LECTURE MATH4311: ALGEBRAIC TOPOLOGY

Disclaimer

Nobody is perfect, and I might have written or said something silly. If there is any doubt, then please check the references or contact me. All questions welcome!

What?

Algebraic topology is a twentieth century field of mathematics that is pervasive across mathematics and the sciences. It is unreasonably successful, being one of the newest fields of mathematics.

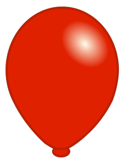
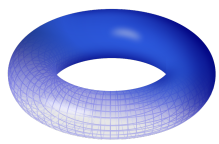
One of the most important aims of algebraic topology is to distinguish or classify topological spaces and maps between them up to homeomorphism.

Invariants (data that stays the same under operations on spaces) and obstructions are key to achieve this aim, meaning that

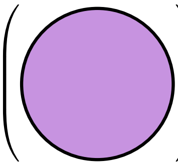
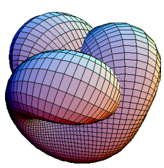
$$\text{invariants different} \Rightarrow \text{spaces different}.$$

The converse is however often false, and invariants are stronger the more often the converse holds. However, strong invariants might but hard or impossible to compute, and a good invariant is an invariant which balances between being strong and computable. The main aim of algebraic topology is to associate algebraic data to topological spaces.

Here is an example. The sphere and the torus illustrated below are certainly not the same. But how can we be sure? The algebraic topology approach is to associate to them, say, numbers χ

$\chi\Bigg($  $\Bigg) = 2 \neq 0 = \chi\Bigg($  $\Bigg),$

so we conclude that the sphere (left, a balloon) is not the torus (right, an empty donut) by simply observing that $2 \neq 0$. However, the invariant χ is not complete in the sense that

$\chi\Bigg($  $\Bigg) = 1 = \chi\Bigg($  $\Bigg),$

but the two topological spaces, the disc and the real projective plane (right, immersed in $\mathbb{R}^3$), are not the same. A crucial aim of algebraic topology is thus to have a big enough backpack of invariants to tackle problems in the wild.

A classical and familiar invariant is the Euler characteristic of a topological space (used in the above examples), which was initially discovered via combinatorial methods and has been rediscovered in many different guises. This invariant associates numbers to topological spaces, and is without doubt a good invariant.

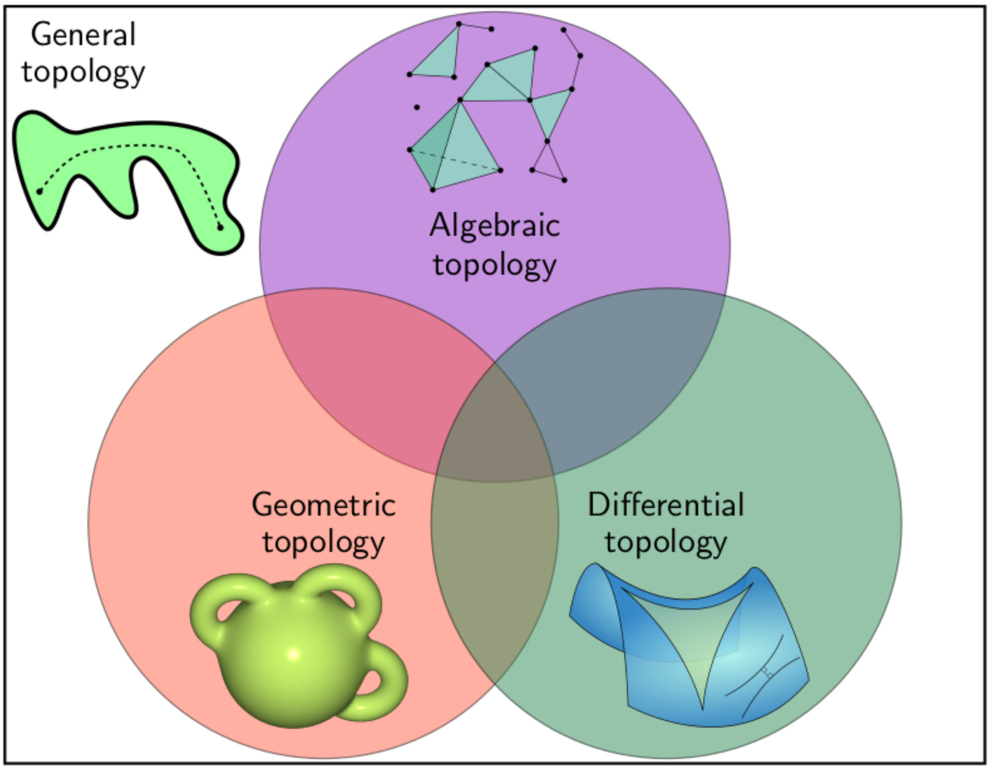
Modern algebraic topology goes one step further, and associates more sophisticated invariants to topological spaces. Algebraic topology allows the solution of complicated geometric problems with algebraic methods.

In a nutshell, imagine a closed loop of string that looks knotted in space. How would you tell if you can wiggle it about to form an unknotted loop without cutting the string? The space of all deformations of the loop is an intractable set. The key idea is to associate algebraic structures, such as groups or vector spaces, with topological objects such as knots, in such a way that complicated topological questions can be phrased as simpler questions about the algebraic structures. In particular, this turns questions about an intractable set into a conceptual or finite, computational framework that allows us to answer these questions with certainty.

In this unit you will learn about fundamental group and covering spaces, homology and cohomology theory. These form the basis for applications in other domains within mathematics and other disciplines, such as physics or biology. In fact, one of the strengths of algebraic topology has always been its wide degree of applicability to other fields.

At the end we will have developed skills to determine whether seemingly intractable problems can be solved with topological methods.

As a final note, there are three main fields of modern topology, with a huge overlap, of course. These fields are algebraic, geometric and differential topology:



All of these are similar in flavor and have the same aims, but use different methods. All of this is embedded into general topology, which provides the underlying language.

The lecture follows [Ha02] (freely available), other always useful textbooks include [StSe78] (general topology), [Tu11] (geometric topology), [Wa68] (differential topology), and many more! Using google and YouTube is also helpful. If you are unfamiliar with the basic ideas of general topology try [Mo20] (freely available), or have a look at the discussion on math.stackexchange.com/questions/7520.

Who?

Fourth semester students in Mathematics interested in a mixture of (linear) algebra, topology and category theory, but everyone is welcome.

Where and when?

- Time and date for the lecture.
  - Every Monday from 12:00–14:00.
  - Online, zoom links can be found on Canvas. Alternatively use 82018001516
  - First lecture: Monday 01.Aug.2021. Last lecture: Monday 01.Nov.2021.
- Time and date for the tutorials.
  - Every Friday from 12:00–14:00.
  - Online, zoom links can be found on Canvas. Alternatively use 81580288984
  - First tutorial: Friday 13.Aug.2021. Last tutorial: Friday 05.Nov.2021.

Material for the lecture

- Main source [Ha02].
- MATH4311 in Canvas.
- My website www.dtubbenhauer.com/lecture-algtop-2021.html
- Website www.maths.usyd.edu.au/u/UG/HM/MATH4311/
- Prerecorded lectures on the "What is...algebraic topology?" playlist here: www.youtube.com/c/VisualMath/playlists
- Exercise sheets are available on Canvas or on the course website.

Assessments

- Assignment 1 (25%), due 23:59 17.Sep.2021.
- Assignment 2 (25%), due 23:59 05.Nov.2021.
- Exam (50%), 24.Nov.2021, start 9:00. For more details see www.dtubbenhauer.com/lecture-algtop-2021.html.

Preliminary Schedule.

- The beginnings – What is...algebraic topology? (09.Aug.2021)
- Some definitions in topology – Cell complexes and alike. (16.Aug.2021)
- The fundamental group I – The first steps. (23.Aug.2021)
- The fundamental group II – The Seifert–van Kampen theorem. (30.Aug.2021)
- The fundamental group III – Covering spaces. (06.Sep.2021)
- The fundamental group IV – Groups, graphs and $K(G,1)$ spaces. (13.Sep.2021)
- Homology and cohomology I – Simplicial, singular and cellular homology. (20.Sep.2021)
- Homology and cohomology II – The axiomatic approach. (04.Oct.2021)
- Homology and cohomology III – Cohomology groups. (11.Oct.2021)
- Homology and cohomology IV – The cohomology ring. (18.Oct.2021)
- Homology and cohomology V – Poincaré duality. (25.Oct.2021)
- Whats next? – Some outlook including homotopy. (01.Nov.2021)

References

[Ha02] A. Hatcher. Algebraic topology. Cambridge University Press, Cambridge, 2002. xii+544 pp. URL: https://pi.math.cornell.edu/~hatcher/AT/AT.pdf

[Mo20] S.A. Morris. Topology without tears. Eprint. URL: https://www.topologywithouttears.net/topbook.pdf

[StSe78] L.A. Steen and J.A. Seebach, Jr. Counterexamples in topology. Reprint of the second (1978) edition. Dover Publications, Inc., Mineola, NY, 1995. xii+244 pp. URL: https://en.wikipedia.org/wiki/Counterexamples_in_Topology

[Tu11] L.W. Tu. An introduction to manifolds. Second edition. Universitext. Springer, New York, 2011. xviii+411 pp.

[Wa68] A. Wallace. Differential topology. First steps. With a foreword by Robert Gunning and Hugo Rossi. Reprint of the 1979 (third) printing of the 1968 original. Dover Publications, Inc., Mineola, NY, 2006. xiv+130 pp. URL: https://pi.math.cornell.edu/~hatcher/AT/AT.pdf

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