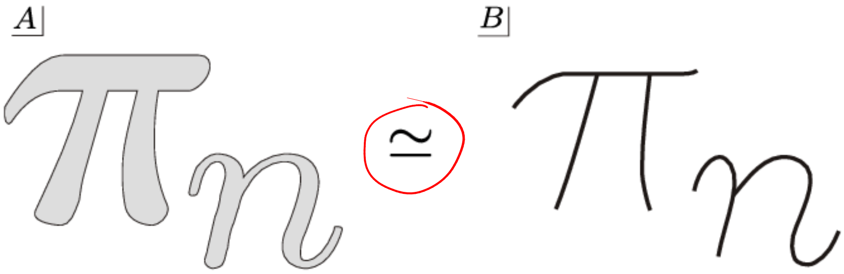
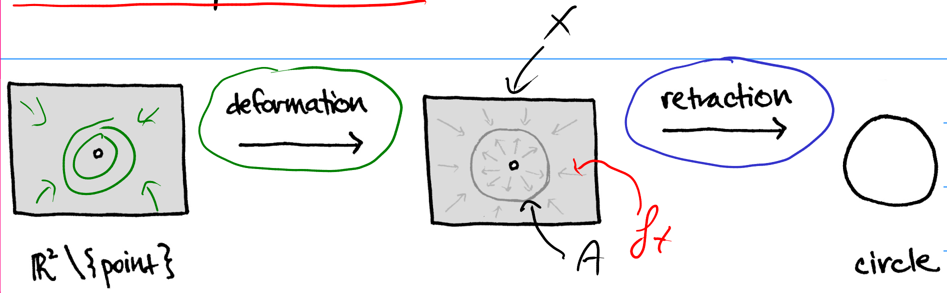
Lecture 2
Part 1 Homotopy equivalence



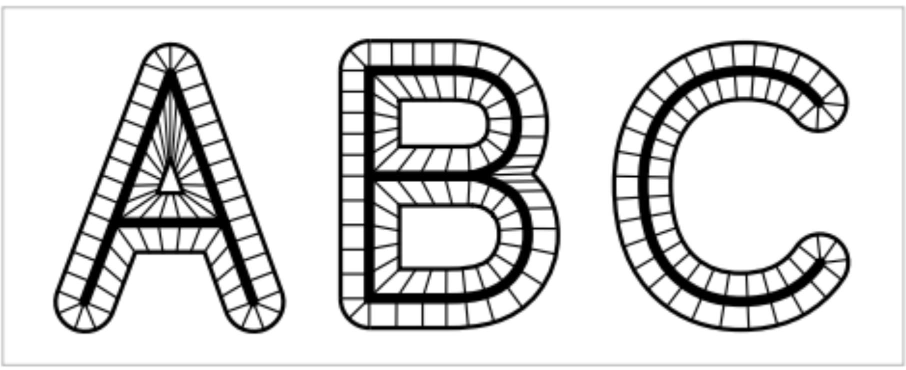


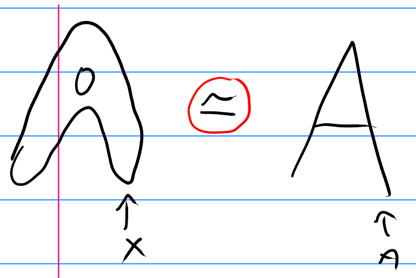
Def X, A $\quad A$ subspace of X

A deformation retraction of X onto A is:

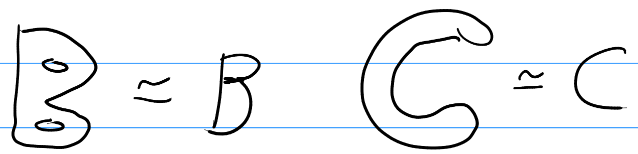
- a family of maps $f_t : X \to X \quad t \in [0,1]$
- $f_0 = id$, $f_1(X) = A$, $f_t|_A = id$

$f_t : X \times [0,1] \to X \quad (x,t) \mapsto f_t(x)$ continuous





X deformation retracts onto A



Def (Homotopy)

X, Y topological spaces

A homotopy is a family of maps $\{g_t : X \to Y, t \in [0,1]\}$ such that $G : X \times I \to Y$ is continuous

$G(x,t) = g_t(x)$

Def (Homotopy part 2)

$f_0, f_1 : X \to Y$ are **homotopic** $\exists \{g_t : X \to Y, t \in [0,1]\}$ homotopy with $g_0 = f_0$, $g_1 = f_1$

We write $f_0 \simeq f_1$ in case they are homotopic

Def (Homotopy part 3)

X, Y, map $f : X \to Y$ is a **homotopy equivalence** if $\exists\, g : Y \to X$ such that $f \circ g \simeq id_Y$, $g \circ f \simeq id_X$

($\neq$, $\neq$: $\exists$ inverse up to homotopy)

Def (Homotopy part 4)

X is **homotopy equivalent** to Y if $\exists\, f : X \to Y$ a homotopy equivalence

$X \simeq Y$ in this case

Homotopy equivalence is an equivalence relation

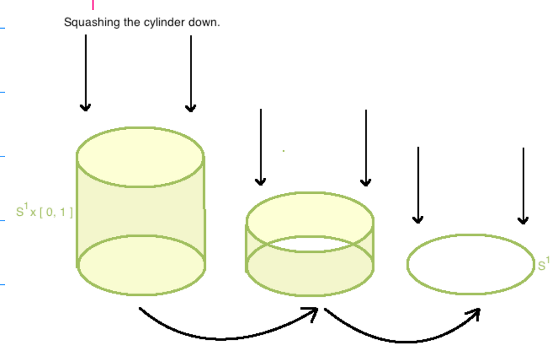
$\rightsquigarrow$ homotopy equivalence classes

Examples

i) $f \simeq f$ by taking $g_t = f \;\; \forall t$

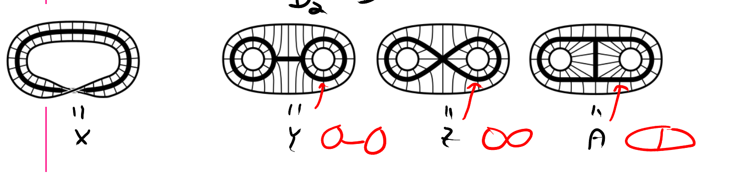
ii) $X \simeq X$ by taking id as a homotopy equivalence

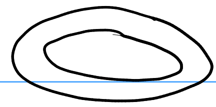
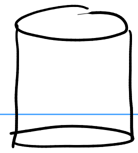
iii) Any $f : X \to Y$ homeomorphism is a homotopy equivalence $\Rightarrow$ $X \cong Y \Rightarrow X \simeq Y$ ($\not\Leftarrow$)

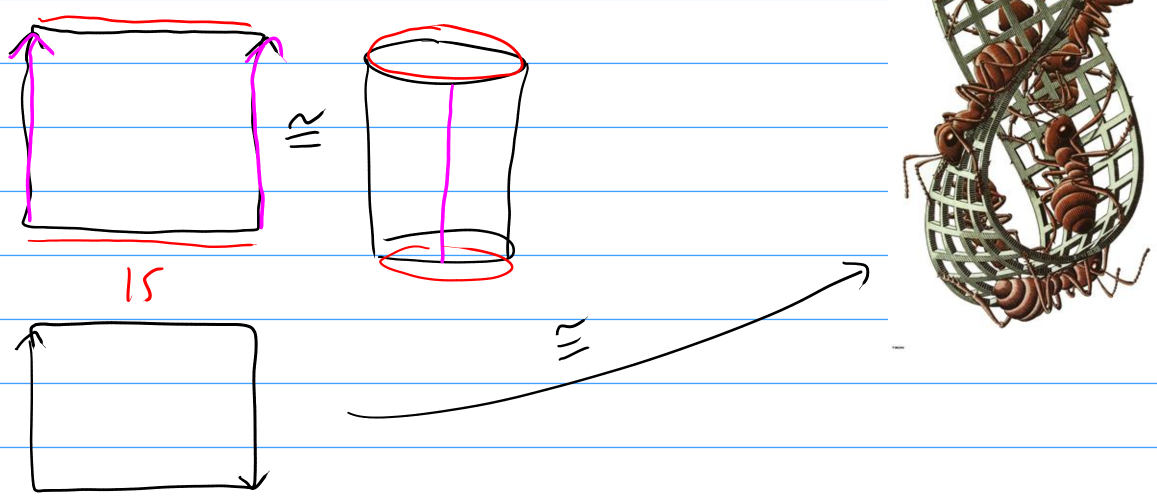


iv) $S^1 \times [0,1] \simeq S^1$

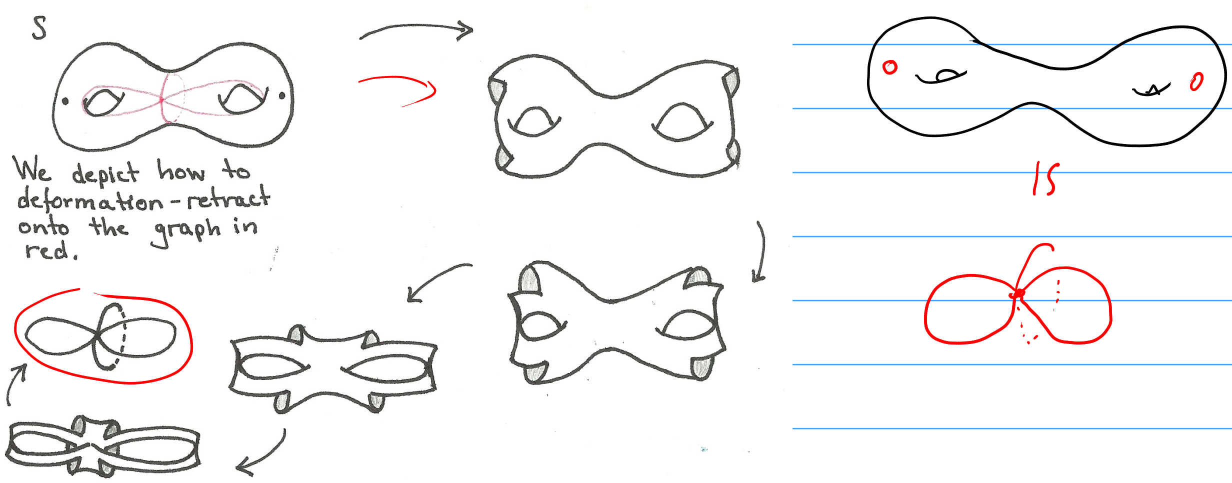
v) $Y \simeq Z \simeq A \simeq D_2$



vi) $X \simeq S^1 \simeq$  $\simeq$ 

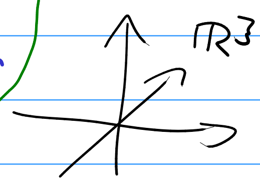


VII)

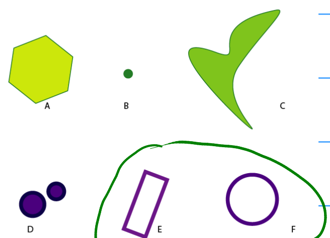


Def A space X is <u>contractible</u> if $X \simeq \{*\}$ ← point

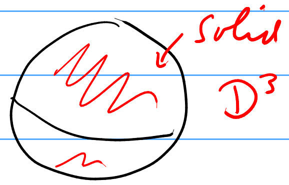
Examples i) $\mathbb{R}^n$ is contractible ($\mathbb{R}^n \simeq \mathbb{R}^m$; $\mathbb{R}^n \cong \mathbb{R}^m \iff n = m$)



ii) $D^n \simeq \{*\}$ contractible
↑ n-dim disc

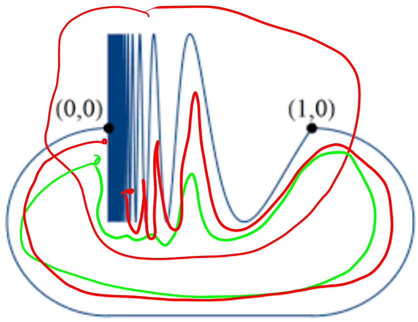
iii) 

A + B + C are all contractible



D is not contractible because $D \simeq \{*, ☆\}$

E, F are not contractible but that is not obvious iv)



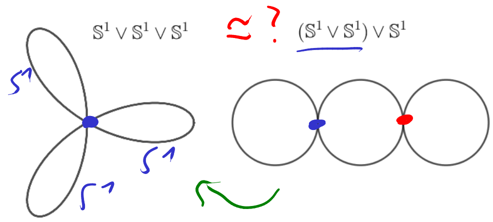
Is this contractible?

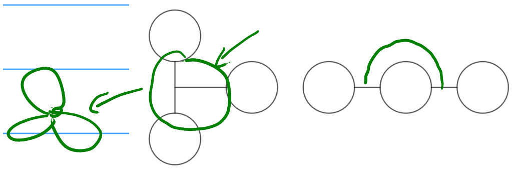
No this is not contractible!

We need some machinery to think about this!

⇝ algebraic invariance!

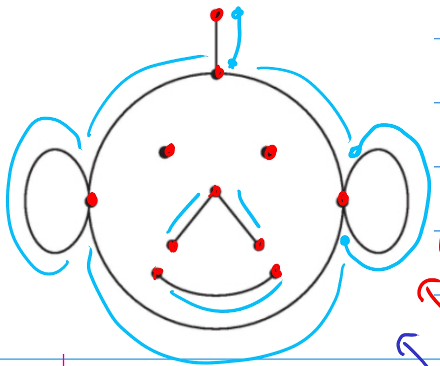
Are these $\simeq$?





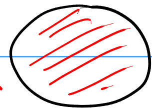
Part 2 CW-complex Cell complex ← basic building blocks

↪ Closure finite weak topology



Idea: Built inductively · ← D^0

- Zero cells aka points ⇝ 0 skeleton
- Glue in 1-cells aka -1 ·—· 1 Intervalls (D^1) ⇝ 1 skeleton
- glue in 2-cell aka



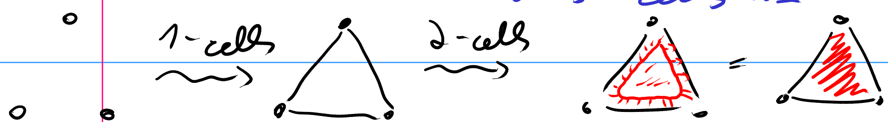
← 11 0-cells

← 9 1-cells

← 0 2-cells

0 3-cells ...

$L_n = \emptyset \quad \forall n > 1$



Def (Cell complex)

X is a cell complex if its constructed inductively as follows:

(1) X^0 finite number of points with discrete topology

X^0 is called 0-skeleton

(2) Assume that we have constructed X^{m-1}

Then choose a family of maps $\{\varphi_\alpha : S^{m-1} \to X^{m-1} \mid \alpha \in L_m\}$

To each $\alpha \in L_m$ we associate a copy of D^m_α. ← m-cells   ↖ δD^m

It's allowed that $L_m = \emptyset$

$\rightsquigarrow \varphi = \coprod_\alpha \varphi_\alpha : \coprod_\alpha \delta D^m_\alpha \to X^{m-1}$

Define X^m (m skeleton) to be

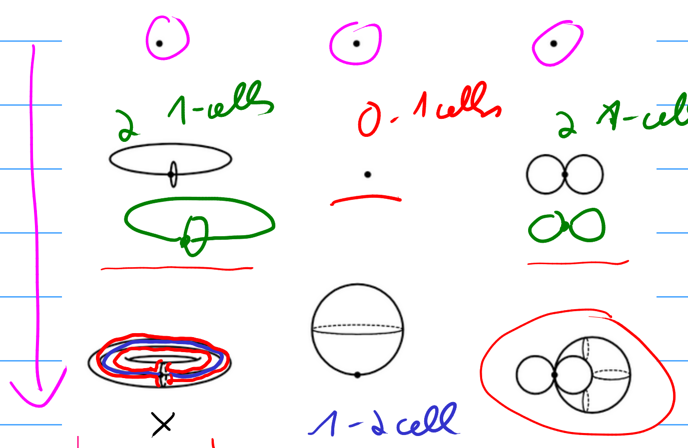
$$X^m = \left(X^{m-1} \amalg \coprod D^m_\alpha\right) / \varphi$$

(2') For $X = \bigcup_n X^n$ ($\coprod$ ✗) use the topology that $A \subset X$ is open $\Leftrightarrow$ $A \cap X^n$ is open in X^n $\forall n$

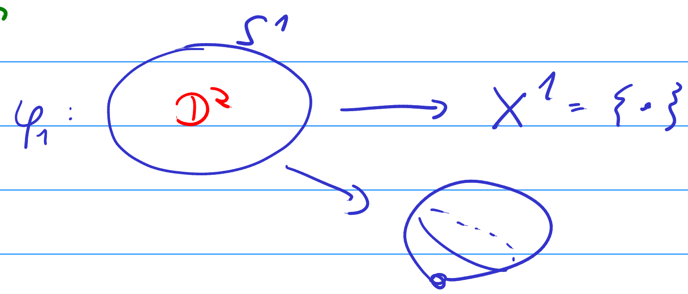
A cell complex is finite if

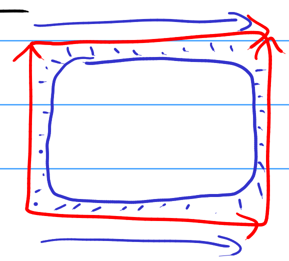
- $|L_m| < \infty$ $\forall m$
- $\exists N$ such that $X_N = X_{N+k}$ $\forall k \in \mathbb{N}$

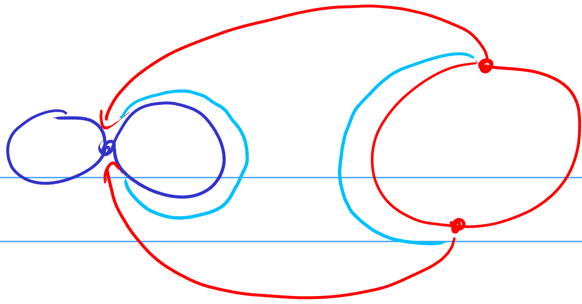
If X is finite, then $X = X_n$ for the last $L_n \neq \emptyset$

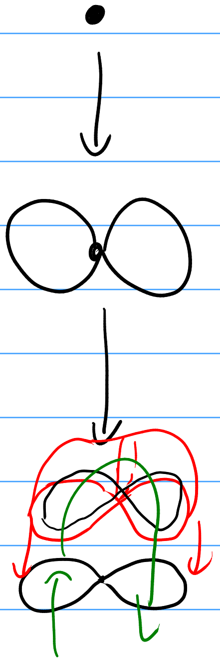


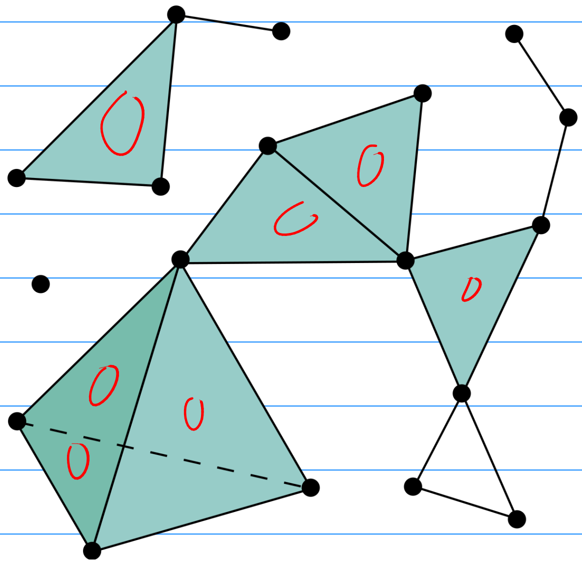
$X = X^2$





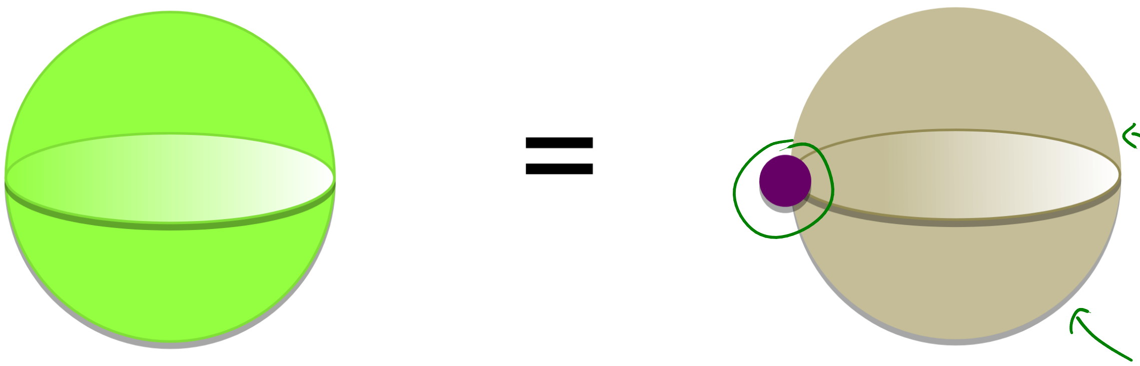






7 2-cells

Careful: Being a cell complex is a structure!



1 - 0 cell
0 1-cells
1 - 2-cell

1 - 0 + 1 = 2

	TETRAHEDRON	CUBE	OCTAHEDRON	DODECAHEDRON	ICOSAHEDRON
2-cells →	4 FACES	6 FACES	8 FACES	12 FACES	20 FACES
0-cells →	4 VERTICES	8 VERTICES	6 VERTICES	20 VERTICES	12 VERTICES
1-cells →	6 EDGES	12 EDGES	12 EDGES	30 FACES	~~30 FACES~~ Edges

12 - 30 + 20 = 2

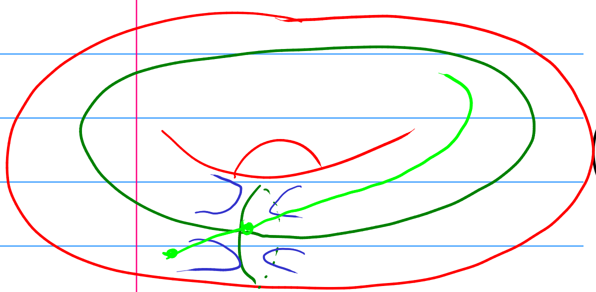
finite cell complex

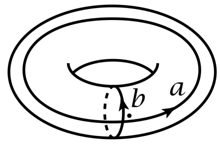
Def The Euler characteristic $\chi(X)$ is defined as $\chi(X) = \#\,0\text{-cells} - \#\,1\text{-cells} + \#\,2\text{-cells}$

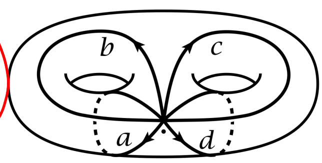
$+ \dots$

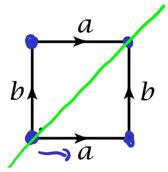
Theorem χ is a homotopy invariant, that is $X \simeq Y \Rightarrow \chi(X) = \chi(Y)$ for any choice of cell structures on X and Y

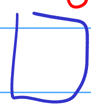
Examples: i)



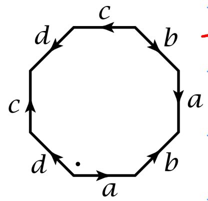




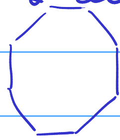


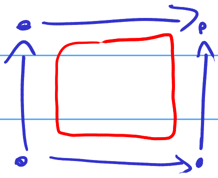


0 { 1 0-cell, 2 1-cells, 1 2-cell }



−2 { 1 0-cell, 4 1-cells, 1 2-cell }





−4 { 1 0-cell, 6 1-cells, 1 2-cell }

12-gon

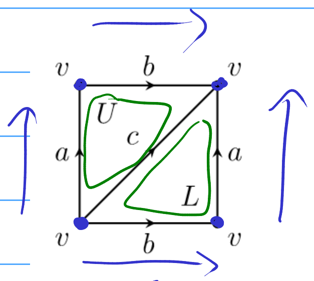
16-gon

−6

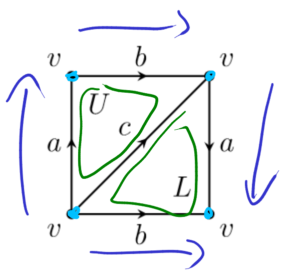
−8

⇒ all of these are not homotopy equivalent!

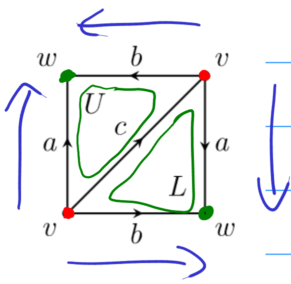
ii)



T



K



$\mathbb{R}P^2$

2 0-cells

3 a+b+c 1-cells

U, L 2-cells

2

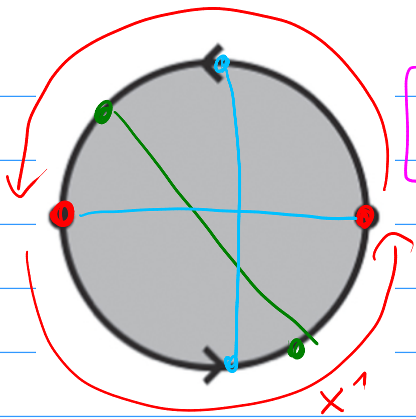
→ $\chi(\mathbb{R}P^2) = 1$

$\chi(T) = 0$ [1 0-cell, 3 1-cells, 2 2-cells]

$\chi(K) = 0$

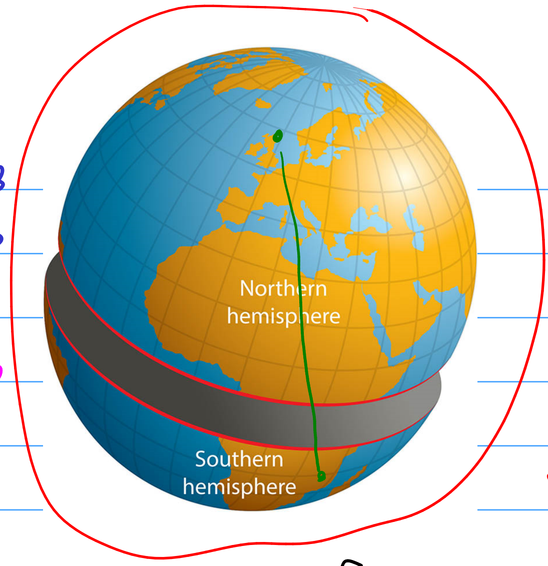
⇒ $T, K \not\simeq \mathbb{R}P^2$ but we can not say anything about $T \not\simeq K$

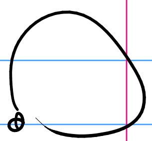
(ii)

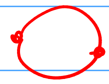


$\begin{cases} 2 \text{ 0-cells} \\ 2 \text{ 1-cells} \end{cases} \rightsquigarrow 0$

$\chi(S^1) = 0$







$\uparrow S^1$ $\qquad$ $\uparrow S^2$

$\chi(S^2) = 2 \leftarrow \begin{cases} 2 \text{ 0 cells} \\ 2 \text{ 1 cells} \\ 2 \text{ 2 cells} \end{cases}$

$\mathbb{R}^{n+1}$
$\cup$

$$S^n/(v\sim -v) \cong \mathbb{RP}^n$$

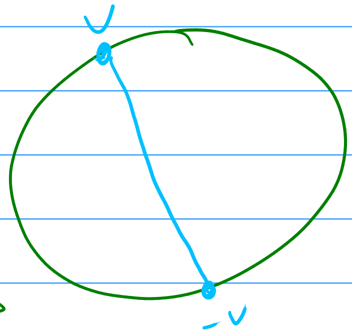
$\mathbb{RP}^1 = S^1/v\sim -v$ $\quad$ 1 0-cell, 1 1-cell

$\mathbb{RP}^2 = S^2/v\sim -v$ $\quad$ 1 0-cell, 1 1-cell, 1 2-cell

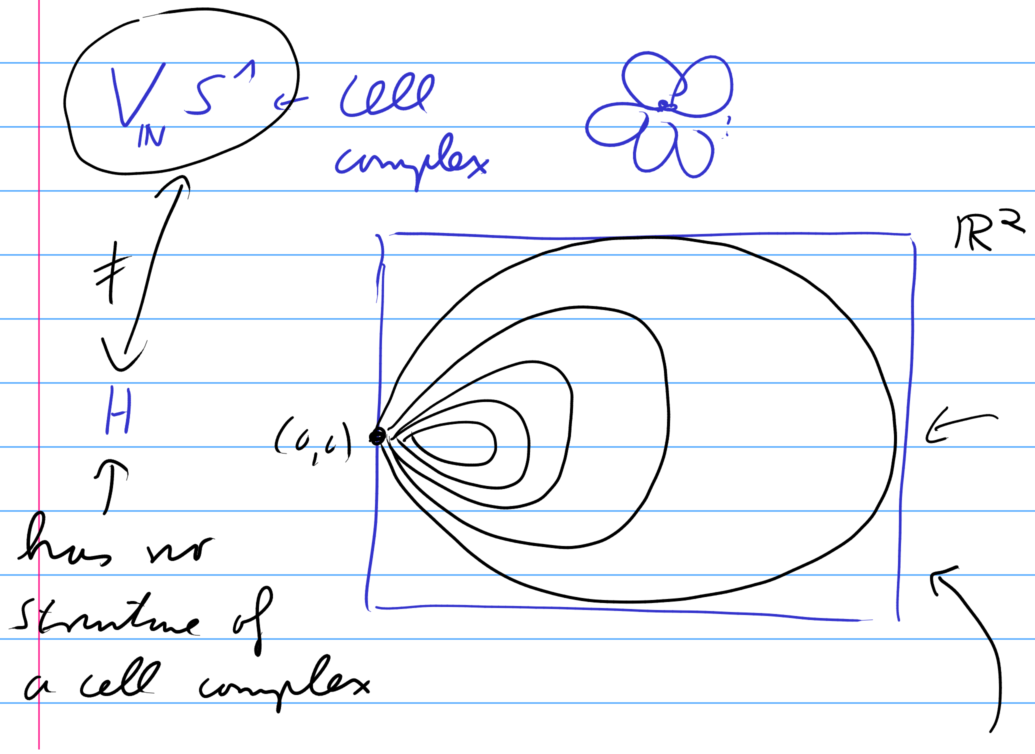
$\mathbb{RP}^n = S^n/v\sim -v$ $\quad$ 1 k-cell for all $k = 0, \ldots, n$

$\mathbb{C}$

$S^1 = \{v \in \mathbb{C}, |v| = 1\}$



$v \sim -v$



Proposition A.1. *A compact subspace of a CW complex is contained in a finite subcomplex.*

Now we can explain the mysterious letters 'CW', which refer to the following two properties satisfied by CW complexes:

(1) Closure-finiteness: The closure of each cell meets only finitely many other cells. This follows from the preceding proposition since the closure of a cell is compact, being the image of a characteristic map.

(2) Weak topology: A set is closed iff it meets the closure of each cell in a closed set. For if a set meets the closure of each cell in a closed set, it pulls back to a closed set under each characteristic map, hence is closed by an earlier remark.

Proposition A.4. *Each point in a CW complex has arbitrarily small contractible open neighborhoods, so CW complexes are locally contractible.*

Corollary A.12. *A compact manifold is homotopy equivalent to a CW complex.*