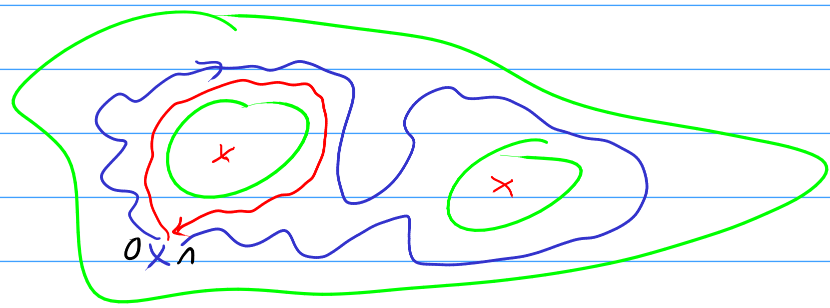
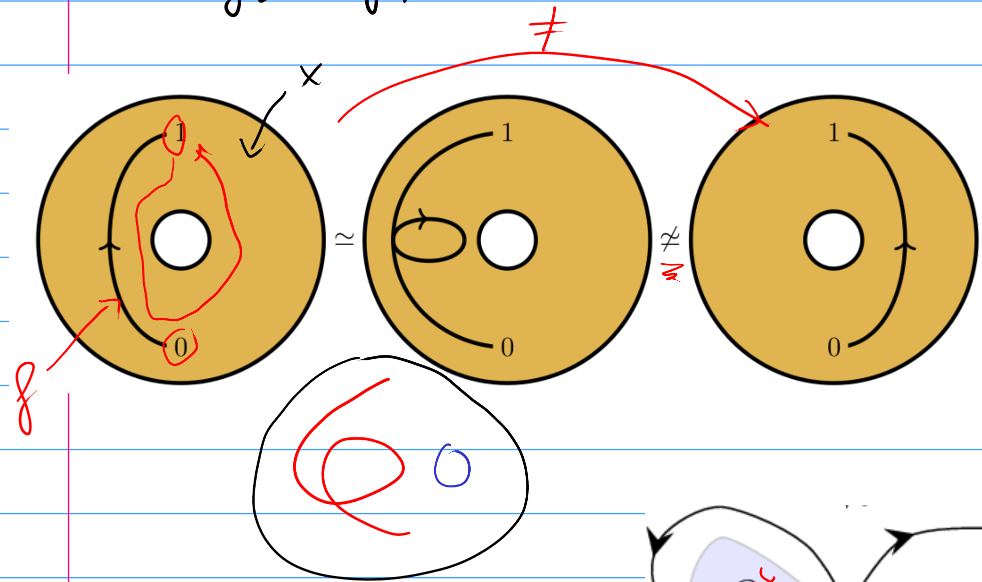
3. Fundametal group $\pi_1(X)$

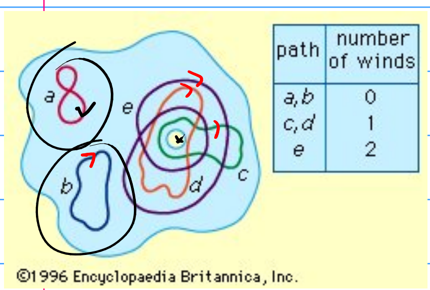
X =

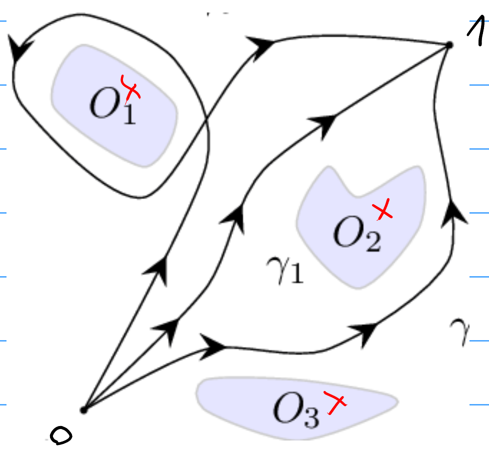


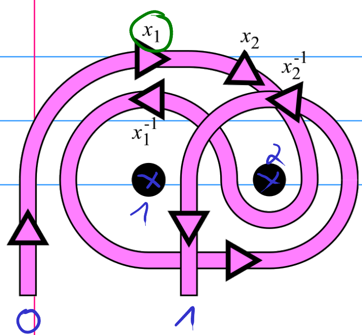
Def A path f in X is a map $f : [0,1] \to X$.
A homotopy of path in X $\{ f_t : [0,1] \to X \mid t \in [0,1] \}$ with $\underset{f_0}{\underset{\shortparallel}{f_0(t)}}$ and $\underset{f_1}{\underset{\shortparallel}{f_1(t)}}$ are independent of t.

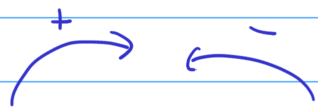
$\rightsquigarrow f_0 \simeq f_1$



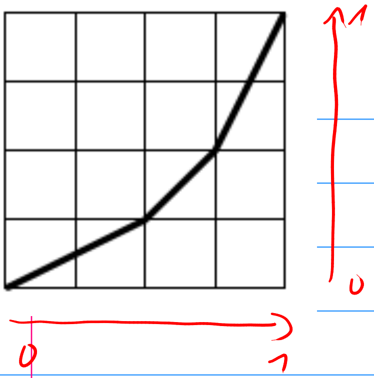




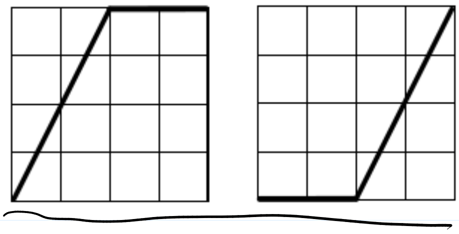




$x_1 x_2 x_1^{-1} x_2^{-1}$

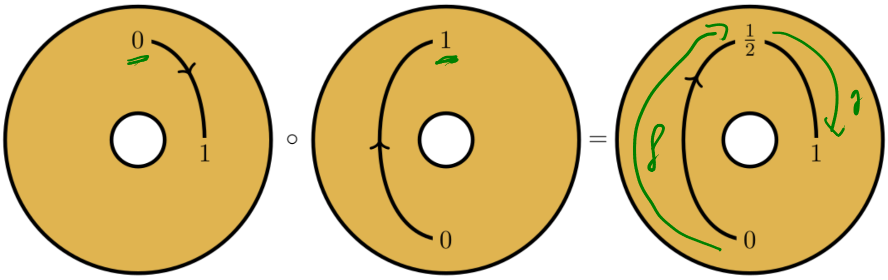


Def A reparametrization of a path $f: I \to X$ (with $I = [0,1]$) it is the path $f\circ\varphi : I \to X$ for $\varphi : I \to I$

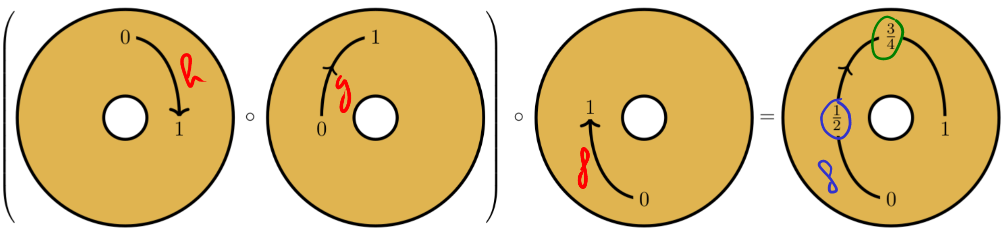


Def: $f, g : I \to X$ with $f(1) = g(0)$
create $g \circ f : I \to X$

$$g\circ f(s) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

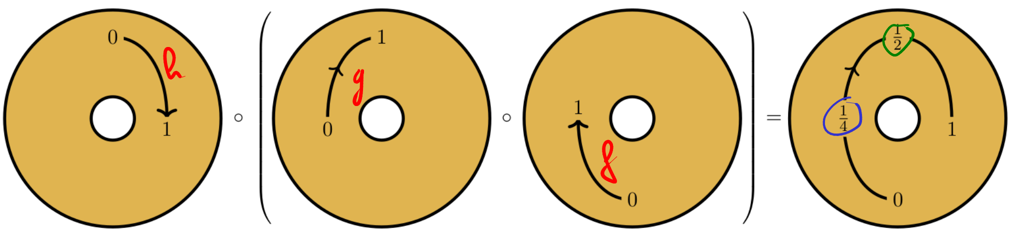


$g \quad \circ \quad f$



$(h\circ g)\circ f$

$\simeq$



$h\circ (g\circ f)$

Def The fundamental group of X at $x_0 \in X$ (basepoint) is

$\pi_1(X, x_0) = \{[f] \mid f: I \to X,\ f(0) = x_0 = f(1)\}$

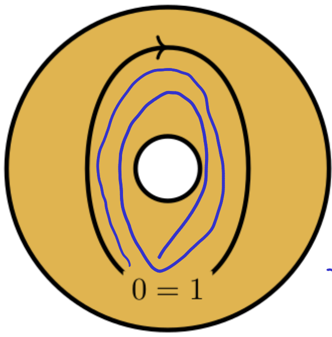
homotopy classes of loops

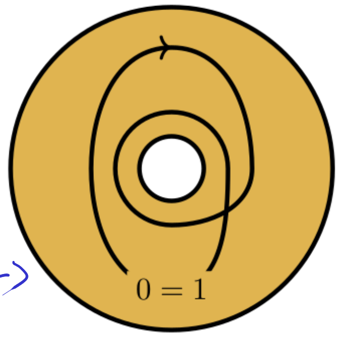
together with path composition $\circ$ as a binary operation

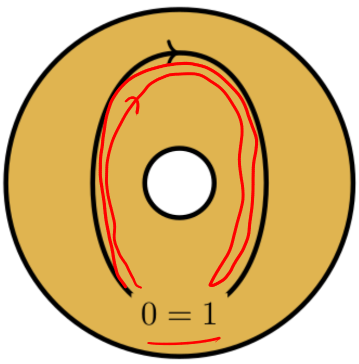
1 = 

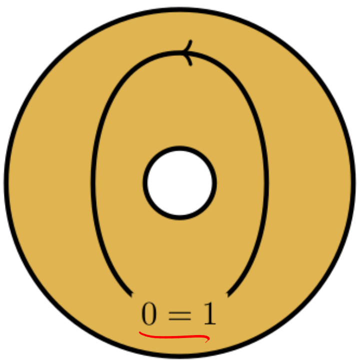
Lemma $\pi_1(X, x_0)$ with $\circ$ is a group!

Proof:
- Well-defined ✓
- Unit element 1
- inverse
- associative

$x =$ 

$x^2 =$ 

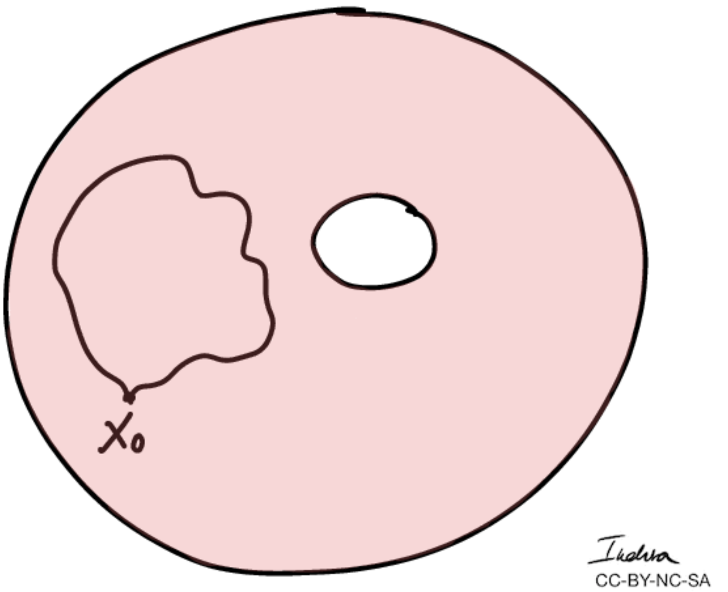
$x =$ 

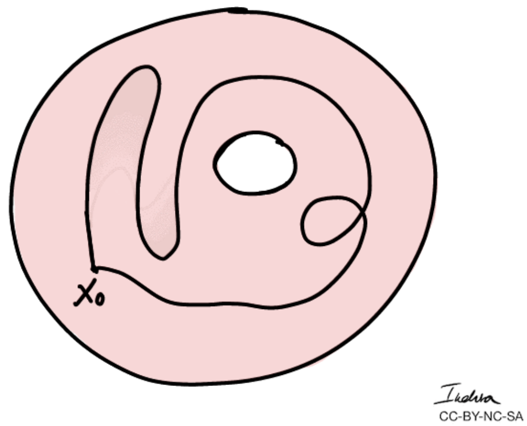
$x^{-1} =$ 

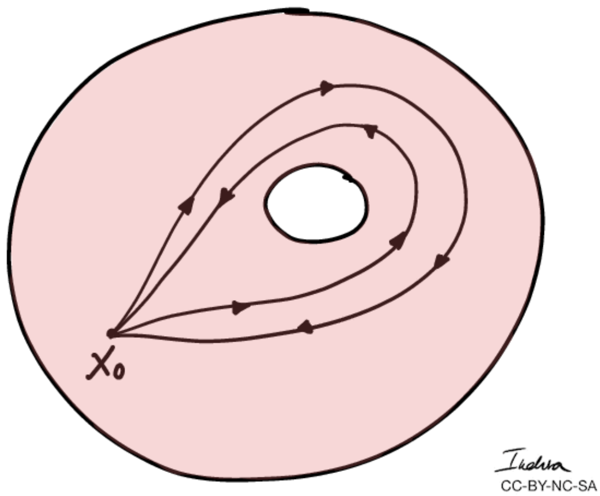
$\square$

Group associated to (X, x_0) ← based topological space

Awesome!







Def (X, x_0), (X, x_1) $\quad \beta_h : \pi_1(X, x_0) \to \pi_1(X, x_1)$

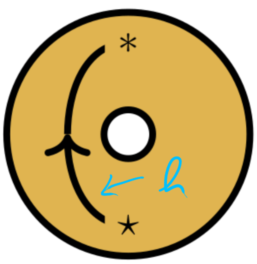
basechange map $\quad h : I \to X \quad h(0) = x_0 \quad h(1) = x_1$

$\rightsquigarrow$ map on group: $[f] \longmapsto [h \circ (f \circ \bar{h})]$

$\bar{h}$ = running h in opposite direction

Lemma: β_h is a group isomorphism!

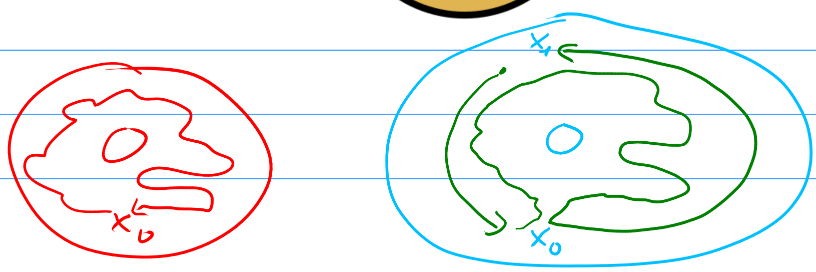
Remark: If X is path connected we can write $\pi_1(X)$.



$\circ$ path in $\pi_1(X, \star) \circ$ 

$=$ path in $\pi_1(X, *)$

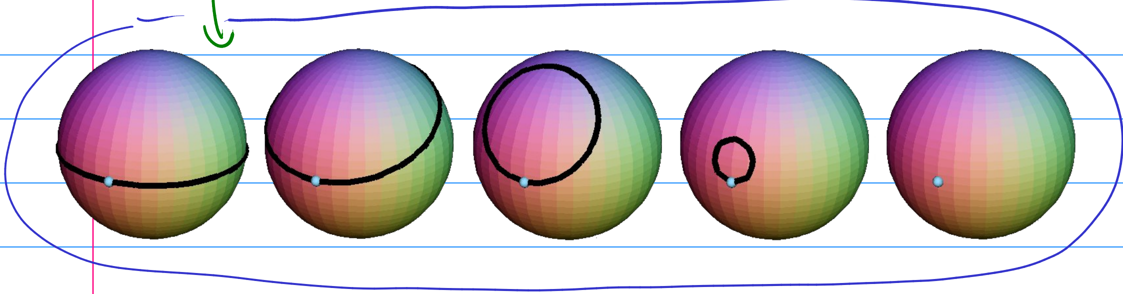
$\star \downarrow x_0 \qquad * \downarrow x_1$

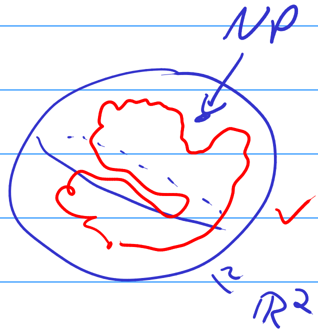


Examples: - $\pi_1(\{*\}) \underset{\text{group iso}}{\simeq} 1$ ← trivial group

- $\pi_1(\mathbb{R}^n) \simeq 1$

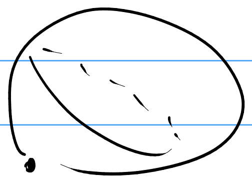
- $\pi_1(S^2) \simeq 1$ $\quad$ ($\pi_1(S^n) \simeq 1$ unless $n=1$)





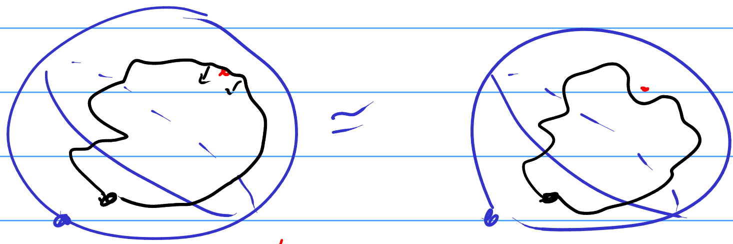
Bad news: π_1 can't detect spheres

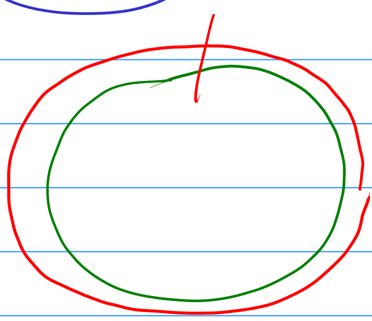
Will contrast $\chi(S^n) = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$ is doing a bit better

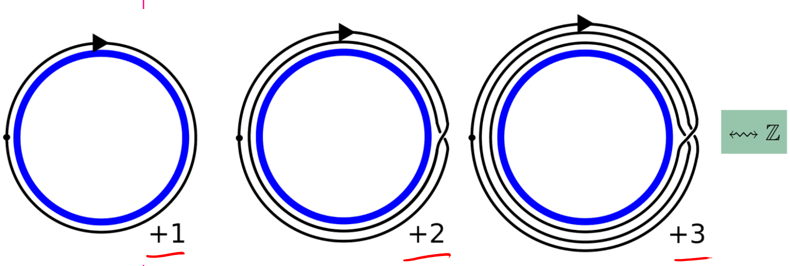


0 - cell

← n - cell



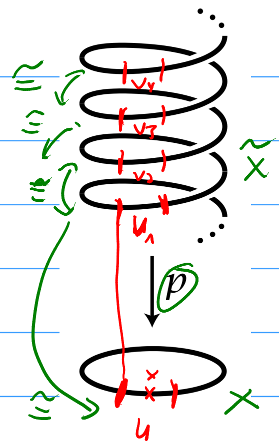




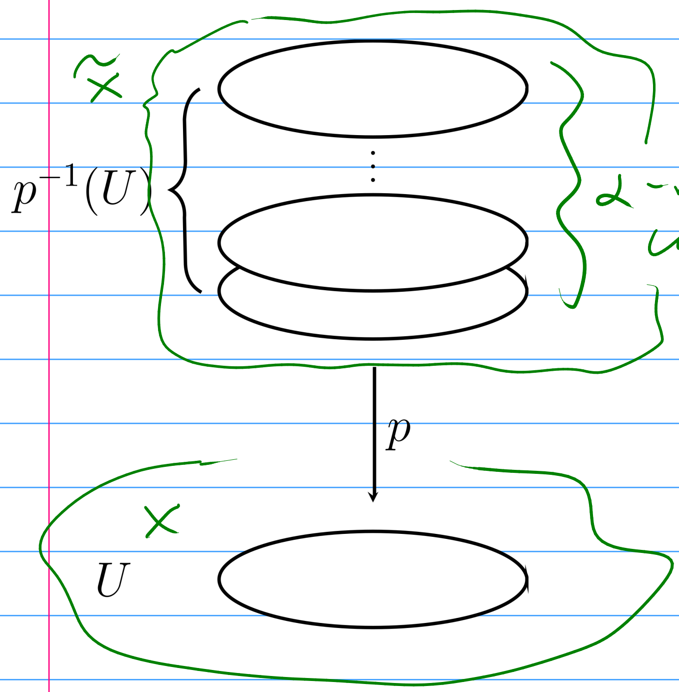
$\pi_1(S^1) \simeq \mathbb{Z}$

Theorem: $\pi_1(S^1) \simeq \mathbb{Z}$

Proof idea: Use covering spaces



Def A cover (covering space) $\tilde{X}$ is a pair of $\tilde{X}$ top space and a map $p: \tilde{X} \to X$ such $\forall x \in X \; \exists$ open $x \in U \subset X$ such that $p^{-1}(U) = \coprod V_\alpha$ with $V_\alpha \cong U$ induced

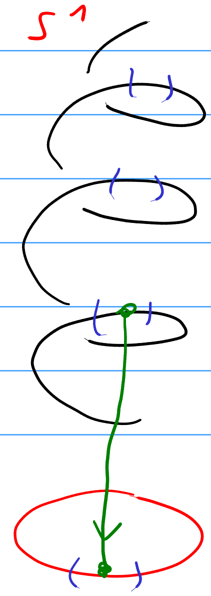


Example: – $X \xrightarrow{p} X$ $p = id$ is a cover ($S^1 \to S^1$)

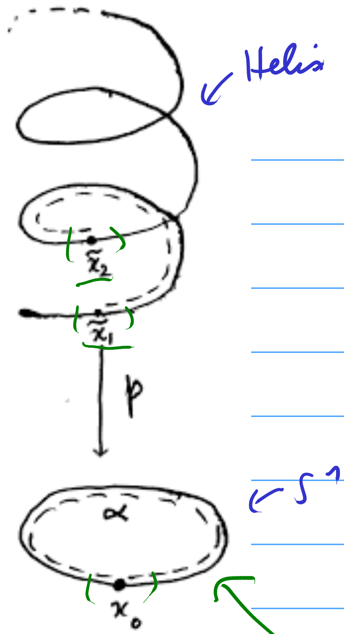
– $p: \mathbb{R} \to S^1 \subset \mathbb{C}$ is a cover of S^1

$p(s) = e^{2i\pi s} \subset \mathbb{C}$
$= (\cos(2\pi s), \sin(2\pi s)) \subset \mathbb{R}^2$

$\mathbb{R} \cong \text{Helix} \subset \mathbb{R}^3$



$\text{Helix} = \{(\cos(2\pi s), \sin(2\pi s), s)\} \subset \mathbb{R}^3$



Fixed $(\tilde X, p)$

Def: A lift of $f : Y \to X$ to $\tilde X$ is a $\tilde f : Y \to \tilde X$ such that

$$\begin{tikzcd} & \tilde X \arrow[d, "p"] \\ Y \arrow[ur, dotted, "\tilde f"] \arrow[r, "f"'] & X \end{tikzcd}$$

commutes

$f = p \circ \tilde f$

A lift of a path $f : I \to S^1$ to the helix

A lift of $f : I \to S^1$ to $\mathbb{R}$ is a path $\tilde f : I \to \mathbb{R}$ starting at $\tilde x_1 \in p^{-1}(x_0)$ to $\tilde x_2 \in p^{-1}(x_0)$ such that $f = p \circ \tilde f$

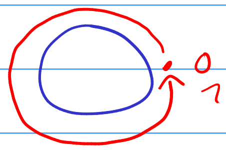
$X = S^1$
$\tilde X = \mathbb{R}$

(a) For each path $f : I \to X$ starting at a point $x_0 \in X$ and each $\tilde x_0 \in p^{-1}(x_0)$ there is a unique lift $\tilde f : I \to \tilde X$ starting at $\tilde x_0$.

(b) For each homotopy $f_t : I \to X$ of paths starting at x_0 and each $\tilde x_0 \in p^{-1}(x_0)$ there is a unique lifted homotopy $\tilde f_t : I \to \tilde X$ of paths starting at $\tilde x_0$.

(a) lifts exists and are unique
(b) Homotopy can be detected in $\tilde X$

$1 \longmapsto$

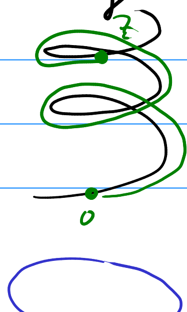


Proof continued

$t \in [0,1]$

We can write down a map $\mathbb{Z} \longrightarrow \pi_1(S^1)$
- by basic properties of exp this is a group homo. $\quad z \longmapsto [\exp(2\pi i z t)]$, $= \omega_z$
- To show injective + surjective use (a) + (b)

$\tilde \omega_z$



In $\mathbb{R}$ it is easy to see that any path starting in 0 and ending in z is $\simeq$ to $\tilde\omega_z$.
Now use (a) + (b).

□

Point now: π_1: topology $\rightsquigarrow$ algebra

$$
\begin{tikzcd}
\text{topology} \arrow[d, "\pi_1"] \\
\text{algebra}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\pi_1"'] & Y \arrow[d, "\pi_1"] \\
\pi_1(X) \arrow[r, "f_* = \pi_1(f)"'] & \pi_1(Y)
\end{tikzcd}
$$

induced group homo

<u>Def</u>: For $f:(X,x_0) \to (Y,y_0)$ of based spaces we get $f_*: \pi_1(X,x_0) \to \pi_1(Y,y_0)$ by

$$f_*: [g] \longrightarrow [f \circ g] \quad \text{Postcomposition}$$

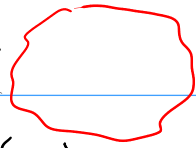
$g: I \xrightarrow{g} X \xrightarrow{f} Y$

<u>Lemma</u>: f_* is a well-defined group homomorphism

"Everything" in topology is send by π_1 to a "nice" notion in algebra

$$
\begin{tikzcd}
\text{topology} \arrow[d, "\pi_1"] & \text{map } X \to Y \arrow[d, "\pi_1"] & \text{homotopy equivalence } X \simeq Y \arrow[d, "\pi_1"] & \text{deformation retract } A \to X \arrow[d, "\pi_1"] \\
\text{algebra} & \text{group homo } \pi_1(X) \to \pi_1(Y) & \text{group iso } \pi_1(X) \cong \pi_1(Y) & \text{group iso } \pi_1(A) \cong \pi_1(X)
\end{tikzcd}
$$

$$(X \simeq Y) \Rightarrow (\pi_1(X) \cong \pi_1(Y)) \quad \textbf{Invariance}$$

Direct application: - $\pi_1\Big($  $\Big) \simeq \pi_1(\{*\}) \simeq 1$ ($\mathbb{R}^2$)

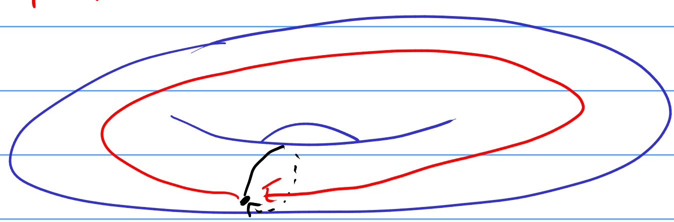
- $\pi_1(S^1) \simeq \mathbb{Z} \Rightarrow S^1 \not\simeq \{*\}$
- $S^1 \not\simeq S^n$ for $n > 1$ $\quad \pi_1(S^n) \simeq 1$

But we can't tell whether $S^2 \simeq S^4$?!

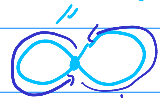
topology	product $X \times Y$	coproduct $X \vee Y$
$\downarrow \pi_1$	$\downarrow \pi_1$	$\downarrow \pi_1$
algebra	group homo $\pi_1(X \times Y) \overset{\simeq}{\cancel{\ncong}} \pi_1(X) \times \pi_1(Y)$	group iso $\pi_1(X \vee Y) \overset{\simeq}{\cancel{\ncong}} \pi_1(X) * \pi_1(Y)$

$\pi_1(T) \simeq \pi_1(S^1 \times S^1) \simeq \pi_1(S^1) \times \pi_1(S^1)$
$\simeq \mathbb{Z}^2$ commutative

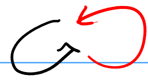
$\Rightarrow T \not\simeq S^2$



$S^1 \vee S^1$

$\pi_1\Big($  $\Big)$

$\simeq \mathbb{Z} * \mathbb{Z}$

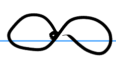


free product

Elements $G * H$ are of the form $g_1 h_1 g_2 h_2 g_3 h_3 \cdots g_k h_k$
$g_i \in G,\ h_i \in H$

$|\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}| = 4$

$|\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}| = \infty$
$s \qquad t$
ststst...

-  $\simeq T$
-  $\simeq S^2$

$\vdots$

$$\begin{array}{ccc} \text{topology} & \cancel{\text{disjoint union } X \cup Y} & \text{bad operation on pointed spaces!} \\ \big\downarrow \pi_1 & \cancel{\big\downarrow \pi_1} & \\ \text{algebra} & \cancel{\text{nothing really}} & \end{array}$$

$\mathbb{R}^3 \not\cong \mathbb{R}^5$??

Corollary 1.16. *$\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.*

Proof: Suppose $f : \mathbb{R}^2 \to \mathbb{R}^n$ is a homeomorphism. The case $n = 1$ is easily disposed of since $\mathbb{R}^2 - \{0\}$ is path-connected but the homeomorphic space $\mathbb{R}^n - \{f(0)\}$ is not path-connected when $n = 1$. When $n > 2$ we cannot distinguish $\mathbb{R}^2 - \{0\}$ from $\mathbb{R}^n - \{f(0)\}$ by the number of path-components, but we can distinguish them by their fundamental groups. Namely, for a point x in $\mathbb{R}^n$, the complement $\mathbb{R}^n - \{x\}$ is homeomorphic to $S^{n-1} \times \mathbb{R}$, so Proposition 1.12 implies that $\pi_1(\mathbb{R}^n - \{x\})$ is isomorphic to $\pi_1(S^{n-1}) \times \pi_1(\mathbb{R}) \approx \pi_1(S^{n-1})$. Hence $\pi_1(\mathbb{R}^n - \{x\})$ is $\mathbb{Z}$ for $n = 2$ and trivial for $n > 2$, using Proposition 1.14 in the latter case. $\square$

Theorem 1.9. *Every continuous map $h : D^2 \to D^2$ has a fixed point, that is, a point $x \in D^2$ with $h(x) = x$.*

$$\begin{array}{ccc} \text{topology} & \text{retraction } A \hookrightarrow X & (r\colon X \to A \text{ and } \iota\colon A \to X \text{ with } r \circ \iota = id_A) \\ \big\downarrow \pi_1 & \big\downarrow \pi_1 & \\ \text{algebra} & \text{group mono. } \pi_1(A) \hookrightarrow \pi_1(X) & \end{array}$$

$D^2 = X$

$\leftarrow S^1 = A$

$\mathbb{Z} = \pi_1(S^1) \hookrightarrow \pi_1(D^2) = 1$

No way $\Rightarrow$ no retraction $S^1 \hookrightarrow D^2$

monomorphism $\leftrightarrow$ injection
epimorphism $\leftrightarrow$ surjection

!?

Theorem 1.10. *For every continuous map $f : S^2 \to \mathbb{R}^2$ there exists a pair of antipodal points x and $-x$ in S^2 with $f(x) = f(-x)$.*

---

based spaces

π_1 "functor" from $Top_\bullet \longrightarrow Gr$

groups

such that: $X \simeq Y \xmapsto{\pi_1} \pi_1(X) \simeq \pi_1(Y)$

$f : X \to Y \xmapsto{\pi_1} f_* : \pi_1(X) \xrightarrow{\simeq} \pi_1(Y)$

Main point: Take "Statement A in $Top_\bullet$"

$\xmapsto{\pi_1}$ "Statement $\pi_1(A)$ in Gr"