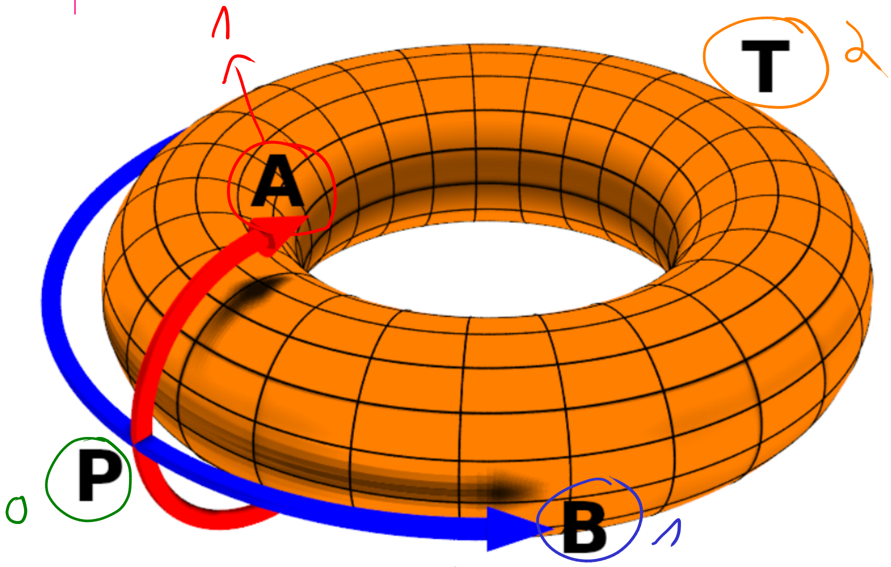
Cohomology ring



$H_0(T) \cong \mathbb{Z} \leftrightsquigarrow [P]$, $H_1(T) \cong \mathbb{Z}^2 \leftrightsquigarrow [A], [B]$, $H_2(T) \cong \mathbb{Z} \leftrightsquigarrow [T]$

In good cases: k-dim submanifolds $/\simeq$ $\Leftrightarrow$ gen. of H_k

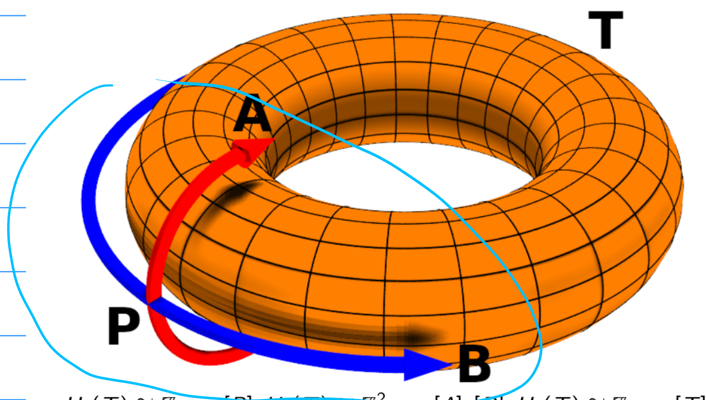
basis of H_k elements

- In good cases generators of $H_k(X)$ correspond to k-dimensional submanifolds
- We should be able to use this information to say more about X

$[T \cap A]$

$[T] \cap [A] = [A]$

$[T] \cap [B] = [B] = [T \cap B]$



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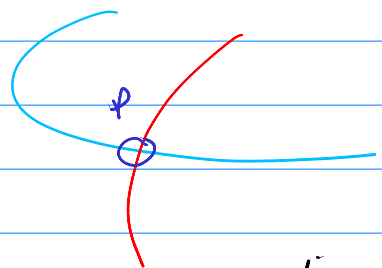


intersection product

$$[A] \cap [B] = \pm [P] = [B] \cap [A]$$

algebra

$T \rightsquigarrow 2$ $\quad$ $A \cap B = P$ $\quad$ $2-1-1=0$

$A \rightsquigarrow 1$

$B \rightsquigarrow 1$

$[A \cap B] =: [A] \cap [B]$

$\rightsquigarrow$ Here $[T]$ is the unit of $\cap$.

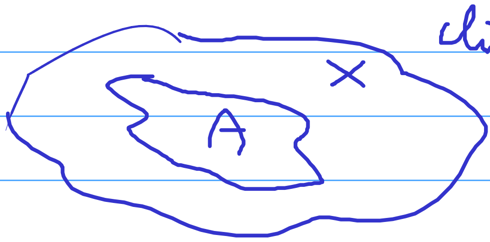
Idea Submanifolds generically intersect in submanifolds $\Rightarrow$ get a product

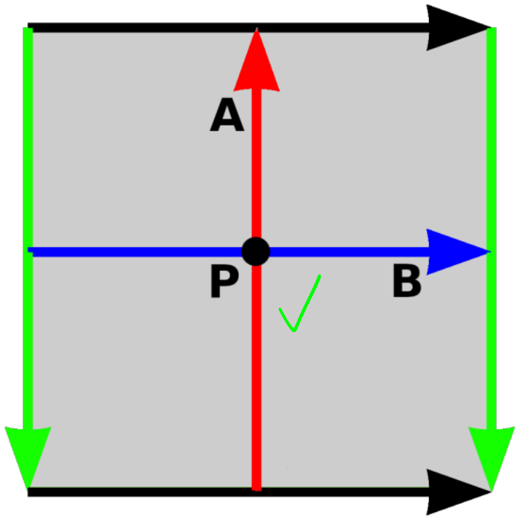
$$\cap\colon H_{n-k}(X) \times H_{n-l}(X) \to H_{n-k-l}(X)$$

(dim $X = n$) and (Codimension k $\cap$ codimension l = codimension $k+l$)

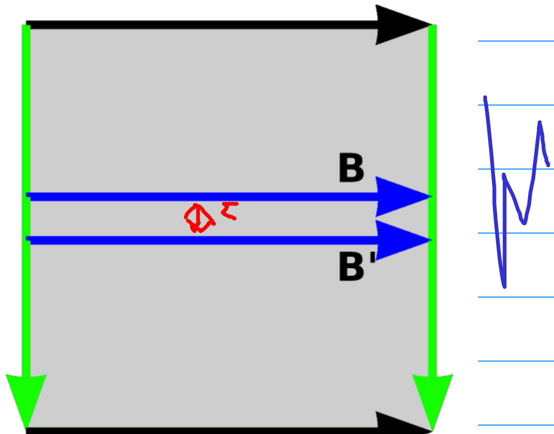
dim $A = k$ dim $B = l$

$A \cap B$ codim $n-k-l$





$[A] \cap [B] = \pm[P]$



$[B] \cap [B] = 0$

$\cap$ gives $H_*(T)$ a ring structure $H_\bullet(T)$:

$$H_\bullet(T) \xrightarrow[{[A],[B]\mapsto A,B}]{\cong} \mathbb{Z}\langle A, B\rangle/(A^2 = B^2 = 0, AB = -BA)$$

$\cong \Lambda \mathbb{Z}^2$ dim T $\Lambda V = \Lambda^0 V \oplus \Lambda^1 V \oplus \Lambda^{\dim V}$

$\to A, B$

$\mathbb{Z} \wedge \mathbb{Z} = 1$

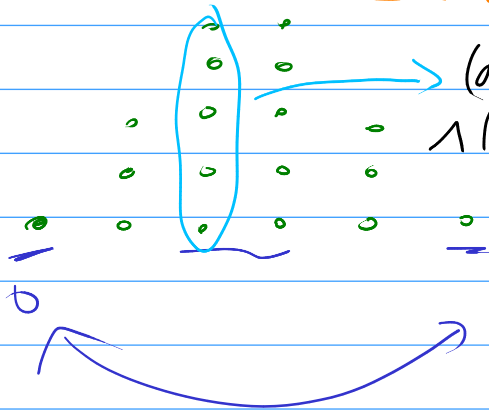
$\emptyset, A, B, A \wedge B = -B \wedge A$

$A \wedge A = 0 = B \wedge B$

$$H_\bullet(T^n) \cong \bigwedge \mathbb{Z}^n, \quad \operatorname{rank} H_k(T^n) = \binom{n}{k} = \operatorname{rank} \bigwedge^k \mathbb{Z}^n$$

$\Lambda \mathbb{Z}^n = \bigoplus \Lambda^k \mathbb{Z}^n \to a_1 \wedge \dots \wedge a_k$

$2^n = \sum \binom{n}{k}$



$\binom{n}{n/2}$

$(a_1 \wedge \dots \wedge a_k) \wedge (b_1 \wedge \dots \wedge b_\ell) = a_1 \wedge \dots \wedge b_\ell$

"||"

$a_1 \dots b_\ell$

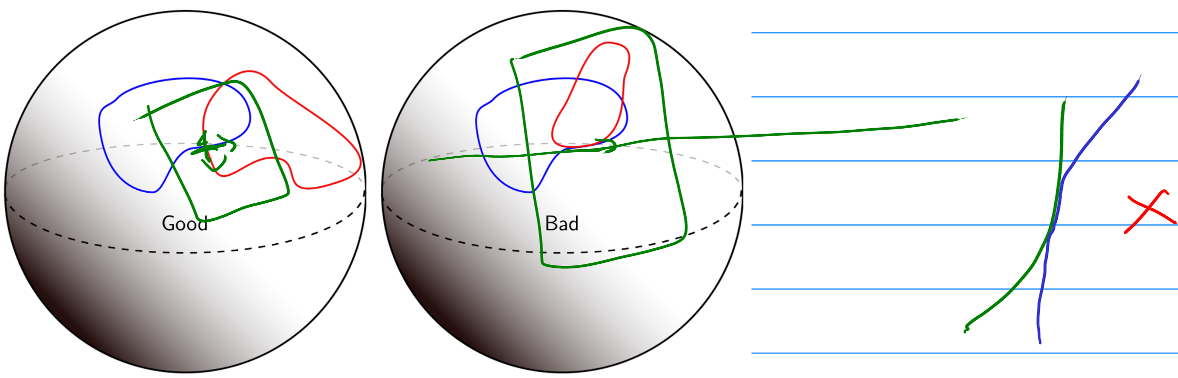
$\boxed{ab = -ba}$!

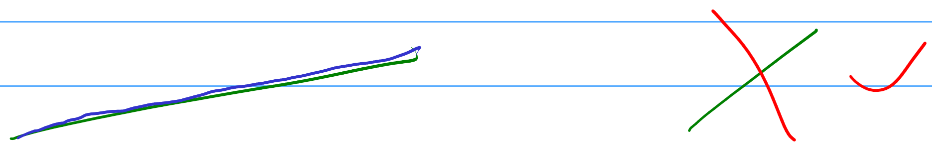
$\Lambda \mathbb{Z}^n$ "=" anti. com. polys in n-gens

$\Lambda^k \mathbb{Z}^n$ "=" " " of deg k

$A \cap B$ = empty is transversal

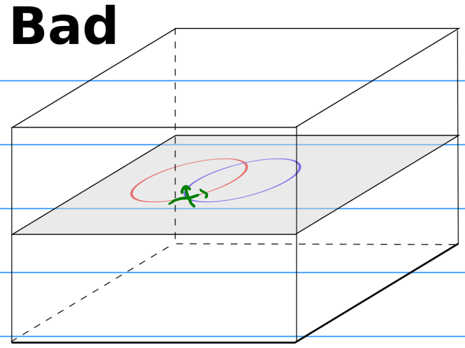
Transverse intersection For every $p \in A \cap B$ the map of tangent bundles $T_pA \oplus T_pB \to T_pX$ induced by the inclusions is surjective (This is only defined under appropriate smoothness conditions)

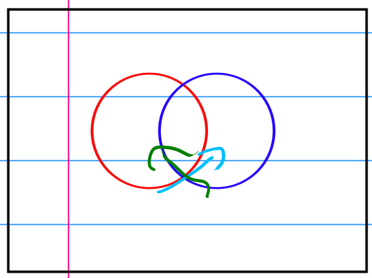




$\mathbb{R}^2$ $\mathbb{R}^3$

Bad





Good

If A, B intersect transversely in X, then $A \cap B$ is a submanifold of the expected dimension:

$$\dim(A \cap B) = \dim A + \dim B - \dim X$$

$$-(k + l) + n$$

Some flaws, fixed by the "correct definition":

- This does not work for all spaces X e.g. X not a manifold ...
- $\cap$ has a non-intuitive multiplication direction
- One needs to be careful what generically (transversely) intersect means

Prototypical ring: $\mathbb{Z}[x]$ polynomial

↳ deg d $\quad x_1 = (1, 0, \ldots)$ $\quad x_2 = (0, 1, \ldots)$

$x^2 \rightsquigarrow$ deg $2d$ $\qquad \Lambda\mathbb{Z}^d$ ↳ "signed" polynomial in d variables

$\rightsquigarrow$ correct setups are "graded commutative":

$$ab = (-1)^{\deg a + \deg b}\, ba \qquad \deg a = d,\ \deg b = e$$

Slogan Cohomology rings "are" polynomial rings

$\mathbb{Z}[x]$, $\quad \mathbb{Z}[x]/(x^2 = 0)$

$\mathbb{Z}[x]/(x^2 - 1 = 0)$

as VS these are the same

$\mathbb{Q}[X]/(X^2) \rightsquigarrow \{1, X\} \leftarrow$ Basis of $\mathbb{Q}^2 \cong$

$\mathbb{Q}[X]/(X^2-1) \rightsquigarrow \{1, X\} \leftarrow$ Basis of $\mathbb{Q}^2 \cong$

$\mathbb{Q}^2$ $\qquad$ $\mathbb{Q}^2$

$H^*(X) \rightarrow \mathbb{Q}[X]/(X^2) \underset{\text{rings}}{\not\cong} \mathbb{Q}[X]/(X^2-1) \leftarrow H^*(Y)$

$\underline{X^2=0}$ $\qquad$ $X \longmapsto ??$ $\qquad$ no nil elements

nilpotent

$\Rightarrow X \not\cong Y$

Want: $H^*(X)$ + ring structure being $\cong$ invariant

!! Also $H^*(X)$ turn out to be "nice" rings and "nice" rings come from nice spaces

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Slogan Cohomology rings "are" polynomial rings

- Polynomial can be multiplied:

$$f(X) = X + 1, g(X) = X - 1 \Rightarrow (fg)(X) = X^2 - 1$$

- This immediately generalizes to functions with values in a ring R:

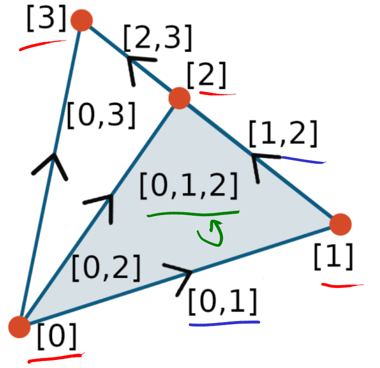
$$(fg)(r) = f(r)g(r) \text{ (product in } R)$$

- Cochains are functions on chains with values in $\mathbb{Z}$, so

$\mathrm{Hom}(C_*, \mathbb{Z}) = C^*$

$$\text{"}(f \smile g)(\sigma) = f(\sigma)g(\sigma)\text{"}$$

This is almost the definition of the cup product $\smile$

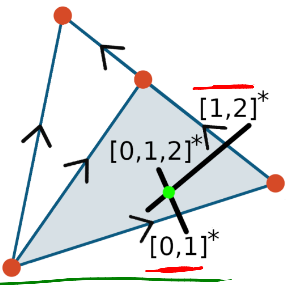


$$[0,1]^*([a,b]) = \begin{cases} 1 \text{ if } [a,b] = [0,1] \\ 0 \text{ else} \end{cases}$$

- n-chains $\rightsquigarrow$ n-simplices, *e.g.* $[0,1]$ Basis
- n-cochains $\rightsquigarrow$ n-cosimplices, *e.g.* $[0,1]^*$ The dual basis

$[0,1]^* \smile [1,2]^* = [0,1,2]^* \rightsquigarrow$

(σ) (v)



$[0]^* \in C^0$

$[0,1]^* \in C^1$

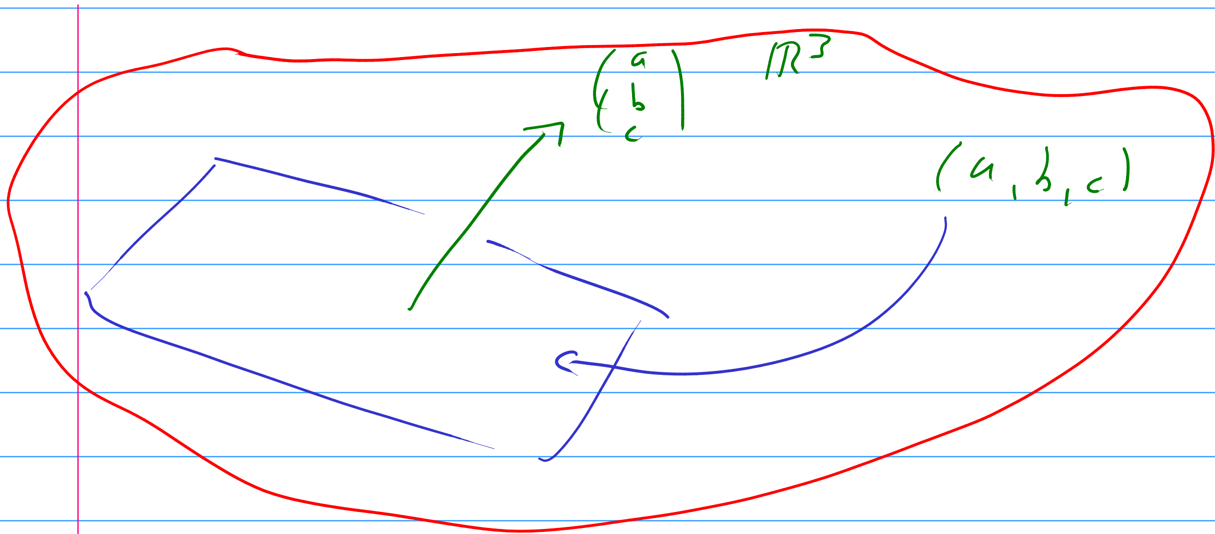
$\smile : C^k \times C^l \to C^{k+l}$

- Multiply $f \in C^k(X)$ $k+1$ inputs and $g \in C^l(X)$ $l+1$ inputs:

C_{k+l+1}

$$(f \smile g)(\sigma) = f(\sigma|[v_0 \ldots, v_k])g(\sigma|[v_k \ldots, v_{k+l}])$$

- Note that they "dual-intersect" in v_k $\smile$ measures dual-intersections



Let X be any topological space

- The cup product on singular chains is

Algebra

$$\smile : C^k(X) \times C^l(X) \to C^{k+l}(X)$$
$$(f \smile g)(\sigma) = f(\sigma|[v_0 \dots, v_k])\, g(\sigma|[v_k \dots, v_{k+l}])$$

$\rightsquigarrow$ explicit?? Not so clean?

$(\cap \rightsquigarrow \cup)$ $\smile$

- The cup product descents to cohomology

$$\smile : H^k(X) \times H^l(X) \to H^{k+l}(X)$$

- This defines a graded commutative ring structure $H^\bullet(X) = (H^*(X), \smile)$

$$f \smile g = (-1)^{k+l}(g \smile f)$$

- This structure itself is a homotopy/homeomorphism invariant

$\Rightarrow$ the whole point

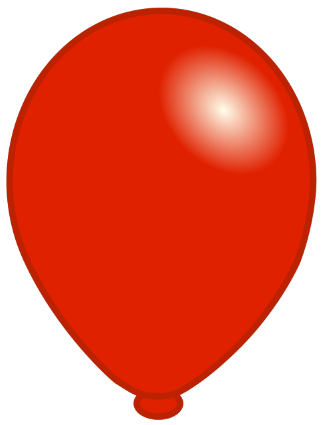
easier

For any reasonable space X:

$H_\bullet(X)$ is "Poincaré dual" to $H^\bullet(X)$

$\cap$-ring

top.

$H^\bullet \left(S^n = \right.$  $\left. \right) \cong \mathbb{Z}[X]/(X^2), \deg X = n$

$\dots 0 \quad \mathbb{Z} \quad \dots \quad \mathbb{Z}$ (n, 0)

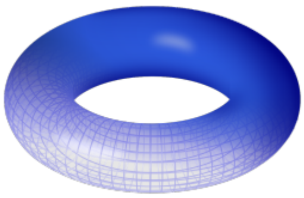
- This can be computed directly from the cell structure Balloon
- For n odd $\mathbb{Z}[X]/(X^2)$ isn't quite right to write as a graded commutative ring

$\deg X$

$\mathbb{Z}[X] \rightsquigarrow XX = (-1) XX$

$\mathbb{Z}[X]/(X^2)$

$$1 \leftrightsquigarrow [S^n], X \leftrightsquigarrow [\text{point}]$$

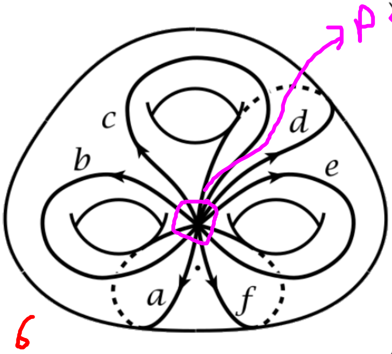


$$H^\bullet(T^d) \cong \frac{\mathbb{Z}\langle X_1, \dots, X_d\rangle}{(X_iX_j = -X_jX_i)} \cong \bigwedge^\bullet \mathbb{Z}^d \qquad \deg X_i = 1$$

$X_iX_i = 0$

$X_iX_j = -X_jX_i$

$$H^\bullet\left(M_{g,0} = \right.$$



$$\left. \right) \cong \frac{\mathbb{Z}[X_1, \dots, X_{2g}]}{(X_iX_j = -X_jX_i = (1-\delta_{i,j})X_1X_2)}$$
$$\deg X_i = 1$$

p

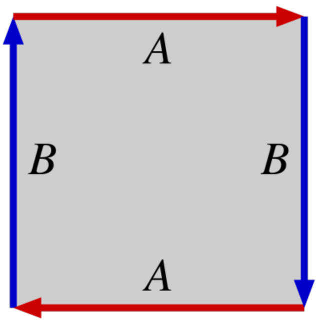
$M_{g,0}$

$g = 3$

$2 \cdot 3 = 6$

- This can be computed intersecting submanifolds Intersection ring
- X_i correspond to the classes $[\alpha_i]$ of the fundamental loops
- X_1X_2 corresponds to the class $[M_{g,0}]$ of the beast itself

$$H^\bullet\left(\mathbb{R}P^n = \right.$$



$$\left. , \mathbb{Z}/2\mathbb{Z} \right) \cong \frac{\mathbb{Z}/2\mathbb{Z}[X]}{(X^{n+1})}, \deg X = 1$$

$\mathbb{Z}/2\mathbb{Z}[x]/(x^{n+1})$

$H^\bullet(\mathbb{R}P^\infty) = \mathbb{Z}[x]$

- This can be computed intersecting submanifolds Intersection ring

$\mathbb{R}P^k$

$\dots \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0$

$[\mathbb{R}P^{n-1}_k \cap \mathbb{R}P^{n-1}_\ell] = [\mathbb{R}P^{n-2}_{n-k-\ell}]$

For $n = 5$:

$$1 \leftrightsquigarrow [\mathbb{R}P^5], X \leftrightsquigarrow [\mathbb{R}P^4], X^2 \leftrightsquigarrow [\mathbb{R}P^3], X^3 \leftrightsquigarrow [\mathbb{R}P^2], X^4 \leftrightsquigarrow [\mathbb{R}P^1], X^5 \leftrightsquigarrow [\mathbb{R}P^0]$$

$(x_0 : \dots : x_5)$

$(\underbrace{x_0 : \dots : x_4}_{X_3}, 0) \cap (\underbrace{x_0, \dots, x_3}_{X_3}, 0, x_5)$

$= (x_0 : \dots : x_3, 0, 0)$

$[\mathbb{R}P^4] \leftrightsquigarrow (x_0, x_1, x_2, x_3, x_4, 0), \quad [\mathbb{R}P^3] \leftrightsquigarrow (x_0, x_1, x_2, 0, 0, x_3)$

$\Rightarrow [\mathbb{R}P^4 \cap \mathbb{R}P^3] = [\mathbb{R}P^2] \leftrightsquigarrow (x_0, x_1, x_2, 0, 0, 0)$

Even nicer $H^\bullet(\mathbb{C}P^n) \cong \mathbb{Z}[X]/(X^{n+1})$, $\deg X = 2$

$H^\bullet(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) = \underset{1}{\mathbb{Z}/2} \oplus \underset{X}{X\mathbb{Z}/2} \oplus \underset{X^2}{X^2\mathbb{Z}/2} \oplus \ldots$

Here is a list of important cohomology rings

- Spheres S^n

$$H^\bullet(S^n) \cong \frac{\mathbb{Z}[X]}{(X^2)}, \deg X = n$$

- Torus T, real projective plane $\mathbb{R}P^2$ and Klein bottle K ($\deg X = \deg Y = 1$)

$$H^\bullet(T) \cong \bigwedge\{X, Y\}, \; H^\bullet(\mathbb{R}P^2, \mathbb{Z}/2\mathbb{Z}) \cong \frac{\mathbb{Z}/2\mathbb{Z}[X]}{(X^3)}, \; H^\bullet(K, \mathbb{Z}/2\mathbb{Z}) \cong \frac{\mathbb{Z}/2\mathbb{Z}[X, Y]}{(X^3, Y^2, X^2Y)}$$

- Orientable surfaces $M_{g,0}$ of genus $g > 0$ without boundary

$$H^\bullet(M_{g,0}) \cong \frac{\mathbb{Z}[X_1, \ldots, X_{2g}]}{(X_iX_j = -X_jX_i = (1 - \delta_{i,j})X_1X_2)}, \deg X_i = 1$$

- Real and complex projective spaces

$$H^\bullet(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z}) \cong \frac{\mathbb{Z}/2\mathbb{Z}[X]}{(X^{n+1})}, \deg X = 1, \quad H^\bullet(\mathbb{C}P^n) \cong \frac{\mathbb{Z}[X]}{(X^{n+1})}, \deg X = 2$$

Let X, Y be any topological spaces, and R a PID

- There are short (non-naturally) splitting exact sequences

$$\bigoplus_{p+q=n} H_p(X, R) \otimes_R H_q(Y, R) \to H_n(X \times Y, R) \to \bigoplus_{p+q=n-1} \operatorname{Tor}^R(H_p(X, R), H_q(Y, R))$$

$$\bigoplus_{p+q=n} H^p(X, R) \otimes_R H^q(Y, R) \to H^n(X \times Y, R) \to \bigoplus_{p+q=n-1} \operatorname{Tor}^R(H^p(X, R), H^q(Y, R))$$

$\mathbb{Z}$

Note the torsion error terms

- There are isomorphism of $\mathbb{Q}$-vector spaces

$$H_*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H_*(Y, \mathbb{Q}) \cong H_*(X \times Y, \mathbb{Q})$$
$$H^*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(Y, \mathbb{Q}) \cong H^*(X \times Y, \mathbb{Q})$$

No error terms

- In particular, $P(X \times Y) = P(X)P(Y)$

If X, Y are finite cell complexes with projections $p, q\colon X \times Y \to X, Y$, then

$$\times\colon H^\bullet(X,\mathbb{Q}) \otimes'_{\mathbb{Q}} H^\bullet(Y,\mathbb{Q}) \xrightarrow{\cong} H^\bullet(X\times Y,\mathbb{Q})$$ as graded commutative rings

$$\times(a,b) = p_*(a) \smile q_*(b)$$

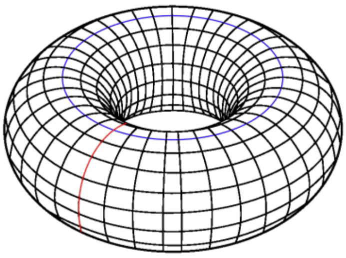
"Product of rings = Ring(Product)"

$\otimes$

"Sum of rings = Ring(Union)"

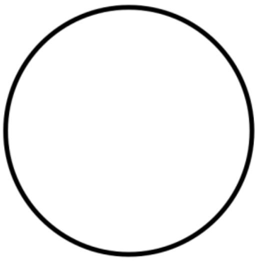
$\oplus$

$\sum \binom{d}{k} t^k$



$H^\bullet(T^d,\mathbb{Q}) \cong \bigwedge \mathbb{Q}^d \quad \deg X_i = 1$

$H^\bullet(T^d) = \bigotimes_d' \mathbb{Q}[x]/(x^2) = \Lambda \mathbb{Q}^d$



$H^\bullet(S^1,\mathbb{Q}) \cong \underline{\mathbb{Q}[X]/(X^2)} \quad \deg X = 1$

$(1+t)^d = \sum \binom{d}{k} t^k$