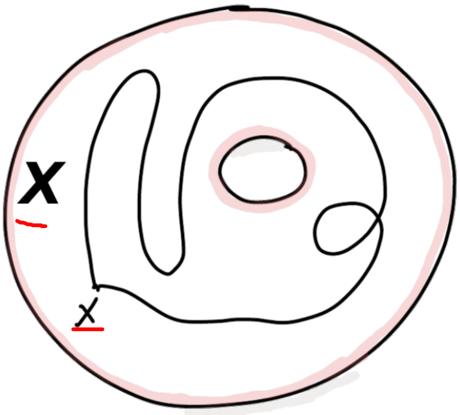
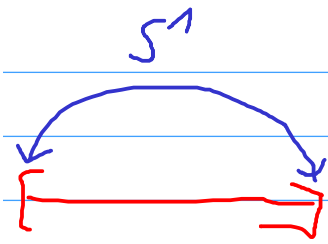
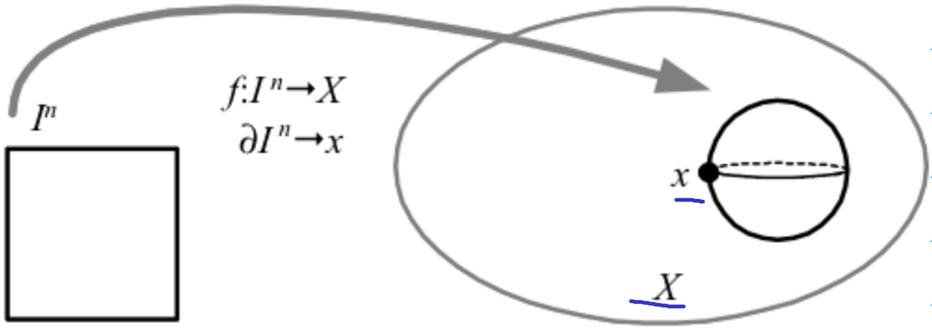
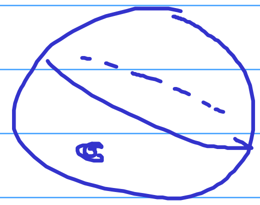




← stop at dim 2

- The fundamental group measures how one can arrange loops in spaces
- Formally, maps $f : [0,1] \to X$ such that $f(0) = f(1)$ Ends glued





π_* π_n $n \in \mathbb{N}$

- The homotopy group π_n measures how one can arrange n-spheres in spaces
- Formally, maps $f : [0,1]^n \to X$ such that $f(\delta[0,1]^n) = x$ Boundary glued
- Note that the fundamental group is the case $n = 1$ S^1 is a loop

Eckmann - Hilton argument "non 1-dim setup"

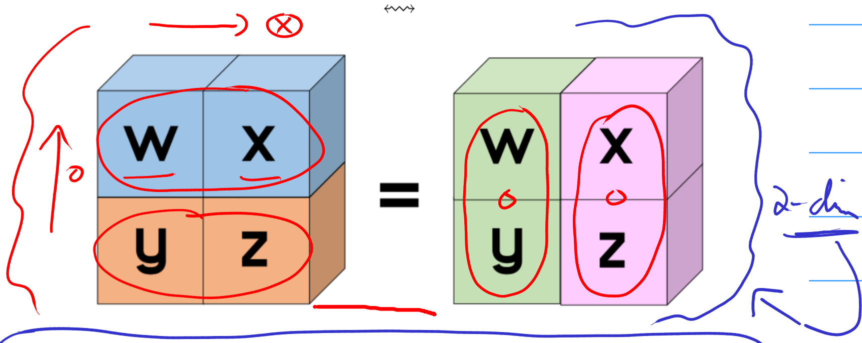
X a set with two binary operations $\circ$ ("vertical") and $\otimes$ ("horizontal") satisfying:

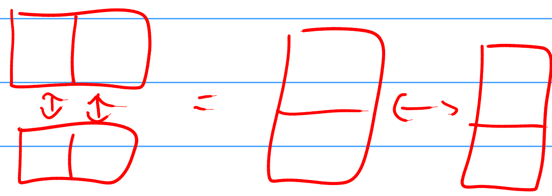
$\circ : X \times X \to X$
$\otimes : X \times X \to X$

(a) They are unital Empty space ←

(b) They satisfy a 2-dimensional compatibility condition

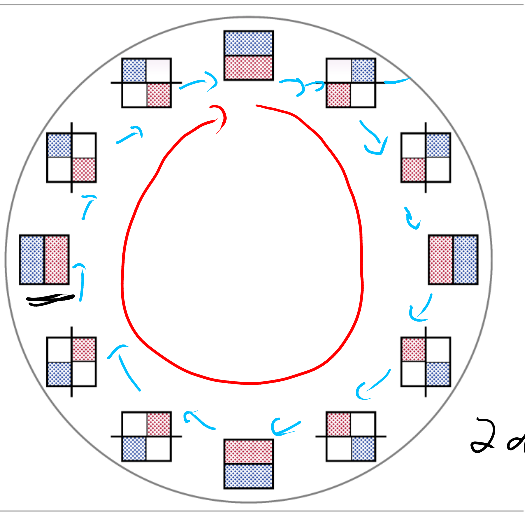
$$(w \otimes x) \circ (y \otimes z) = (w \circ y) \otimes (x \circ z)$$

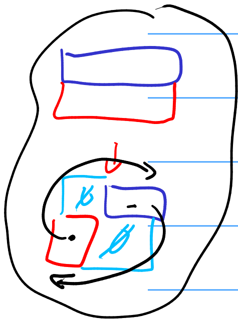




Then $\circ$ and $\otimes$ are the same and in fact commutative and associative

The Eckmann–Hilton clock



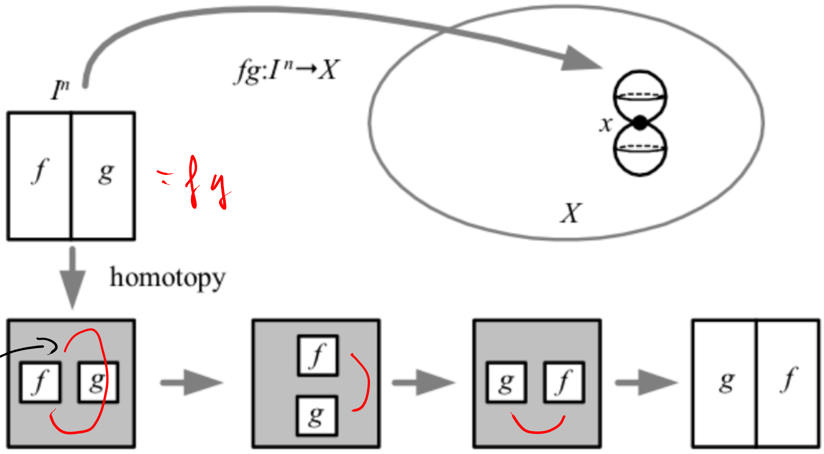


2 dim "trick"

Proof without words

Claim: π_n forms a group ✓

π_n is commutative if $n>1$



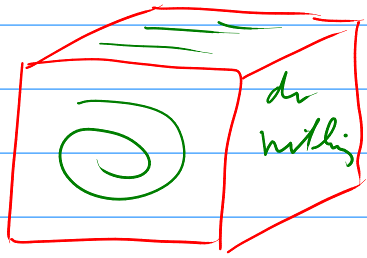
do nothing

$= fg$

$= gf$

$[0,1]\times[0,1]$

$[0,\frac{1}{2}] - [\frac{1}{2},1]$



$\pi_n \underset{n>1}{\rightsquigarrow} \mathbb{Z}^n \oplus \mathbb{Z}/p_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/p_k\mathbb{Z}$

Huge difference to π_1!

- The Eckmann–Hilton argument shows that this is commutative for $n \geq 2$
- "Classical operations are 1-dimensional, and commutativity is lost"

$\hookrightarrow$ really diff from π_1!

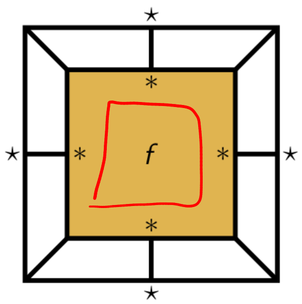
For a topological space X take spheres $f\colon [0,1]^n \to X$ based at $\star \in X$, *i.e.* $f(\delta[0,1]^n) = \star$

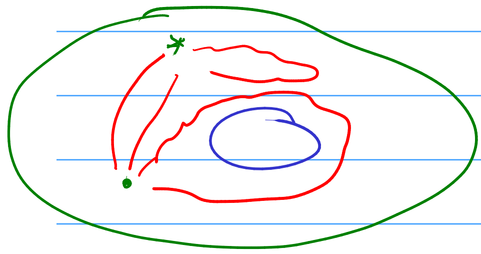
(a) Let $\pi_n(X, \star)$ be the set of equivalence classes of spheres based at $\star$ modulo homotopy

(b) $\pi_n(X, \star)$ has a group structure given by concatenation

same as for π_1

- Slight catch. This is only a group structure by using homotopy
- For path connected X we have $\pi_n(X, \star) \cong \pi_n(X, *)$ Write $\pi_n(X)$





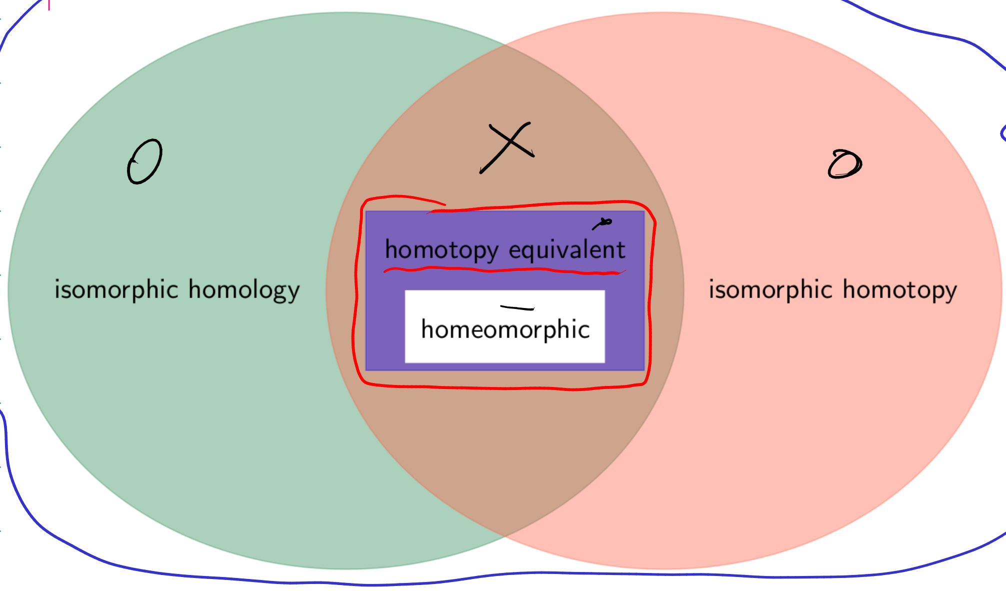
- $(X \simeq Y) \Rightarrow (\pi_n(X) \cong \pi_n(Y)$ for all $n)$ Invariance

~) same as for π_1

Question? Why haven't I explained before H_*?

group?

What is great about homotopy?



spaces

- Homotopy equivalence induce isomorphisms in homotopy/homology
- Question What about the converse?

Whitehead's theorem

For ~~connected~~ cell complexes X, Y and $f\colon X \to Y$ the following are equivalent:

(a) $f\colon X \to Y$ is a homotopy equivalence Topology

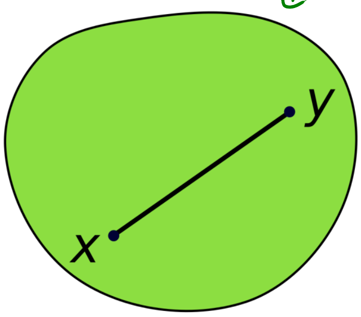
(b) $f_*\colon \pi_*(X) \to \pi_*(Y)$ is an isomorphism Algebra

$\forall n$

$$f \begin{array}{cccc} \pi_0 & \pi_1 & \pi_2 & \pi_3 \\ \downarrow\cong & \downarrow\cong & \downarrow\cong & \downarrow\cong \end{array}$$

The good

cell complex

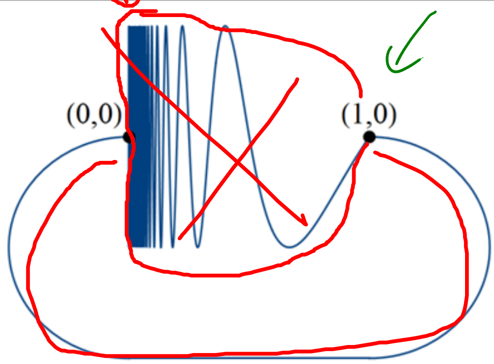
$X =$ 

- The space X has trivial homotopy $\pi_*(X) \cong 0$
- The space X is trivial $X \simeq$ point

The bad

$\sin\left(\frac{1}{x}\right)$

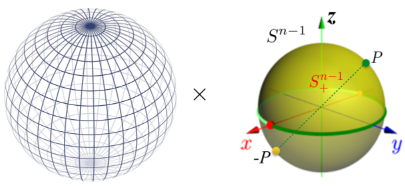
not

$W =$ 

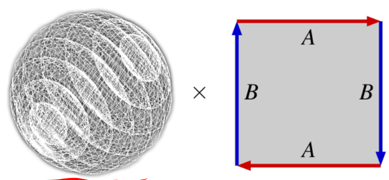
- The Warsaw circle W has trivial homotopy $\pi_*(W) \cong 0$
- The Warsaw circle W is not trivial $W \not\simeq$ point

The ugly

2 3

$X = S^2 \times \mathbb{R}P^3 \cong$ 

3 2

$Y = S^3 \times \mathbb{R}P^2 \cong$ 

- The connected cell complexes X, Y have the same π_* $X \simeq Y$ by Whitehead?
- The connected cell complexes X, Y have different H_* $X \not\simeq Y$!
- What fails? There is no $f\colon X \to Y$ inducing all isomorphisms

For connected cell complexes X, Y and $f\colon X \to Y$ the following are equivalent:

(a) $f\colon X \to Y$ is a homotopy equivalence Topology

(b) $f_*\colon \pi_1(X) \to \pi_1(Y)$ is an isomorphism and (some) $\tilde{f}\colon \tilde{X} \to \tilde{Y}$ gives an isomorphism $\tilde{f}_*\colon H_*(\tilde{X}) \to H_*(\tilde{Y})$ Algebra

What is not great about homotopy?

Easier than the fundamental groups? No!

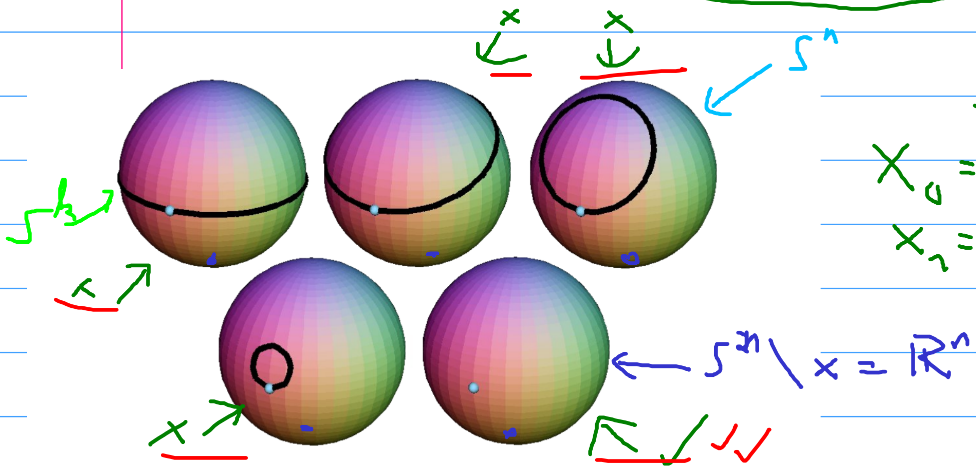
π

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$

S^n known π_1 0 0 $\neq$

- The south west part is "obvious" Not exciting
- The north east part is still mostly unknown
- Not even all $\pi_n(S^2)$ are known Even computing $\pi_3(S^2)$ is a challenge

Hopf



S^1 S^2

$X_0 = *$ $X_1 = S^1$

$X_0 = *$ $X_1 = *$ $X_2 = S^2$

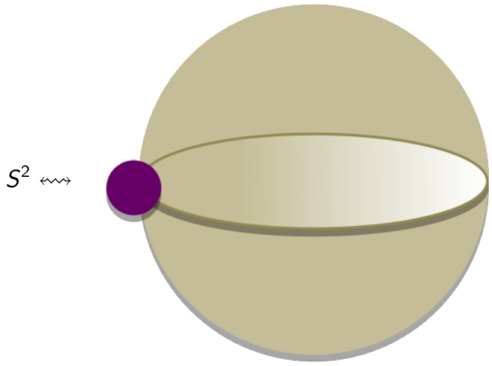
- Goal Show $\pi_k(S^n) \cong 0$ for $k < n$
- Strategy Poke a hole into S^n and contract the rest along with $S^k \to S^n$
- Catch Need to show that any $S^k \to S^n$ misses a point

S^k S^n

A map $f\colon X \to Y$ between cell complexes is called cellular if

$$f(k\text{-skeleton}) \subset k\text{-skeleton} \quad \forall k$$

Every map between cell complex is homotopic to a cellular map



- Take the balloon cell structure on S^n One 0- and one n-cell
- $S^k \to S^n$ can be assumed to end in the k-skeleton of S^n
- The k-skeleton of S^n is trivial for $k < n$ Done!

The idea of homology is pervasive in mathematics!

Group homology [edit]

Main article: Group cohomology

In abstract algebra, one uses homology to define derived functors, for example the Tor functors. Here one starts with some covariant additive functor *F* and some module *X*. The chain complex for *X* is defined as follows: first find a free module F_1 and a surjective homomorphism $p_1 : F_1 \to X$. Then one finds a free module F_2 and a surjective homomorphism $p_2 : F_2 \to \ker(p_1)$. Continuing in this fashion, a sequence of free modules F_n and homomorphisms p_n can be defined. By applying the functor *F* to this sequence, one obtains a chain complex; the homology H_n of this complex depends only on *F* and *X* and is, by definition, the *n*-th derived functor of *F*, applied to *X*.

A common use of group (co)homology $H^2(G, M)$is to classify the possible extension groups *E* which contain a given *G*-module *M* as a normal subgroup and have a given quotient group *G*, so that $G = E/M$.

Other homology theories [edit]

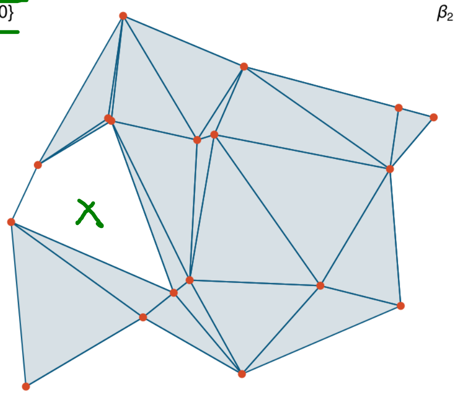
- Borel–Moore homology
- Cellular homology
- Cyclic homology
- Hochschild homology
- Floer homology
- Intersection homology
- K-homology
- Khovanov homology
- Morse homology
- Persistent homology
- Steenrod homology

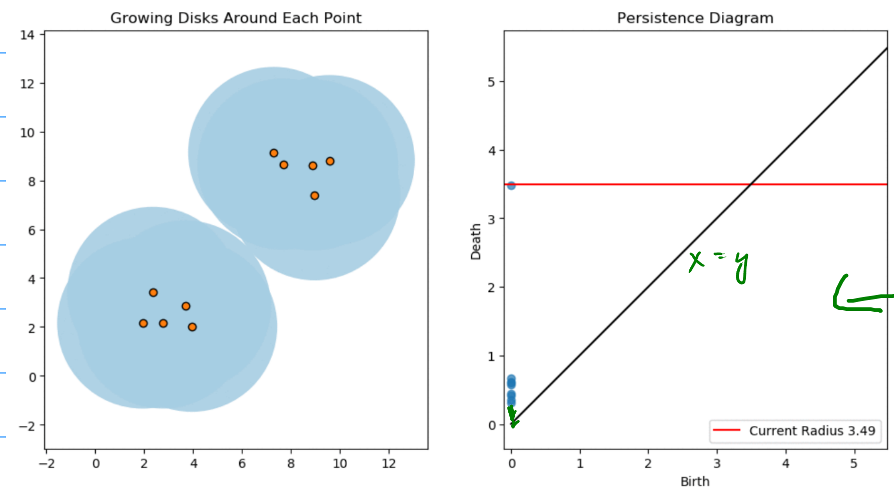
link
knot

- As one increases a threshold, at what scale do we observe changes in data?
- There are many different flavors
- Today Discrete points in $\mathbb{R}^n$

$H_0 \cong \mathbb{Z}$
$H_1 \cong \mathbb{Z}$
$H_2 \cong \{0\}$

$\beta_0 = 1$
$\beta_1 = 1$
$\beta_2 = 0$



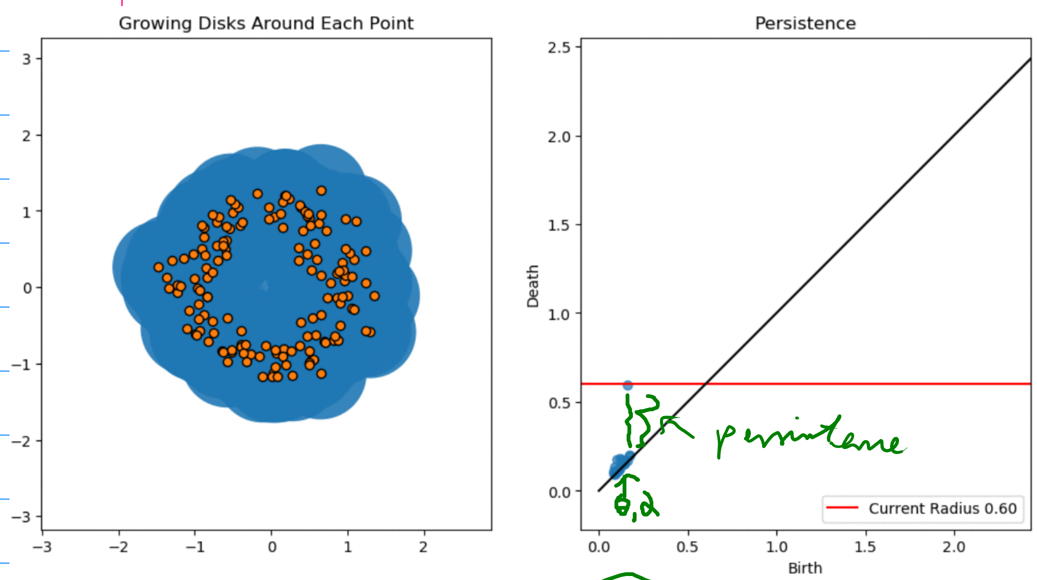


Compute homology

Gen. in H_0 ← H_n

Born

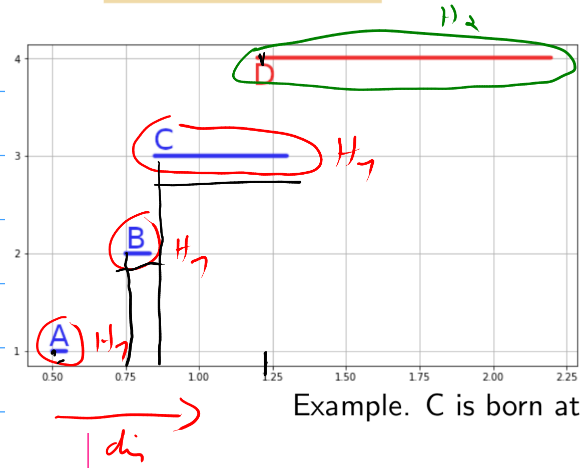
- The 0th persistent homology measures how connected components change
- Birth New 0d holes=connected components (all born at zero at $y = x$)
- Death 0d holes=connected components vanish

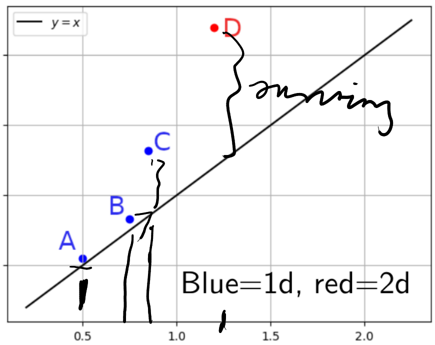


- The 1th persistent homology measures how internal circles change
- Birth New 1d holes=internal circles
- Death 1d holes=internal circles vanish

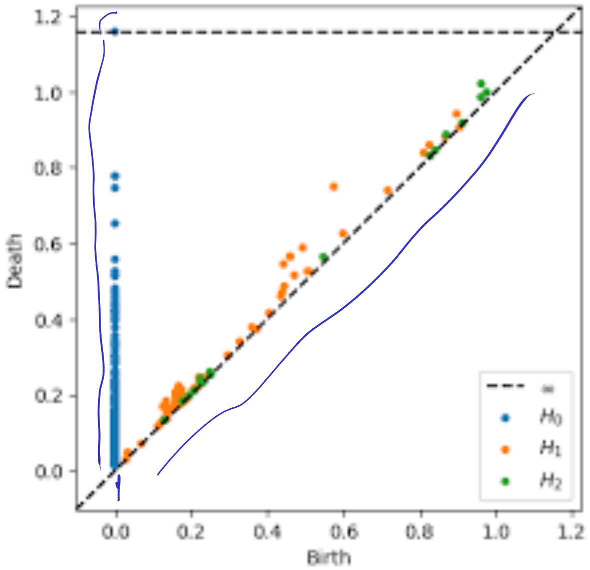
Persistence diagram

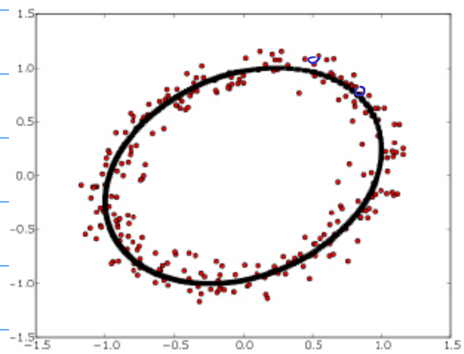
Persistent nd holes are far-away from $y = x$

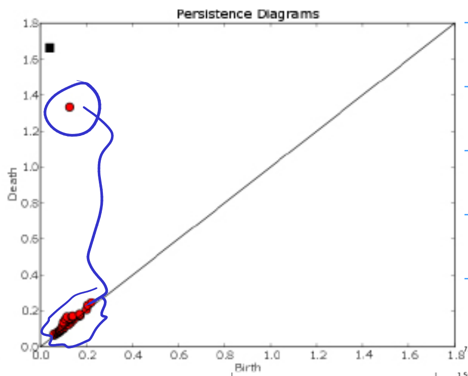


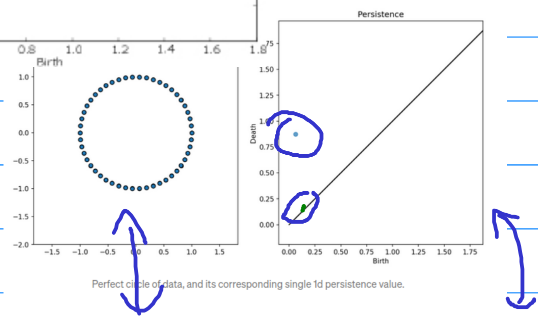


Example. C is born at 0.8 and dies at 1.3

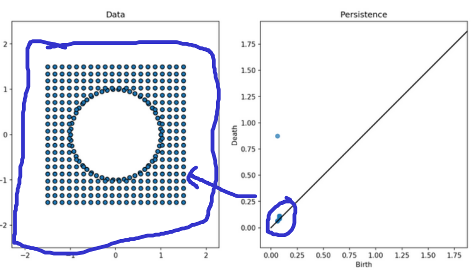




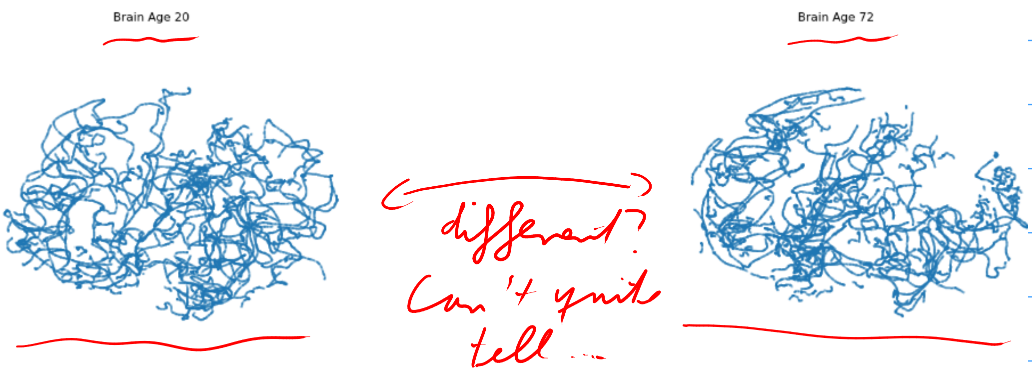




Perfect circle of data, and its corresponding single 1d persistence value.



- **Homology** proved useful in detecting age differences in brain artery trees
- **Idea** Render brain artery trees into point-clouds and use persistent homology
- Differences are subtle – like most differences in human brains – but measurable



2 brain artery trees. On the left, a 20-year old. On the right, a 72-year old.