph $G' = G \cup \{e\}$ and the previous theorem gives us an Euler circuit C in G'. Then $C \setminus \{e\}$ is an Euler path





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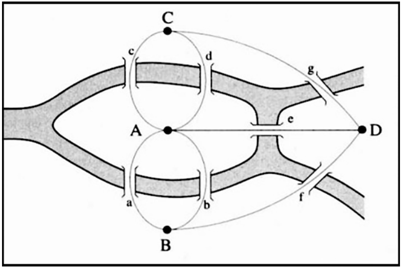

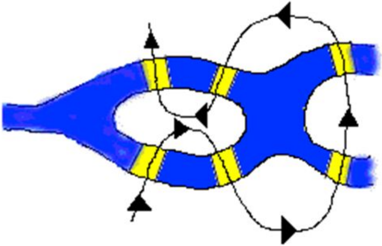

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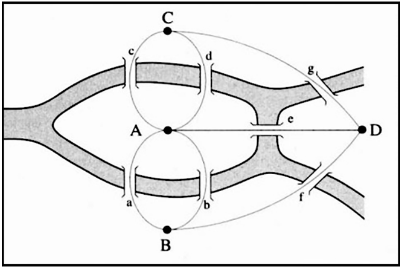

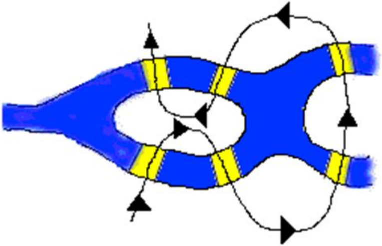

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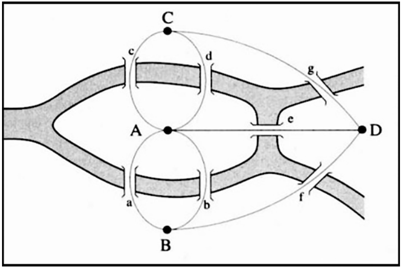

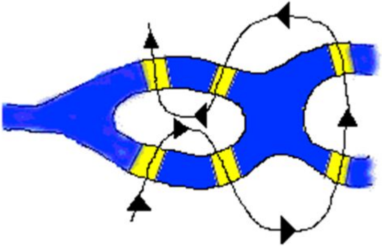

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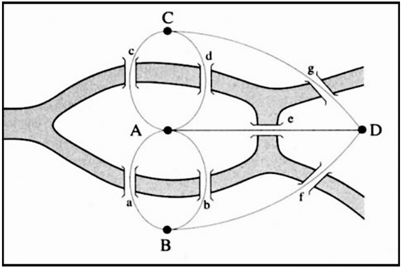

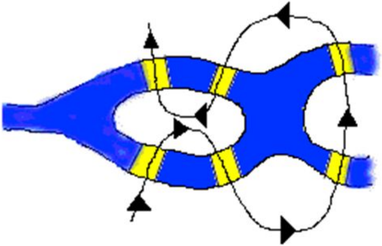

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Solution: Destroy bridge e ;-)

Topological equivalence

Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$, for $m, n \geq 1$

Definition

A homeomorphism $f : X \longrightarrow Y$ is a continuous map that has a continuous inverse $g : Y \longrightarrow X$. The spaces X and Y are homeomorphic if there is a homeomorphism $f : X \longrightarrow Y$

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Examples of homeomorphisms

Proposition

If $a < b$ *and* $c < d$*, then* $[a, b] \cong [c, d]$

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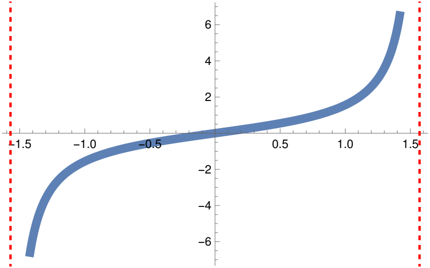
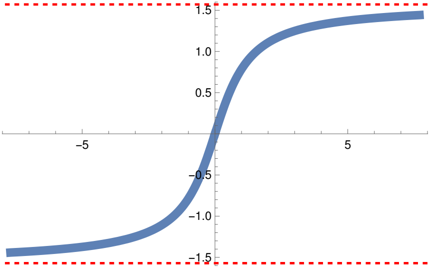
Proof It is enough to show that $(-\frac{\pi}{2}, \frac{\pi}{2}) \cong \mathbb{R}$

Examples of homeomorphisms

Proof continued

Homeomorphisms are given by $f(x) = \tan(x)$ and $g(x) = \tan^{-1}(x)$





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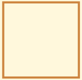
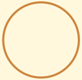
 $\cong$  $= S^1$

Examples of homeomorphisms...

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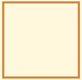
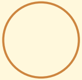
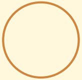
 $\cong$  $= S^1$ *We show that*  $\cong$ 

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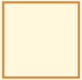
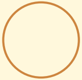
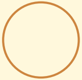
 $\cong$  $= S^1$ *We show that*  $\cong$ 

Proof

The square is $\{ (x, y) \mid |x| + |y| = 1 \}$ and $S^1 = \{ (x, y) \mid x^2 + y^2 = 1 \}$

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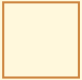
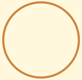
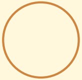
 $\cong$  $= S^1$ *We show that*  $\cong$ 

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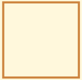
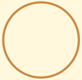
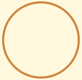
 $\cong$  $= S^1$ *We show that*  $\cong$ 

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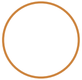
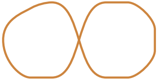
Note that  $\ncong$ 

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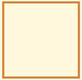
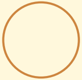
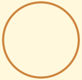
 $\cong$  $= S^1$ *We show that*  $\cong$ 

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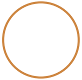
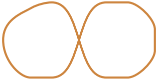
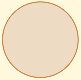
Note that  $\ncong$ 

For free we see that the square and disk are homeomorphic:

Corollary

 $\cong$  $\cong$ 

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Think of the north pole of the circle S^1 as ∞

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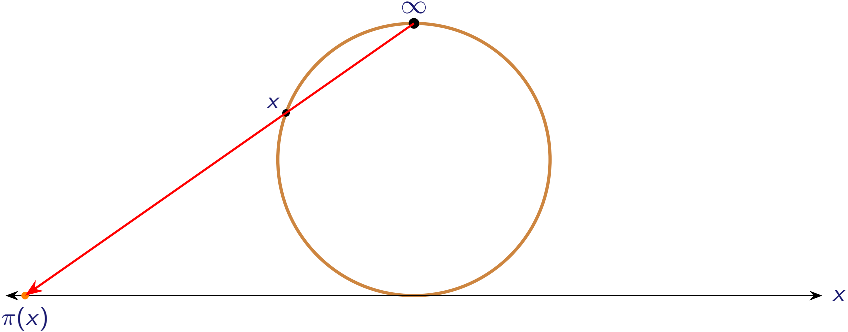


Stereographic projection in three dimensions

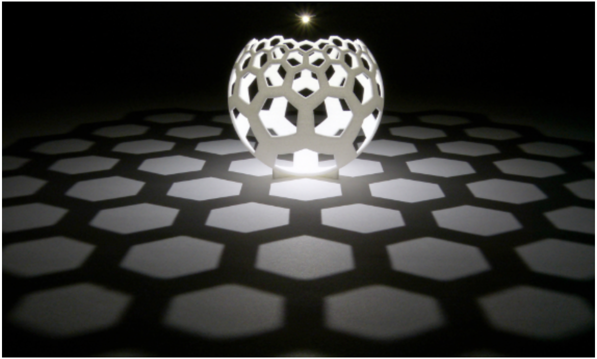
Think of the north pole of the circle S^2 as ∞

Stereographic projection gives a homeomorphism $\pi\colon S^2 \setminus \{\infty\} \to \mathbb{R}^2$:

Stereographic projection in three dimensions

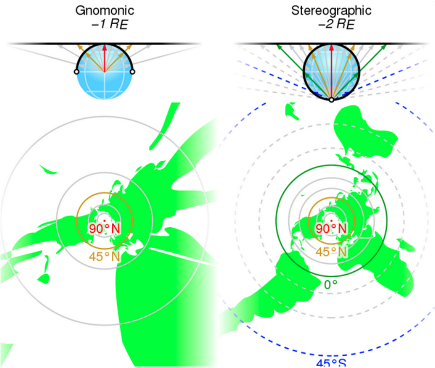
Think of the north pole of the circle S^2 as ∞

Stereographic projection gives a homeomorphism $\pi\colon S^2 \setminus \{\infty\} \to \mathbb{R}^2$:



Maps

Stereographic projection is used to draw maps:



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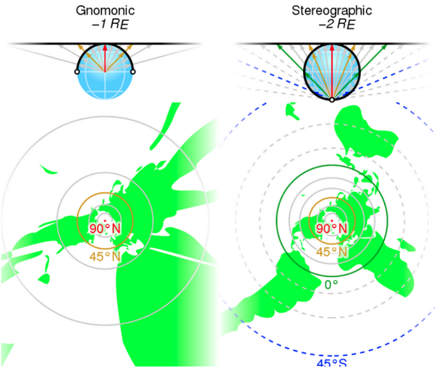

Other projections are also used such as gnomonic projections, conic projections and the Mercator projection, which is a cylindrical projection

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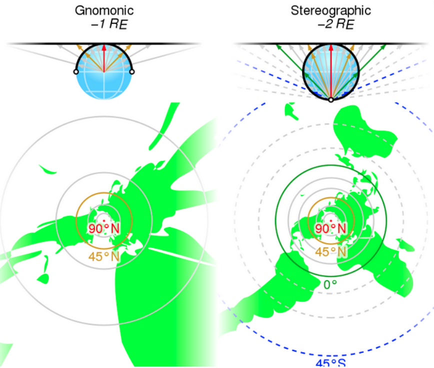

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Now that we have seen homeomorphisms we are ready to define surfaces

Surfaces — informal definition

Definition

A surface is a subset of $\mathbb{R}^n$ that, locally, is homeomorphic to the graph of the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by $f(x, y) = z$ / alternatively to a disc

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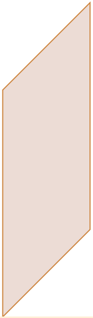
Examples

- A standard xyz-plane in $\mathbb{R}^3$



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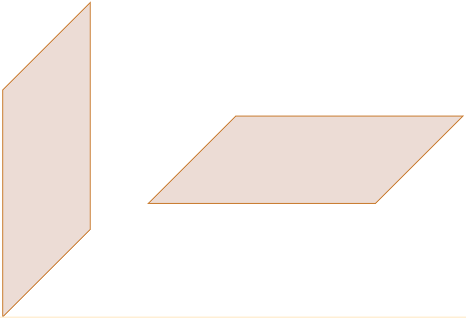

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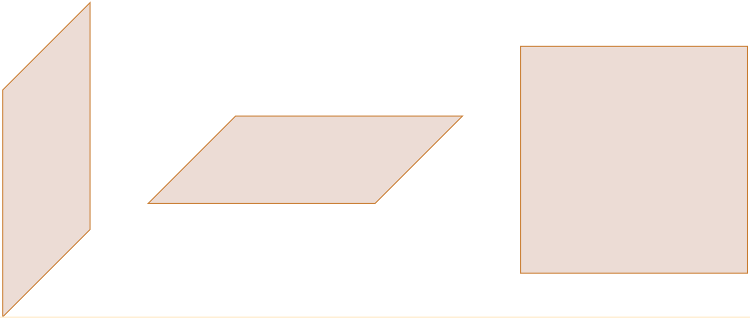
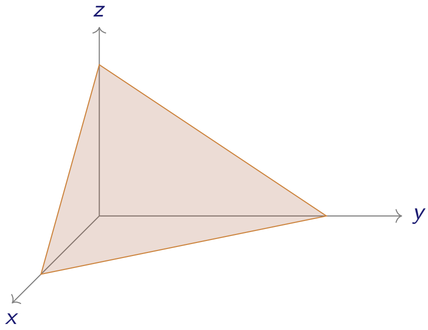


- Non-standard planes in $\mathbb{R}^3$



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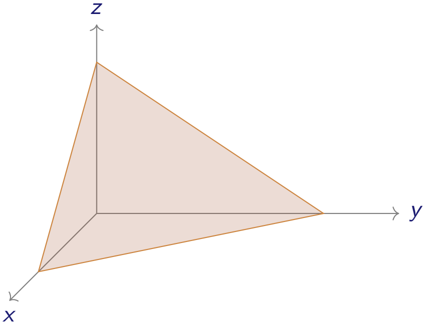
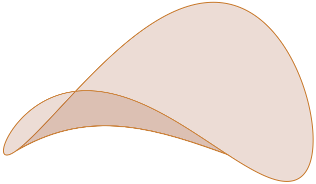
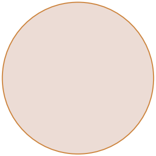


- Curved surfaces in $\mathbb{R}^3$



Surfaces — examples...

- A disk $\mathbb{D}^2$



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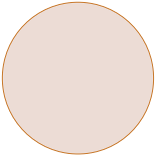
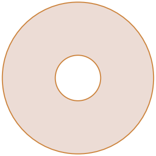


- An annulus $\mathbb{A}$



$\cong$

Surfaces — examples...

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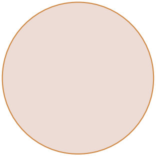
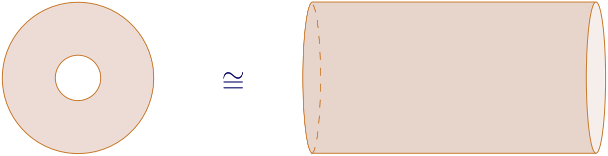


- An annulus $\mathbb{A} \cong$ cylinder

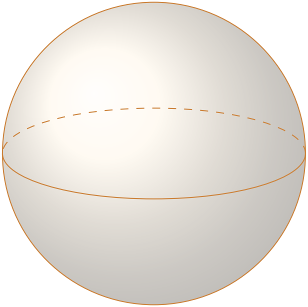


Strictly speaking, these are not surfaces according to our definition because they have a boundary, whereas planes in $\mathbb{R}^2$ do not have boundaries.

Our rigorous definition of a surface will allow surfaces with boundaries

Surfaces — examples...

- A sphere S^2



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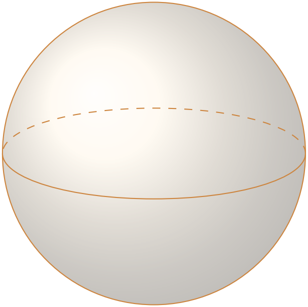
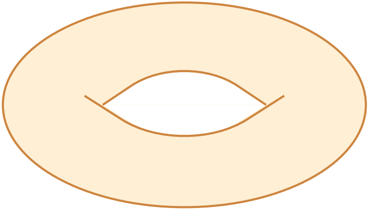


- A torus $\mathbb{T}$



Surfaces — real world examples...

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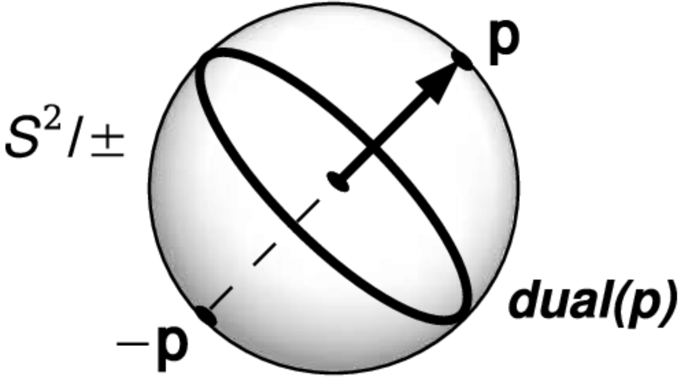
Surfaces — real world example...

- Here is a surface with boundary:

The patches are examples of neighborhoods which are discs

Surfaces — examples...

- The real projective plane $\mathbb{P}^2 = S^2/\text{antipode}$



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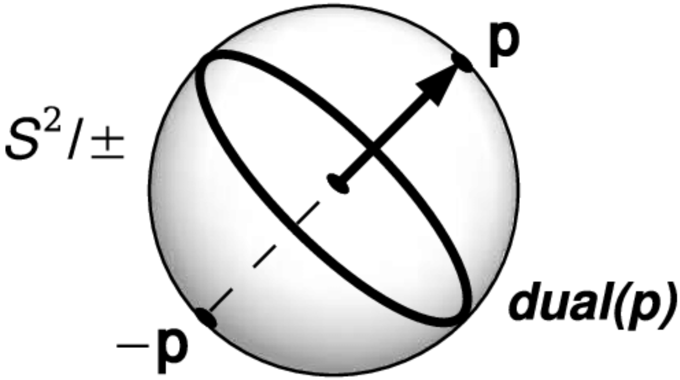
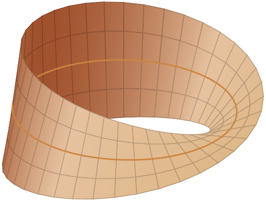


We will see other ways to describe $\mathbb{P}^2$ later

Surfaces — examples...

- A Möbius band, or Möbius strip, $\mathbb{M}$



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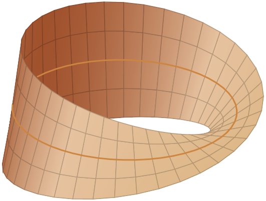

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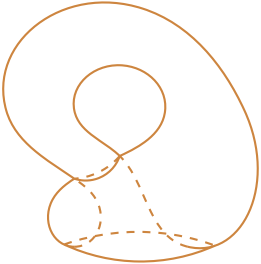

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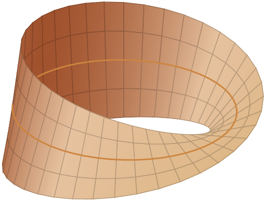

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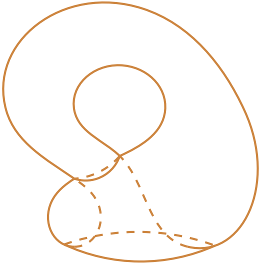
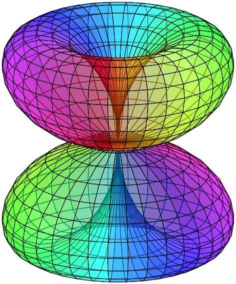


This is a three dimensional "shadow" of a four dimensional object

Surfaces — non-examples

- This is not a surface because of the cusp at the origin



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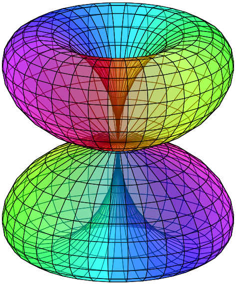
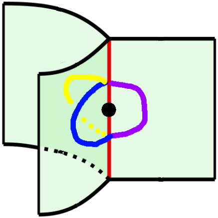


- This is not a surface because the indicated point has not a disc neighborhood



Identification spaces

A partition of a surface $S \subseteq \mathbb{R}^m$ is a collection $X_1, \ldots, X_r$ of subsets of S such that $S = X_1 \cup X_2 \cup \cdots \cup X_r$

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This makes is possible to understand Y in terms of, often, easier spaces $X_1, \ldots, X_r$, which we think of as covering Y like a patchwork quilt

Identification space for a cylinder

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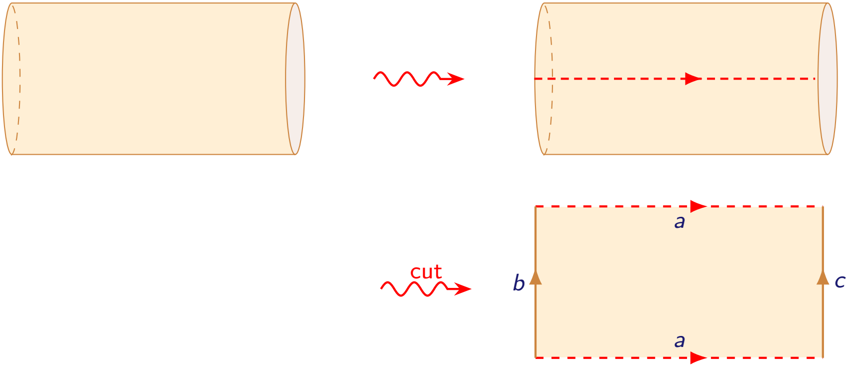

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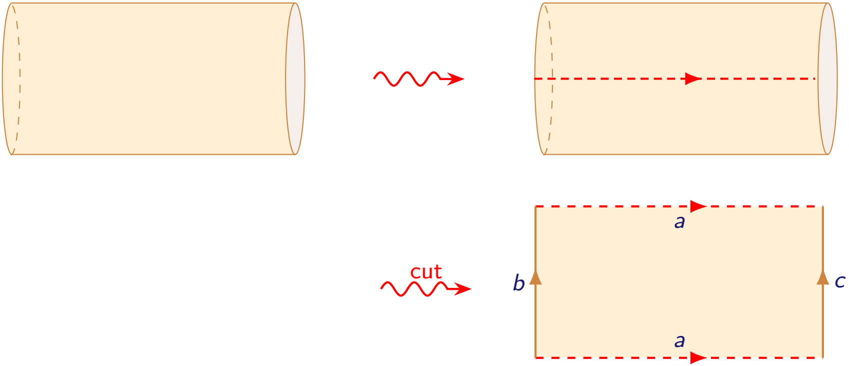

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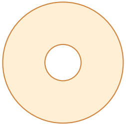
$\mathbb{A} =$ 

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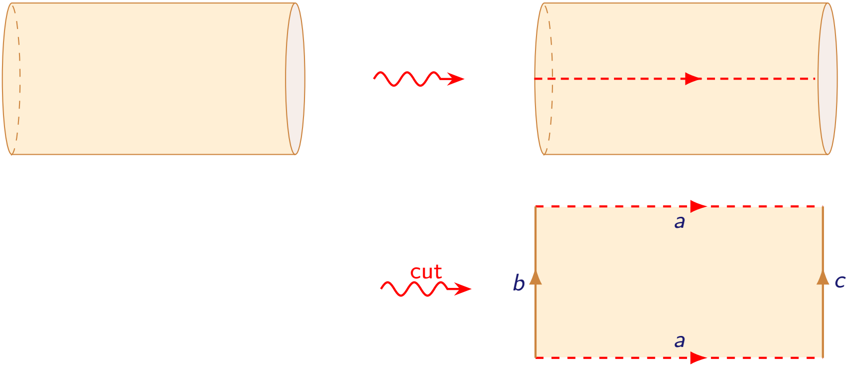

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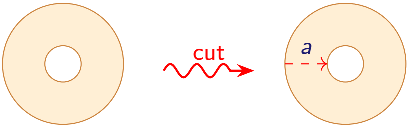
$\mathbb{A} =$ 

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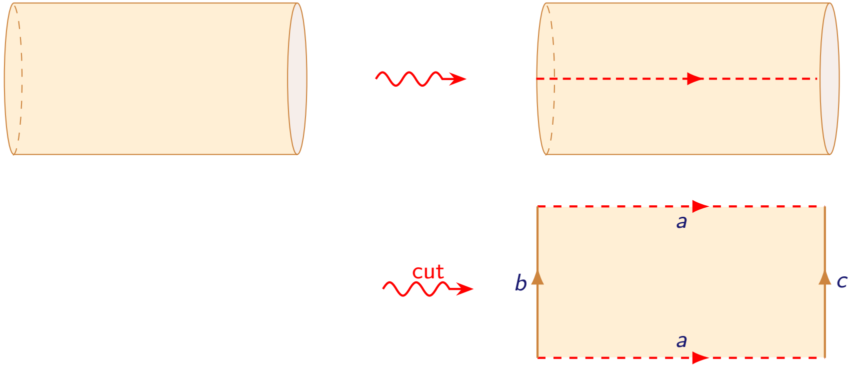

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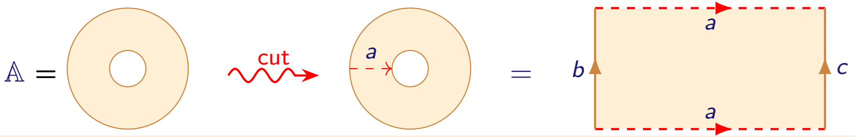
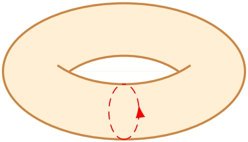


Identification space for a torus



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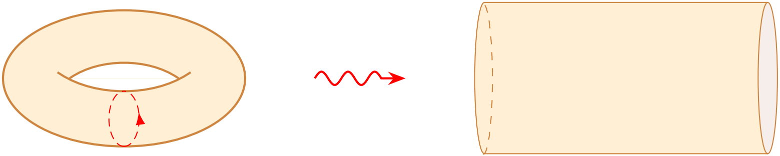

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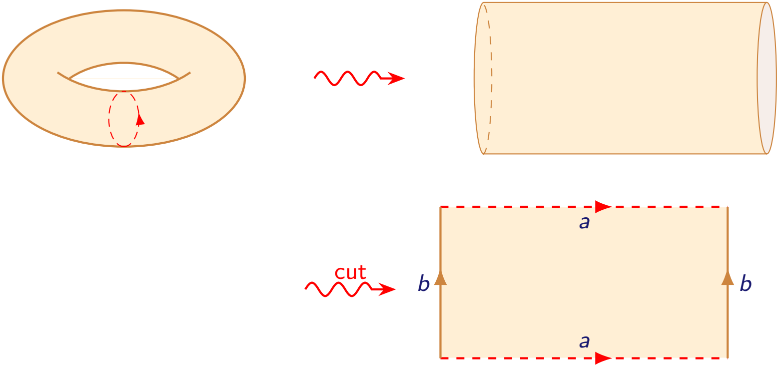
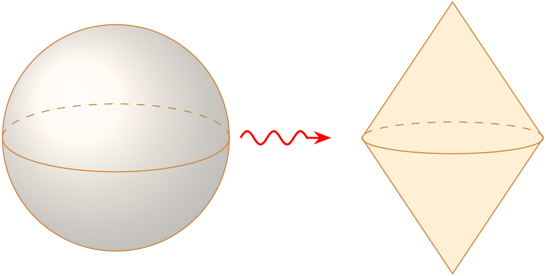


So, the torus $\mathbb{T}$ is obtained by identifying the top and bottom, and the left and right, edges of a rectangle

Identification space for a sphere



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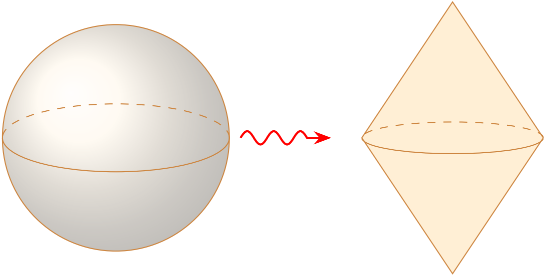

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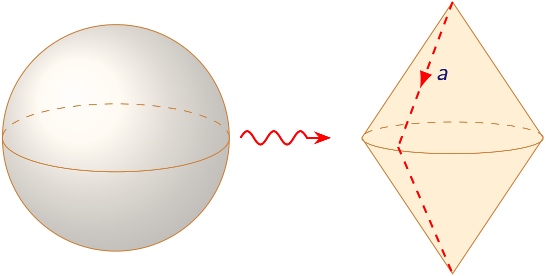

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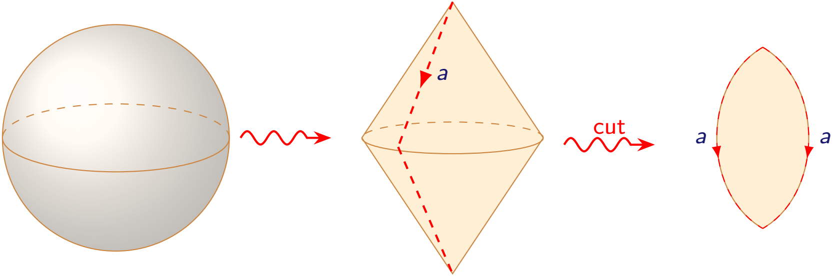

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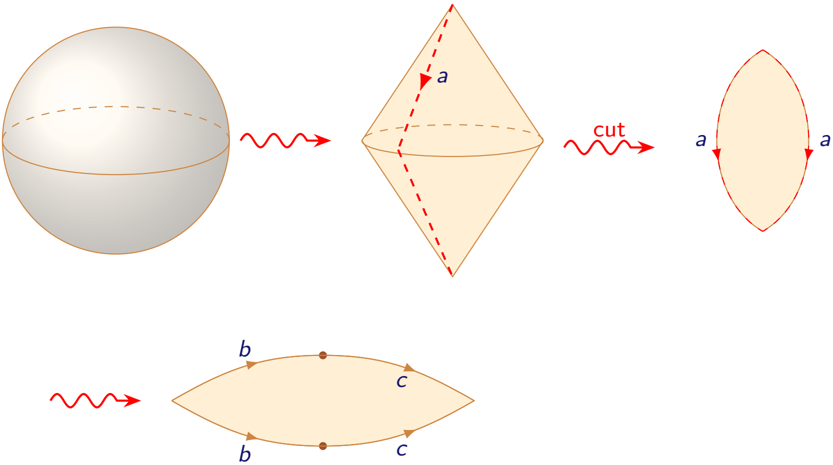

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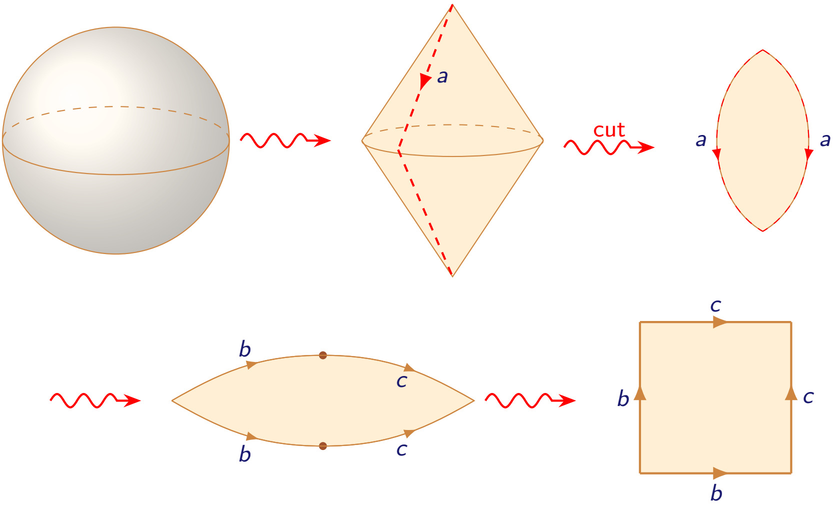

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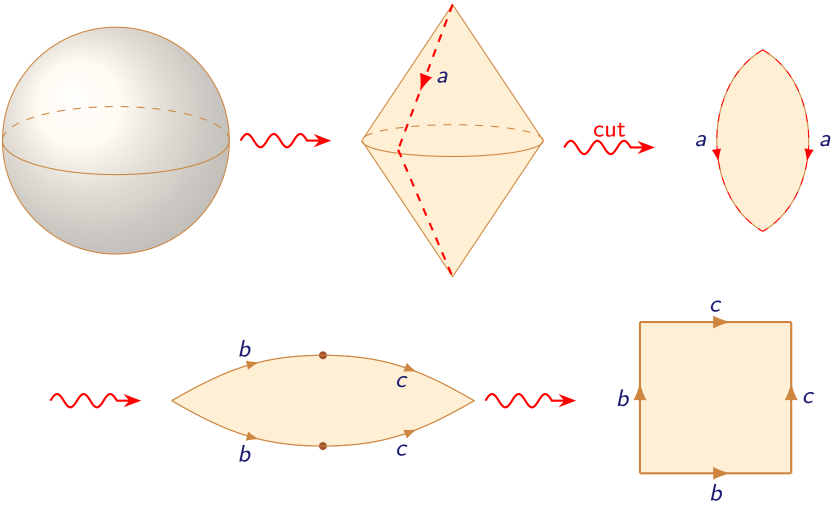
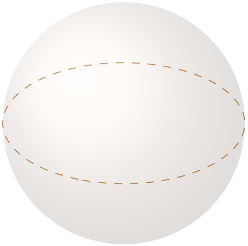


The sphere S^2 is obtained by identifying adjacent sides of a rectangle, or a 2-gon (a polygon with two sides)

Identification space for the projective plane $\mathbb{P}^2$



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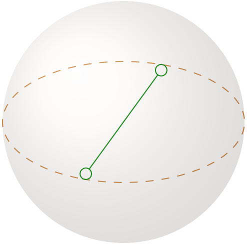

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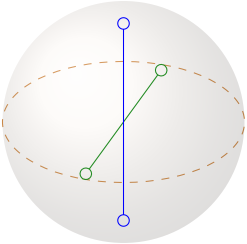

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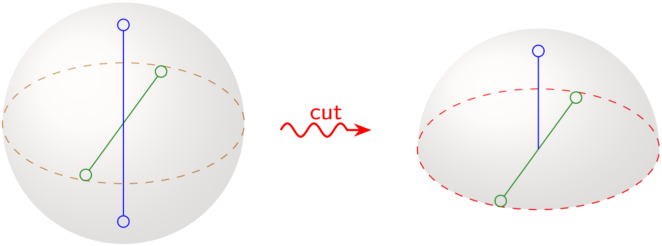

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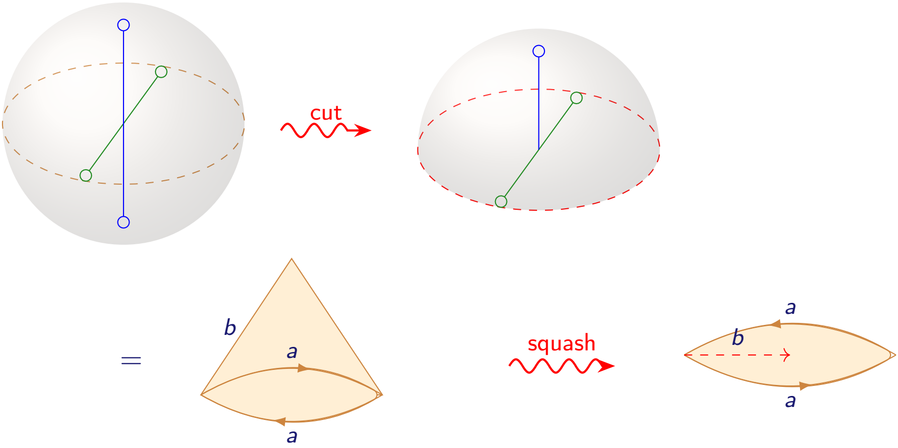

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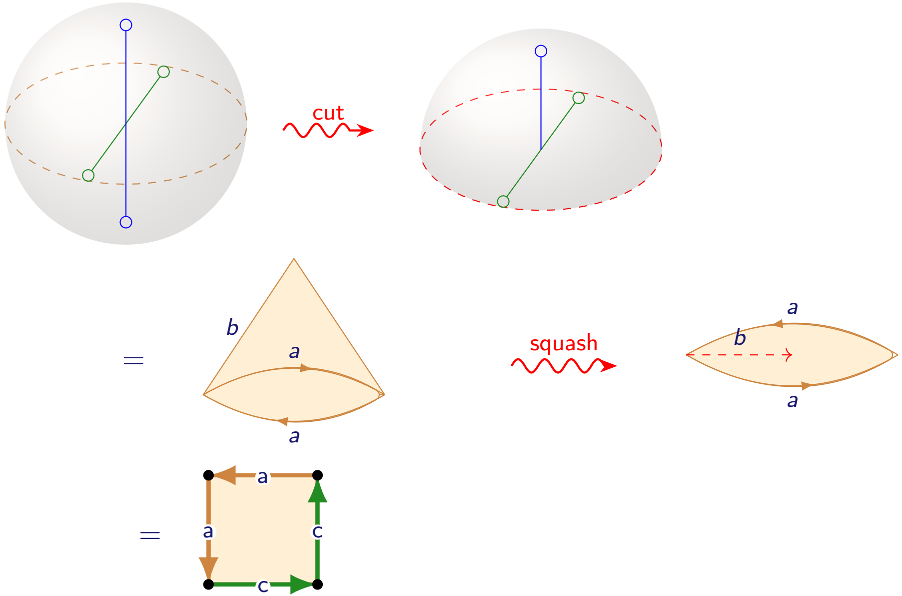
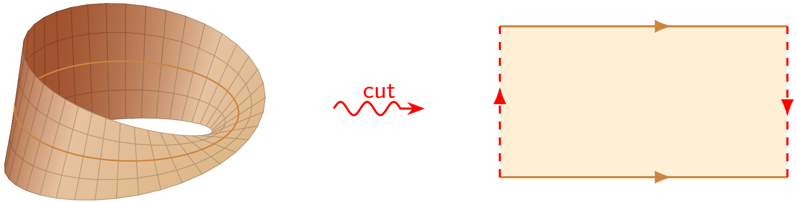
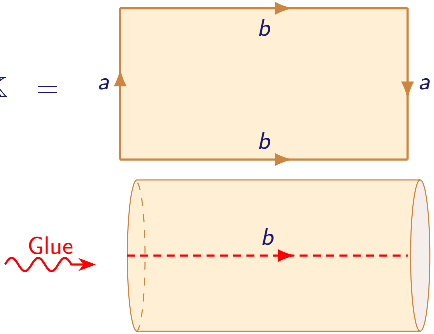


Identification space for a Möbius strip



Identification space for a Klein bottle

The Klein bottle is defined to be the identification space

$\mathbb{K} =$ 

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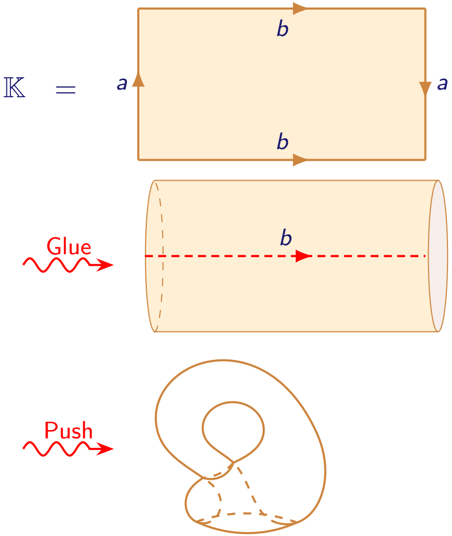

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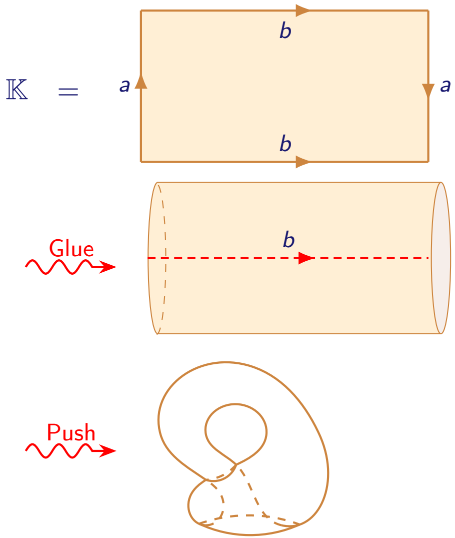

It is not clear how we to do the last step in $\mathbb{R}^3$ and, in fact, we can't!

Polygons in $\mathbb{R}^n$

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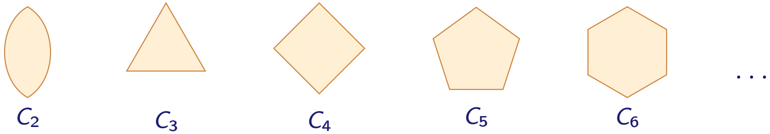

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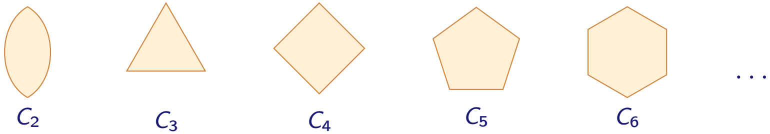

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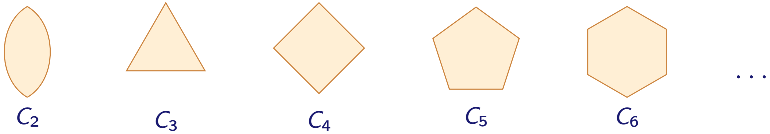

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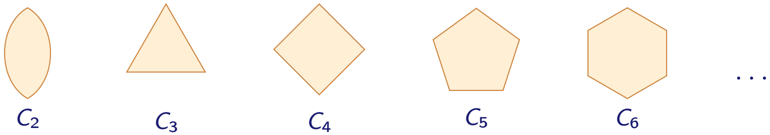

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- The graph C_2 has only one edge. When working with surfaces we think of C_2 as having two edges so that its image in $\mathbb{R}^2$ is a 2-gon

Surfaces and polygonal decompositions

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- We sometimes write $S = (V, E, F)$, where V is the vertex set, edge set E, and face set F

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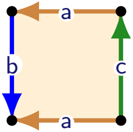
- When doing surgery always double check that you do not accidentally change the orientation of an edge

Examples of polygonal decompositions

We have already seen that:

- Annulus $\mathbb{A} \cong$ 

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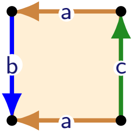

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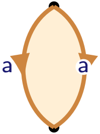
$S^2 \cong$



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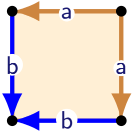

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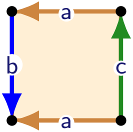

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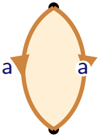

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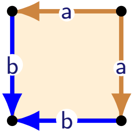

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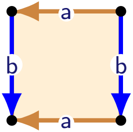

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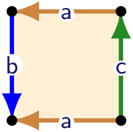

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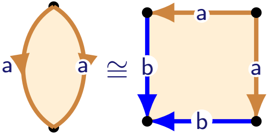

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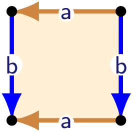

- Projective plane

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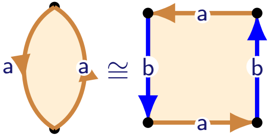

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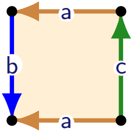
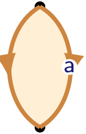
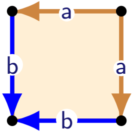
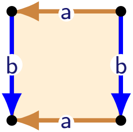
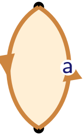
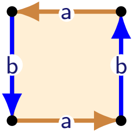
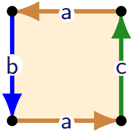
- Annulus $\mathbb{A} \cong$ 

- Sphere $S^2 \cong$ a  a $\cong$ 

- Torus $\mathbb{T} \cong$ 

- Projective plane $\mathbb{P}^2 \cong$ a  a $\cong$ 

- Möbius strip $\mathbb{M} \cong$ 

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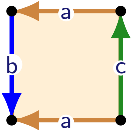

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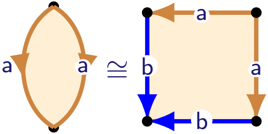

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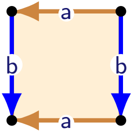

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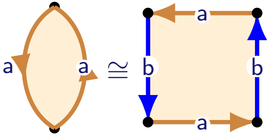

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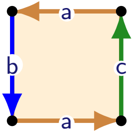
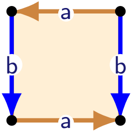


- Klein bottle $\mathbb{K} \cong$



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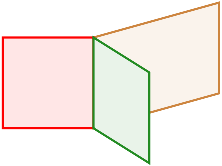

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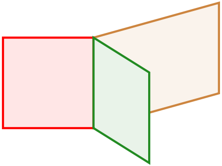

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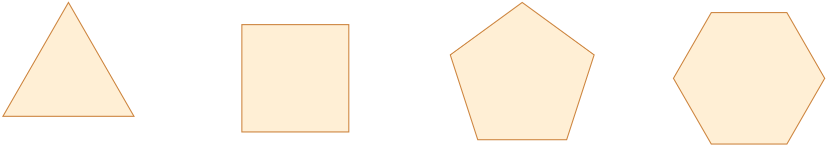

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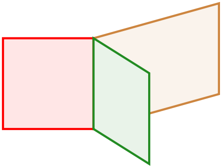

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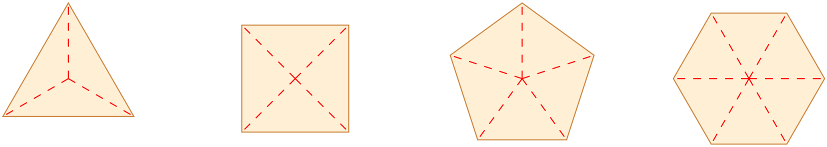

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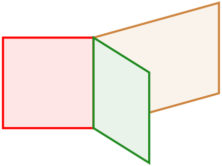

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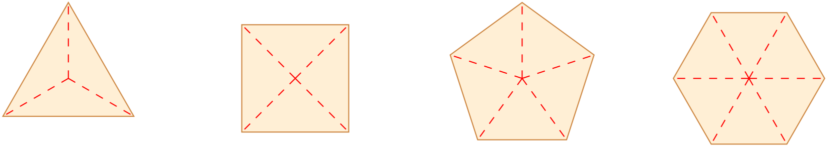
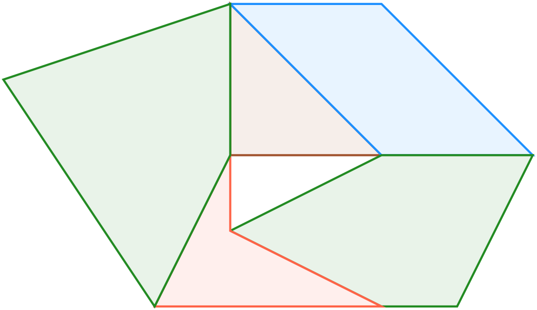


$\Longrightarrow$ Iterating this process, shows that any surface has infinitely many different polygonal decompositions!

Important facts about polygonal decompositions...

- Every connected surface has a polygonal decomposition with one polygon — with identified edges
  (A polygonal surface is connected if the underlying graph is connected)



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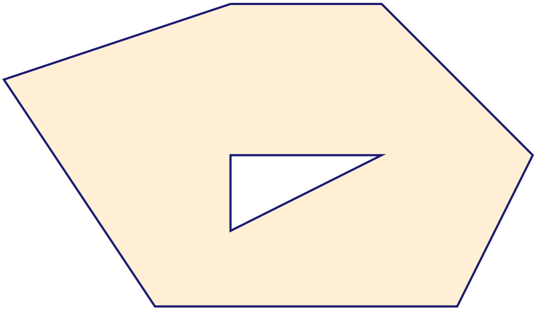

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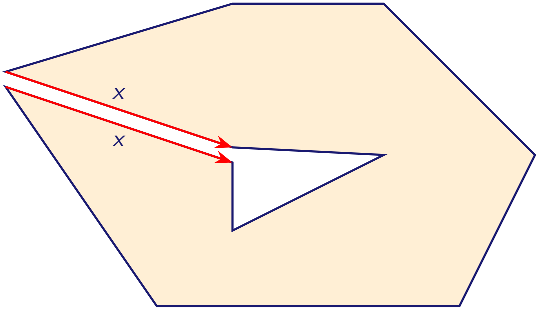

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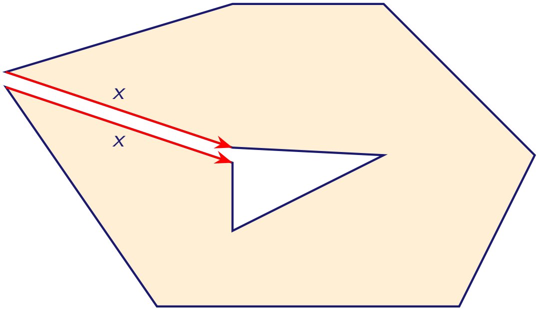

- We have to check that what we are doing does not depend on the choice of polygonal decomposition

Surgery: cutting and gluing

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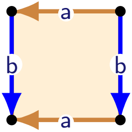
$\mathbb{T} \cong$ 

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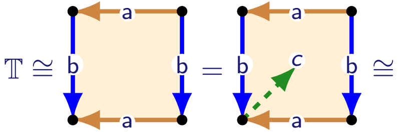

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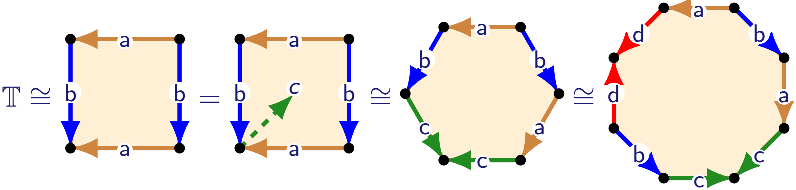

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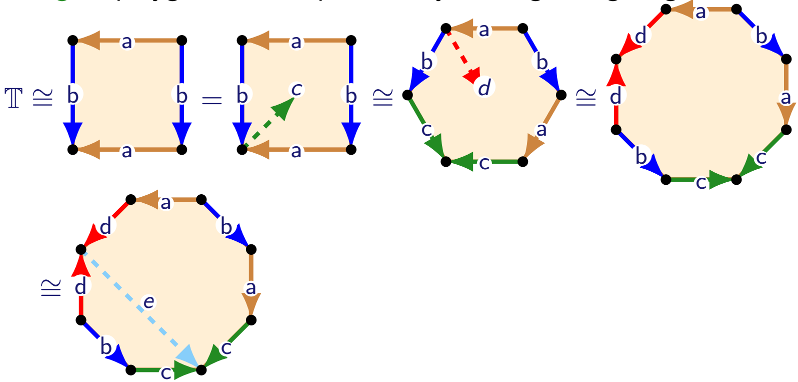

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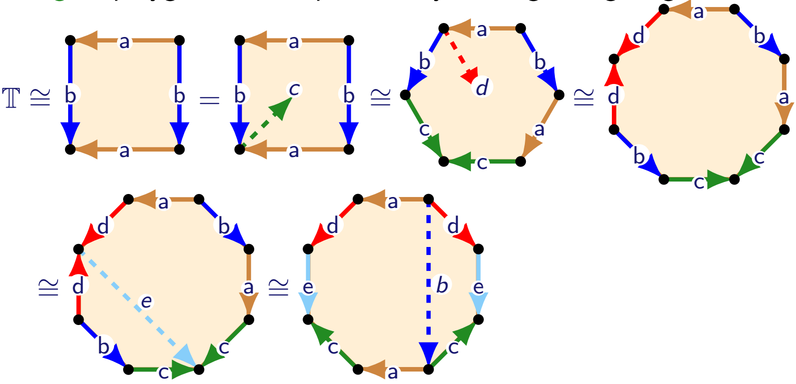

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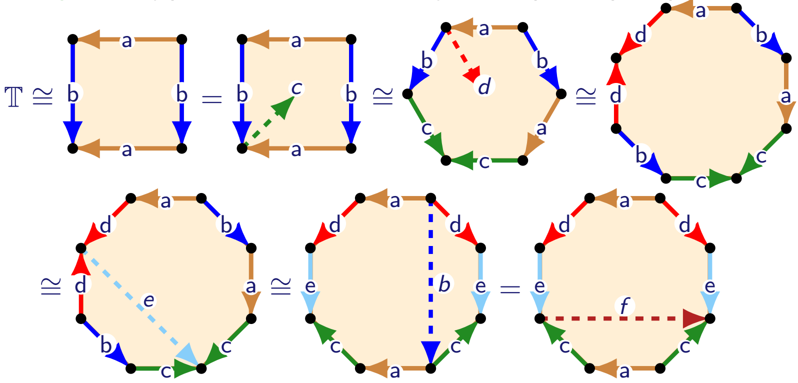

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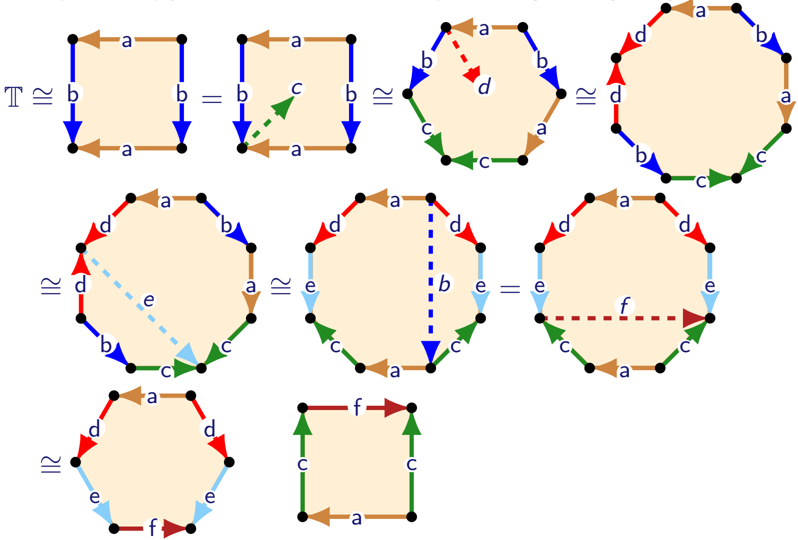

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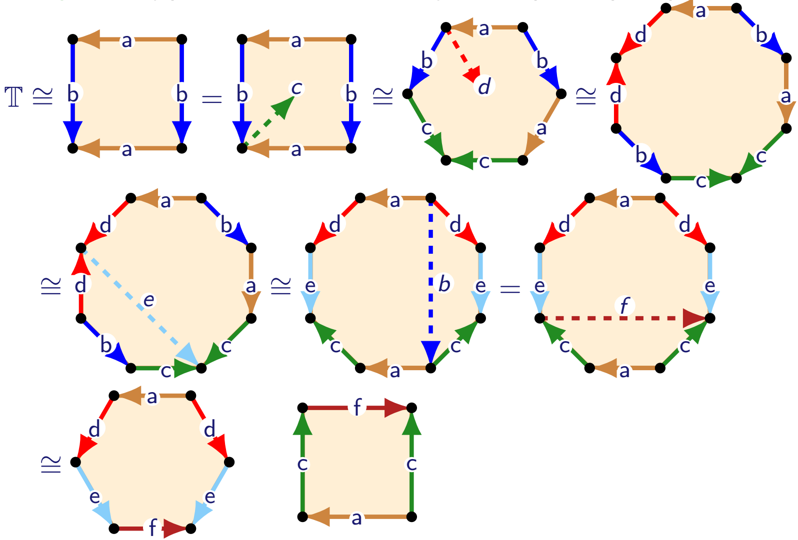
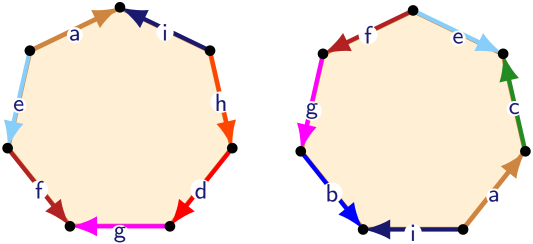


We want an easy way to identify surfaces from polygonal decompositions

Example surface

Exercise Can we describe the following surface?



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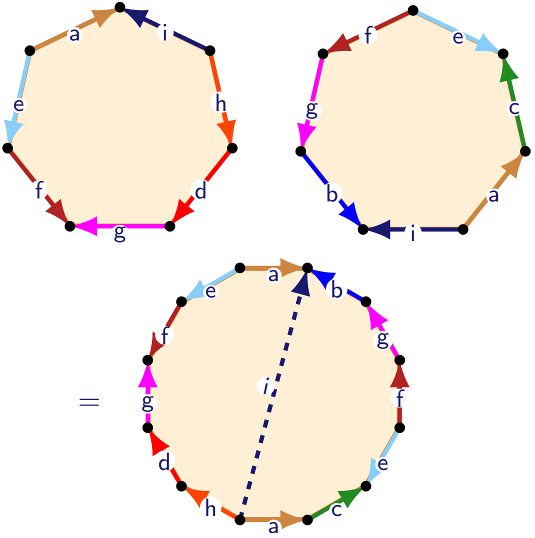

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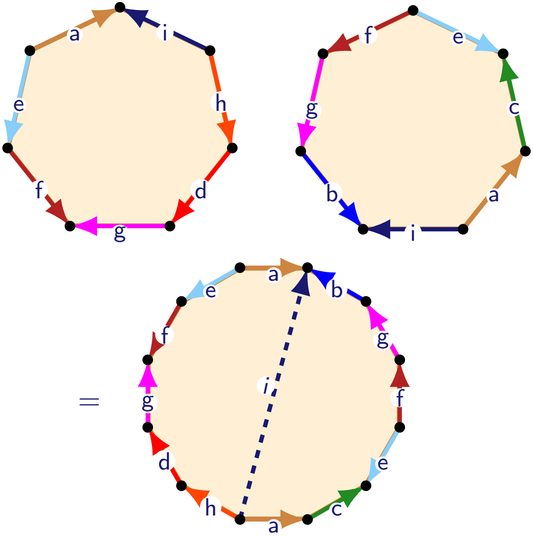

Answer Not yet! First we need more language and technology.

Free and paired edges and the boundary

Let S be a surface with a polygonal decomposition

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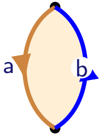
$\mathbb{D}^2 =$  $\cong$ 

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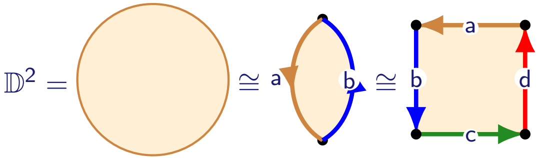

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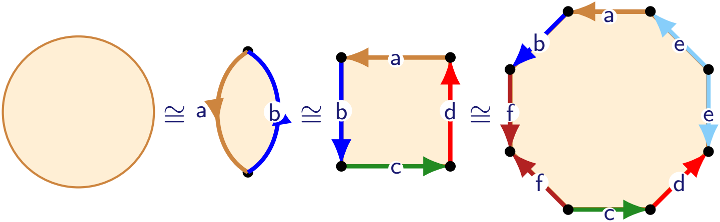
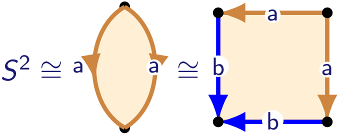
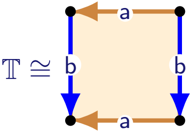


Example boundary circles...

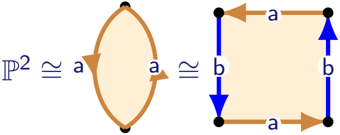
- Sphere



- Torus



- Projective plane



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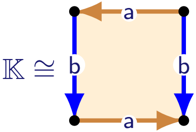

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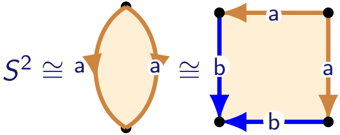

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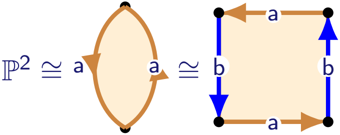

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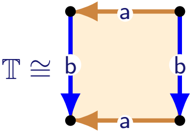

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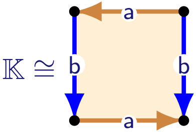

All edges paired $\Longrightarrow$ no boundary

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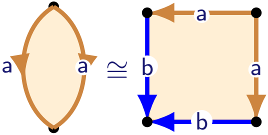
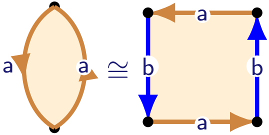
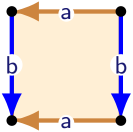
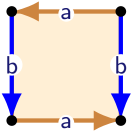
- Sphere $S^2 \cong$ 

- Projective plane $\mathbb{P}^2 \cong$ 

- Torus $\mathbb{T} \cong$ 

- Klein bottle $\mathbb{K} \cong$ 

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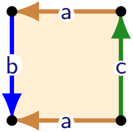
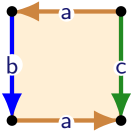
- Annulus $\mathbb{A} \cong$ 

- Möbius $\mathbb{M} \cong$ 

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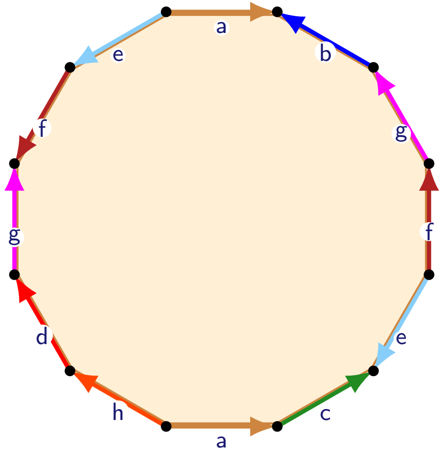

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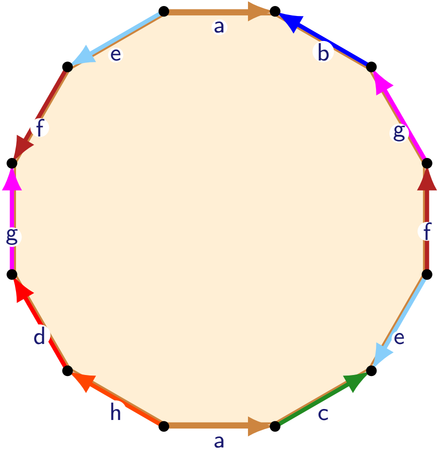

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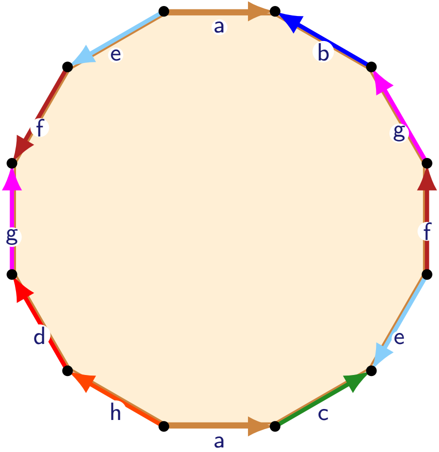

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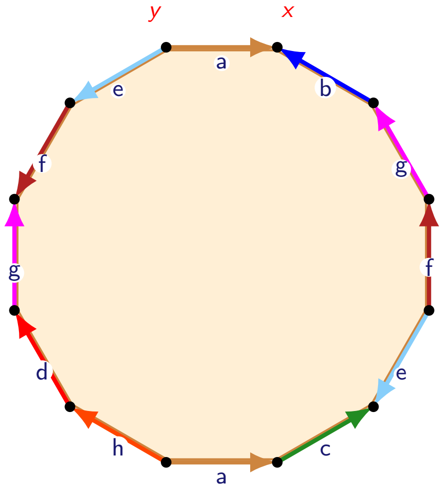

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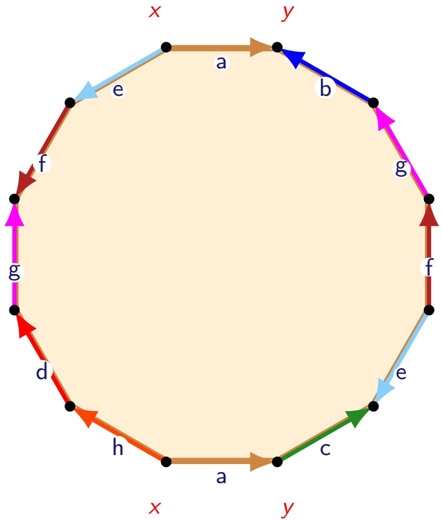

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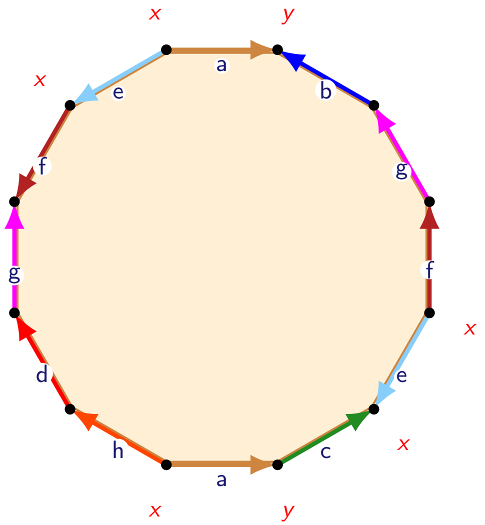

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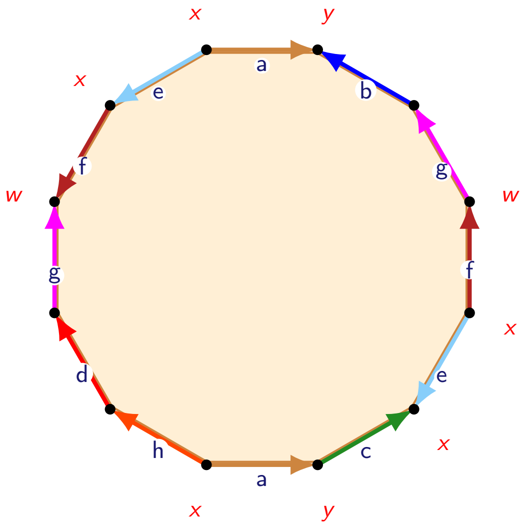

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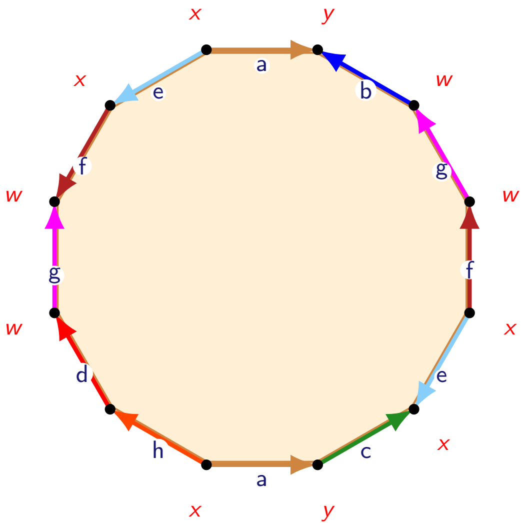

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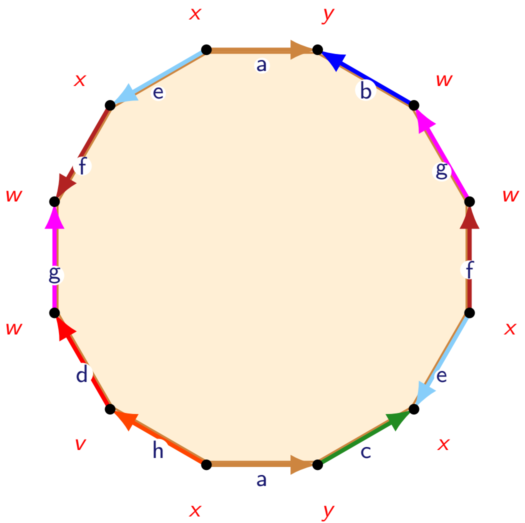

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The Euler characteristic of a surface

Let $S = (V, E, F)$ be a surface with a polygonal decomposition

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Euler characteristic of basic surfaces.

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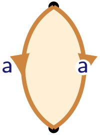
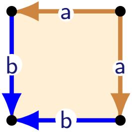
$S^2 \cong$ 

$\cong$ 

$, \chi = 2$

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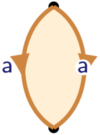
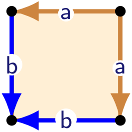
$S^2 \cong$ 

$\cong$ 

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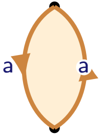
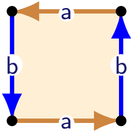
$\mathbb{P}^2 \cong$ 

$\cong$ 

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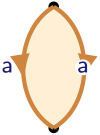

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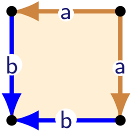

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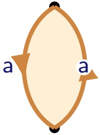

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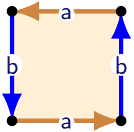

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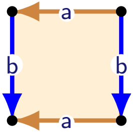

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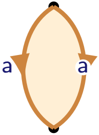
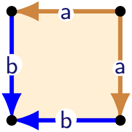
$S^2 \cong$ 

$\cong$ 

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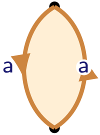
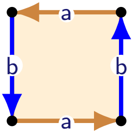
$\mathbb{P}^2 \cong$ 

$\cong$ 

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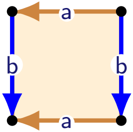
- Torus $\mathbb{T} \cong$ 

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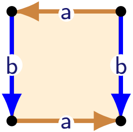
$\mathbb{K} \cong$ 

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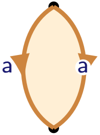
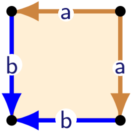
$S^2 \cong$ 

$\cong$ 

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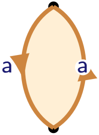
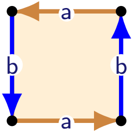
$\mathbb{P}^2 \cong$ 

$\cong$ 

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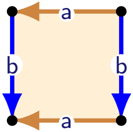
- Torus $\mathbb{T} \cong$ 

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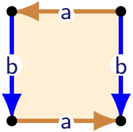
$\mathbb{K} \cong$ 

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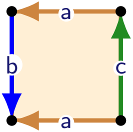
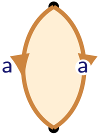
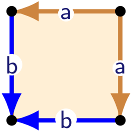
- Annulus $\mathbb{A} \cong$ 

$, \chi = 0$

Euler characteristic of basic surfaces.

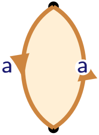
- Sphere

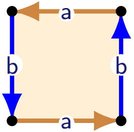
$S^2 \cong$ 

$\cong$ 

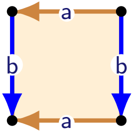
$, \chi = 2$

- Projective plane

$\mathbb{P}^2 \cong$ 

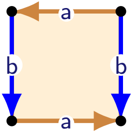
$\cong$ 

$, \chi = 1$

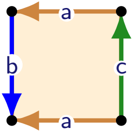
- Torus $\mathbb{T} \cong$ 

$, \chi = 0$

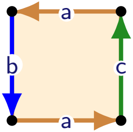
- Klein bottle

$\mathbb{K} \cong$ 

$, \chi = 0$

- Annulus $\mathbb{A} \cong$ 

$, \chi = 0$

- Möbius $\mathbb{M} \cong$ 

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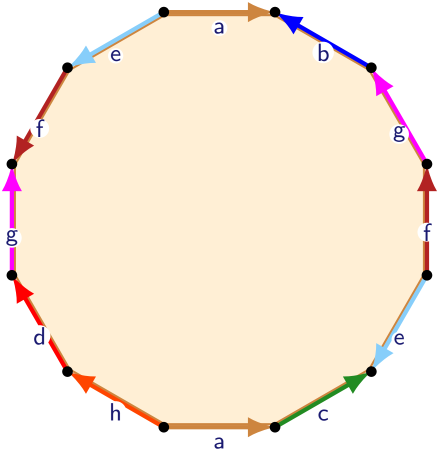
Example What is the Euler characteristic of the surface:

S =



Euler characteristic example

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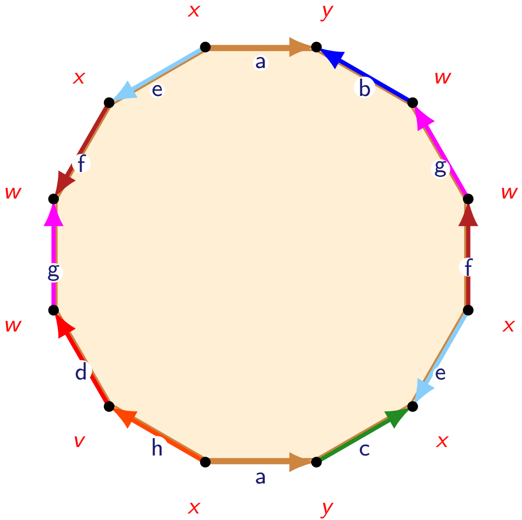
$S =$ 

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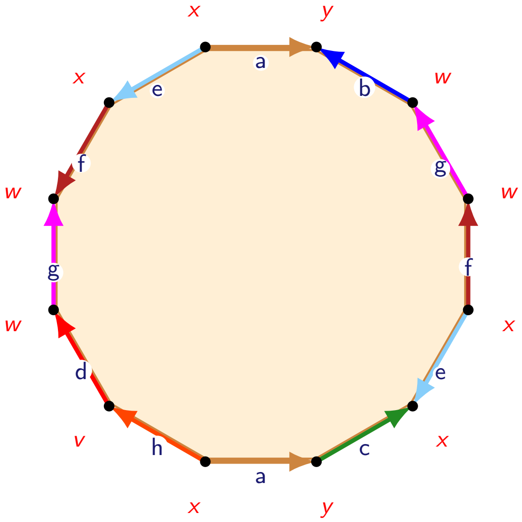

$\Longrightarrow \quad \chi(S) = -3$

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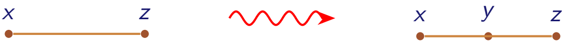

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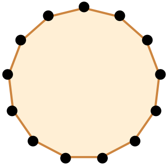

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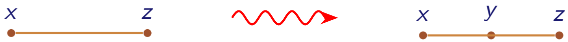

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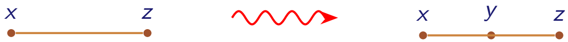

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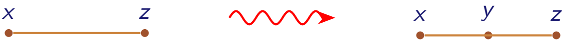

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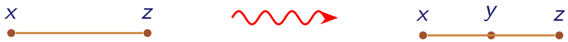

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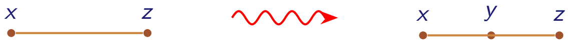

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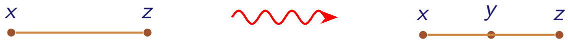

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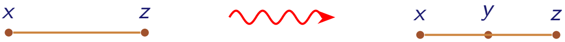

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Let S be a surface and suppose that S has polygonal decomposition $P_1 = (V_1, E_1, F_1)$ and $P_2 = (V_2, E_2, F_2)$. Then S has a polygonal decomposition (V, E, F) that is a common subdivision of P_1 and P_2

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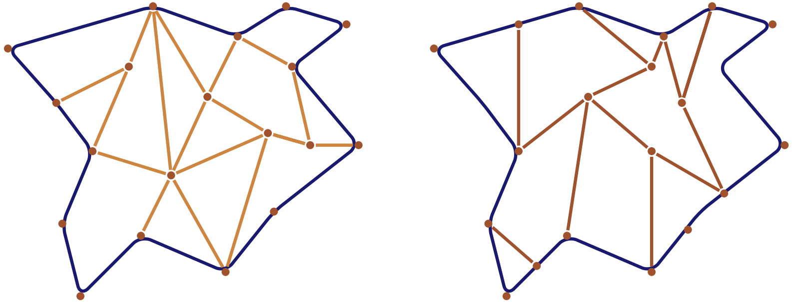

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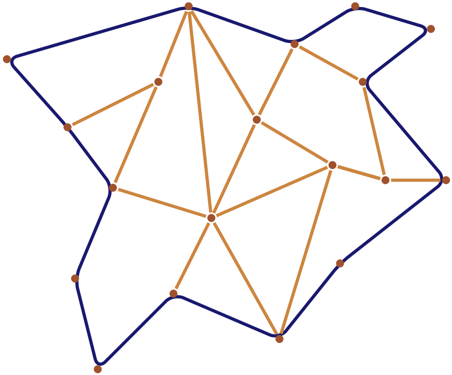

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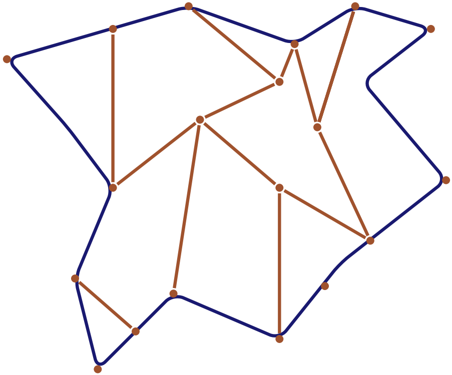

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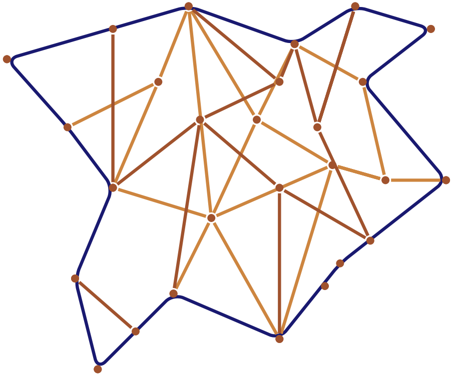

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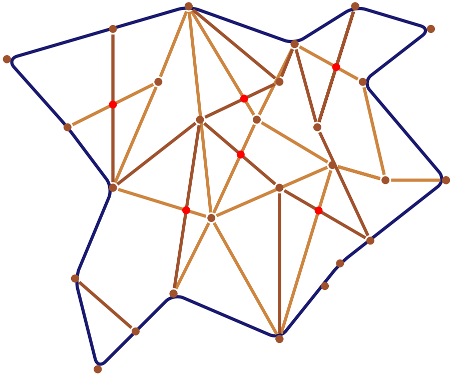
Proof Merge the two subdivisions — adding extra vertices as necessary



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Exercise Using what we know so far, deduce that the surfaces

$$S^2,\ \mathbb{A},\ \mathbb{D}^2,\ \mathbb{K},\ \mathbb{M},\ \mathbb{P}^2$$

are pairwise non-homeomorphic (see Tutorial 9)