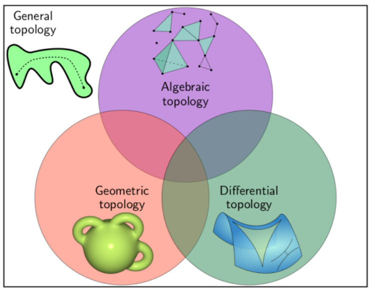
Algebraic topology?

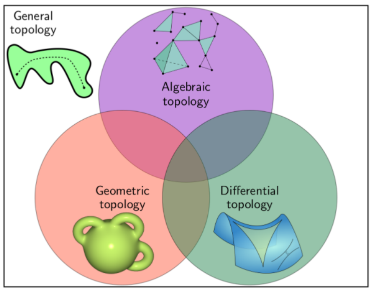
Lecture Algebraic topology

- *Slogan.* Represent whatever is hard to understand using algebra.
- *Who?* Fourth semester students in Mathematics, but everyone is welcome.
- *Preliminaries.* Some linear algebra, algebra and general topology.
- *When?* Monday 12:00-14:00.
- *Website.* http://www.dtubbenhauer.com/lecture-algtop-2021.html
- *Where?* Online via zoom.

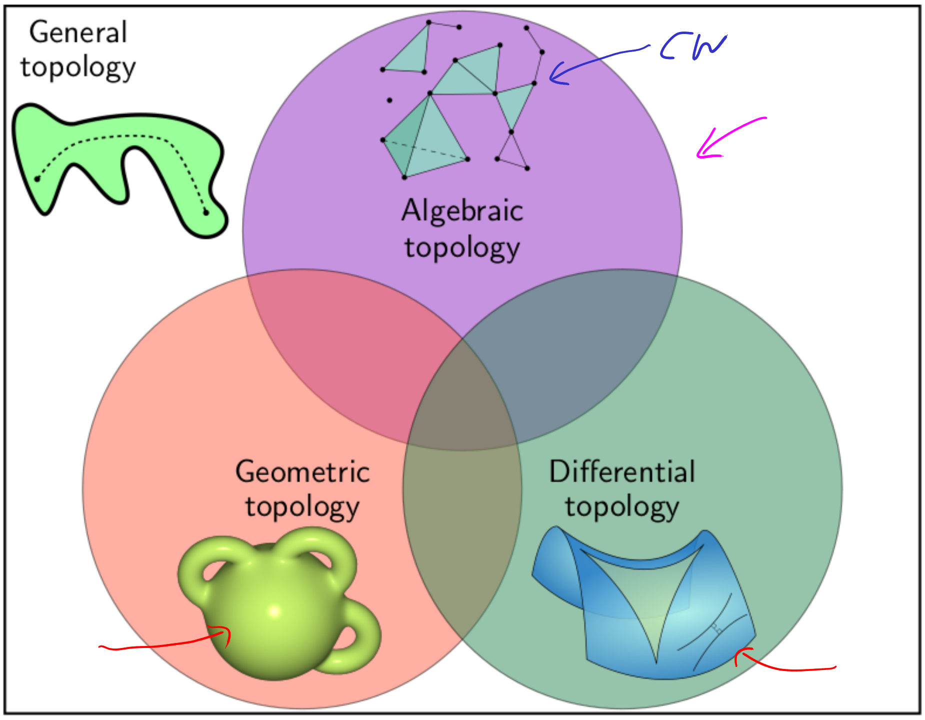


Lecture Algebraic topology

- *Literature.* Hatcher's book http://pi.math.cornell.edu/~hatcher/AT/AT.pdf.
- *Topics.* The fundamental group, homology and cohomology. 4
- *Exercises.* One per lecture, please check before the corresponding Friday.
- *Tutorials.* Friday 12:00-14:00, we discuss the exercises.
- *Assessment.* Two assessments, worth 25% each, 13.Sep.2021 and 01.Nov.2021.
- *Exam.* An oral exam worth 50% to be held at the end.



daniel.tubbenhauer@sydney.edu.au



~1920 ~1800

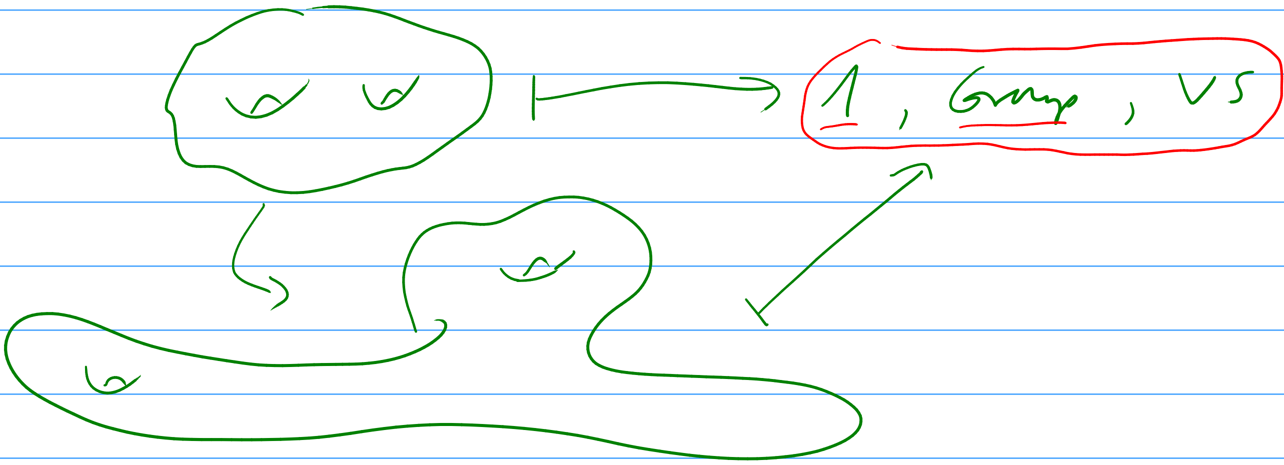
Algebraic topology is a twentieth century field of mathematics that is pervasive across mathematics and the sciences. It is unreasonably successful, being one of the newest fields of mathematics.

One of the most important aims of algebraic topology is to distinguish or classify topological spaces and maps between them up to homeomorphism.

Invariants (data that stays the same under operations on spaces) and obstructions are key to achieve this aim, meaning that

$$\text{invariants different} \Rightarrow \text{spaces different.}$$

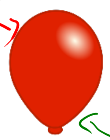
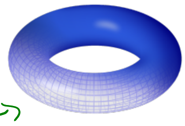
The converse is however often false, and invariants are stronger the more often the converse holds. However, strong invariants might but hard or impossible to compute, and a good invariant is an invariant which balances between being strong and computable. The main aim of algebraic topology is to associate algebraic data to topological spaces.



Here is an example. The sphere and the torus illustrated below are certainly not the same. But how can we be sure? The algebraic topology approach is to associate to them, say, numbers χ

Euler characteristic χ

S^2

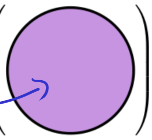
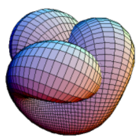
$\chi\Bigg($  $\Bigg) = 2 \neq 0 = \chi \Bigg($  $\Bigg)$,

$\neq$

$\simeq S^1 \vee S^1$

so we conclude that the sphere (left, a balloon) is not the torus (right, an empty donut) by simply observing that $2 \neq 0$. However, the invariant χ is not complete in the sense that

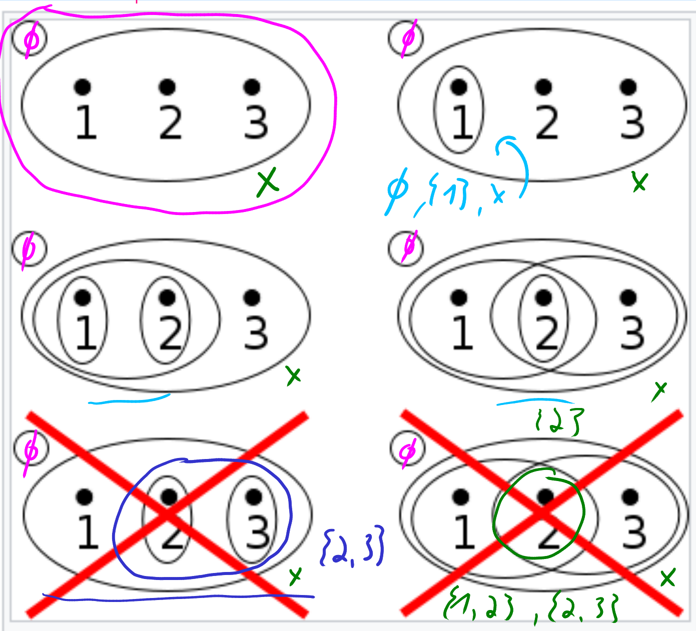
D^2

$\chi\Bigg($  $\Bigg) = 1 = \chi \Bigg($  $\Bigg)$,

$\mathbb{R}P^2$

but the two topological spaces, the disc and the real projective plane (right, immersed in $\mathbb{R}^3$), are not the same. A crucial aim of algebraic topology is thus to have a big enough backpack of invariants to tackle problems in the wild.

Basics from general topology!



Four examples and two non-examples of topologies on the three-point set $\{1, 2, 3\}$. The bottom-left example is not a topology because the union of $\{2\}$ and $\{3\}$ [i.e. $\{2, 3\}$] is missing; the bottom-right example is not a topology because the intersection of $\{1, 2\}$ and $\{2, 3\}$ [i.e. $\{2\}$], is missing

Def 1

A topological space X is a pair $(X, \mathcal{T})$ — points, open sets

$\mathcal{T} \subset P(X)$

- $\emptyset$, X are open
- Every union of elements from $\mathcal{T}$ belongs to $\mathcal{T}$

$U \cup V \in \mathcal{T}$

$U, V \in \mathcal{T}$

- A finite intersection is also in $\mathcal{T}$

$U \cap V \in \mathcal{T}$

$U, V \in \mathcal{T}$

$\mathring{D}^2(x, r) = \{y \in \mathbb{R}^n \mid d(x,y) < r\}$

$\bigcap_i U_i \notin \mathcal{T}$

$\mathbb{R}^2$

$\mathbb{R}^n$

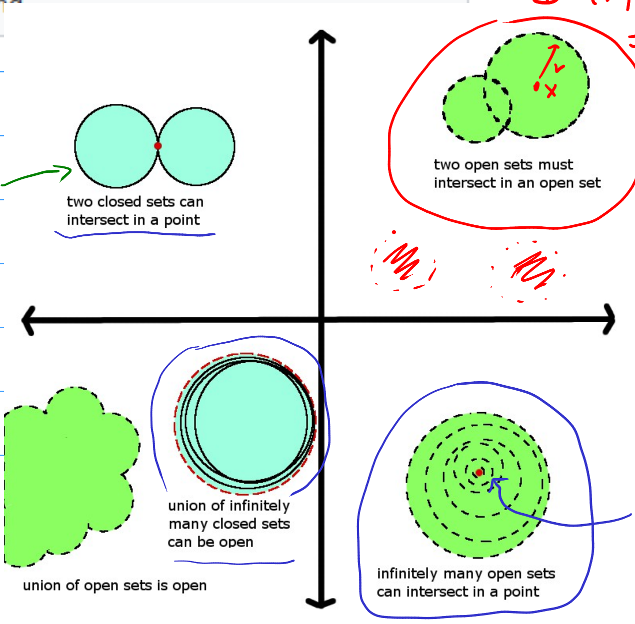
$d(x,y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$

$x = (x_1, \dots, x_n)$

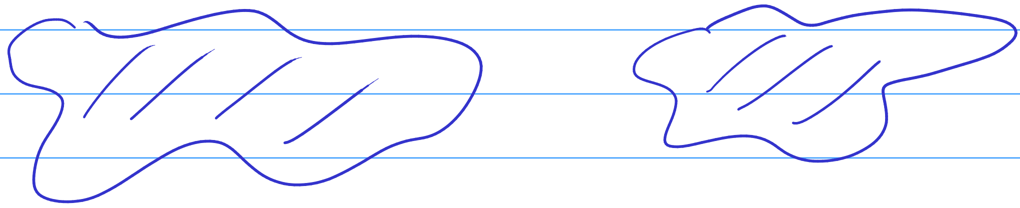
$y = (y_1, \dots, y_n)$

$< \to$ "$\leq$"

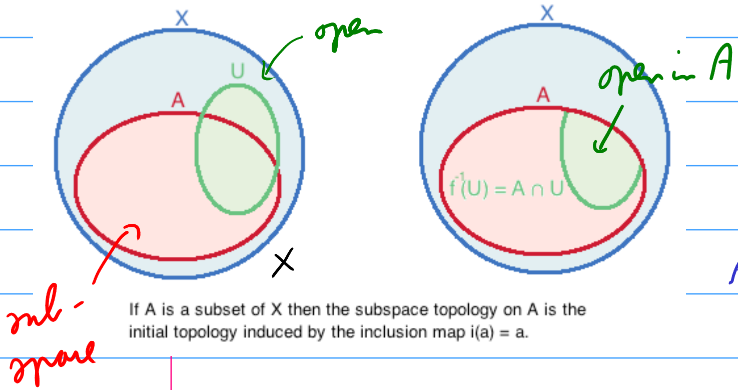
closed



open



closed



If A is a subset of X then the subspace topology on A is the initial topology induced by the inclusion map i(a) = a.

$\mathbb{R}^n \leftarrow$ nice topology given by metric / distance

Euclidean

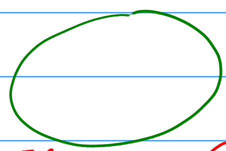
$A \subset X \qquad T_A = \{T \cap A\}$

S^2 2-sphere

$\hookrightarrow S^2 = \{x \in \mathbb{R}^3 \mid d(0,x) = 1\}$

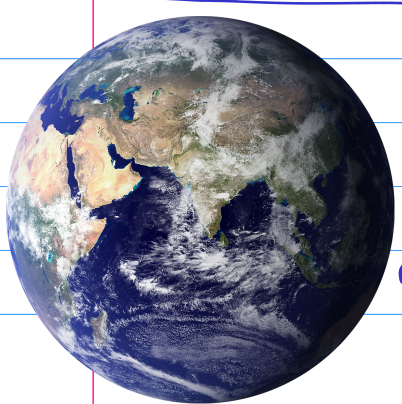
S^n n-sphere

S^1 1-sphere



$\subset \mathbb{R}^2$

S^1 Circle

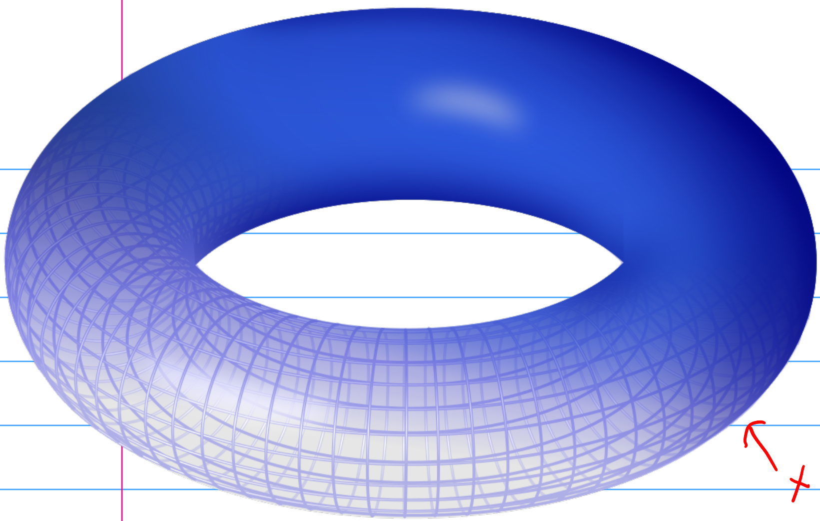


$\leftarrow D^3$

$D^n = \{x \in \mathbb{R}^n \mid d(0,x) \leq 1\}$

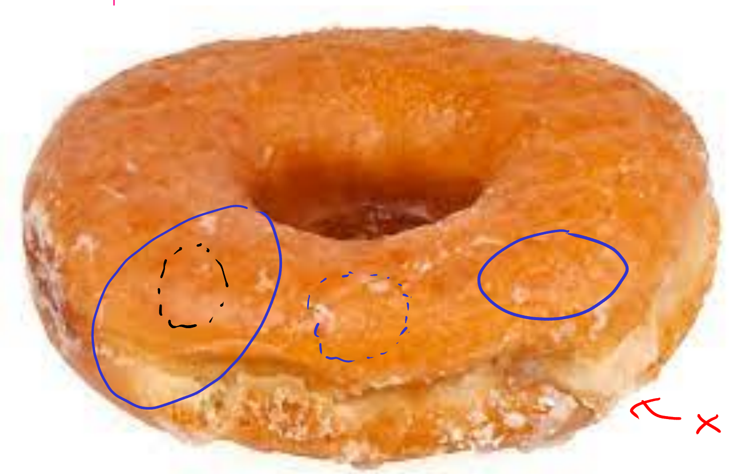
n-ball

n-disc



$\subset \mathbb{R}^3$

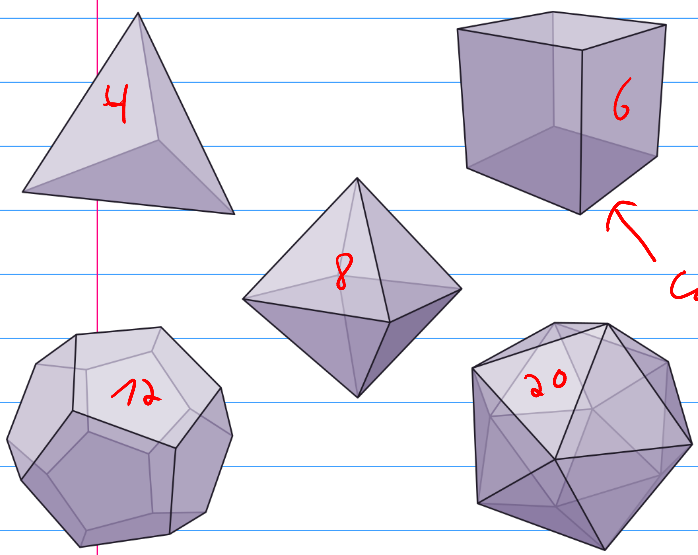
Torus T



Solid torus

$\emptyset, X$

open + closed

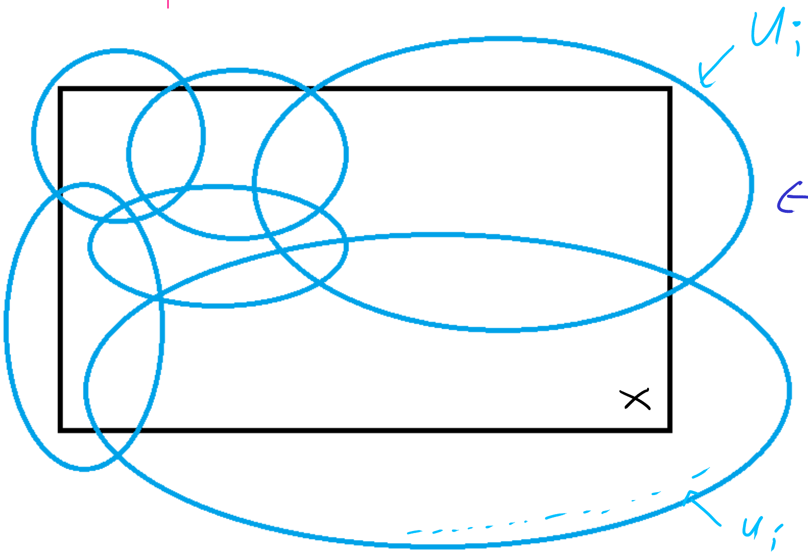


cube

Platonic solids

$\rightsquigarrow$ all 2-spheres

soccer balls

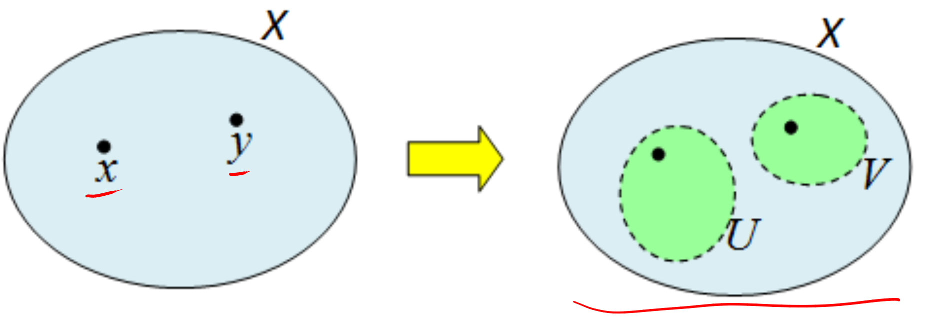


A space X is called <u>compact</u> if there is a <u>finite</u> collection of open sets $\{U_i\}$ such that $\bigcup_i U_i \supset X$

$\mathbb{R}^n$ is not compact

Hausdorff

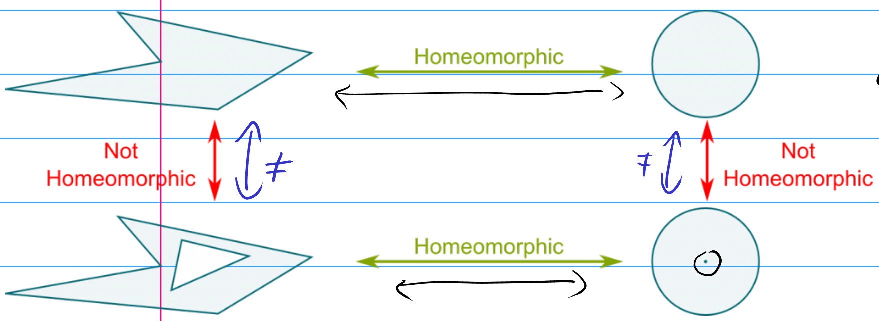
for every cover $\bigcup \tilde{U}_i \supset X$



X is Hausdorff if
$\forall\, x, y \in X$
$\exists\, U, V \in \mathcal{T}$
such that $U \cap V = \emptyset$
$x \in U,\ y \in V$

Hausdorff = separate points :)

<u>Maps between top. spaces:</u>



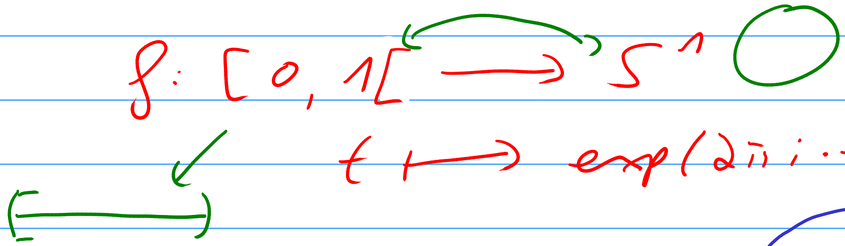
A map $f: X \to Y$ is continuous if for every $V \in \mathcal{T}_Y$
$f^{-1}(U)$ is open in X

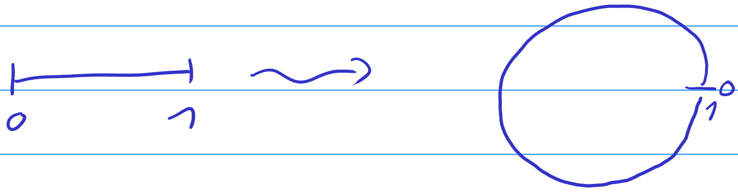
Preimages of open sets are open

<u>Def</u>: $f: X \longrightarrow Y$ is a homeomorphism if
- f is a bijection "The set structure is preserved"
- f^{-1} is continuous

Beware: We need f^{-1} to be continuous:

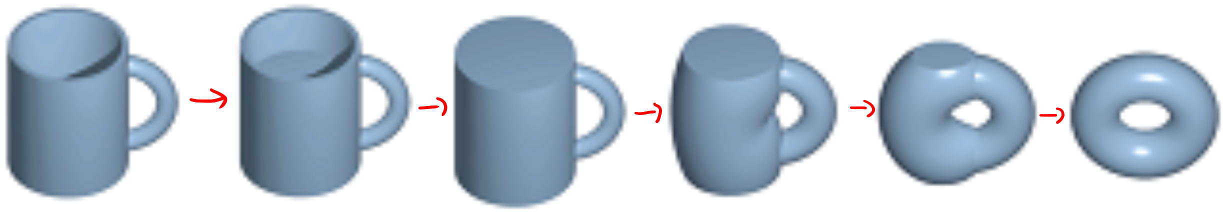
$$f : [0,1[\longrightarrow S^1 \quad \text{is bijective}$$
$$t \longmapsto \exp(2\pi i \cdot t)$$

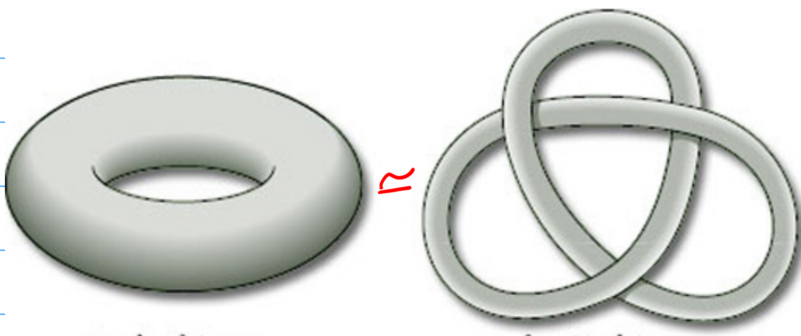




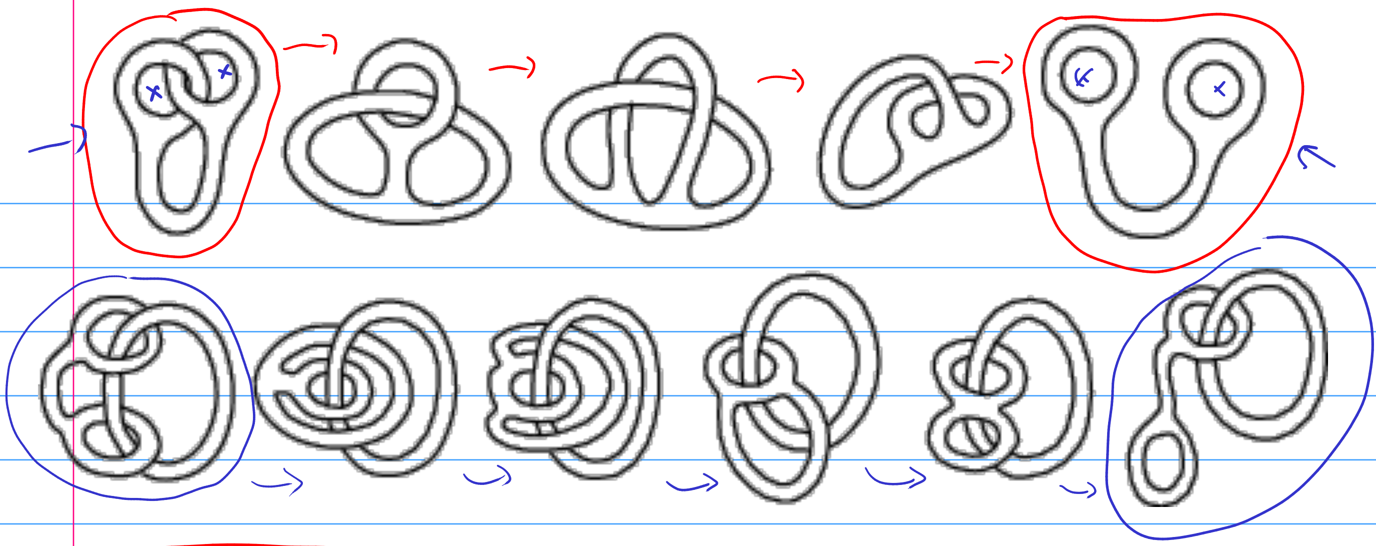
f^{-1} is not continuous!

Study top space up to homeomorphism
$X \simeq Y$ if they are homeomorphic





standard torus $\simeq$ knotted torus



Homeomorphism are hard?

Elementary operations on spaces

Product $X \times Y$ is a top space with $\mathcal{T}_{X\times Y} = \mathcal{T}_X \times \mathcal{T}_Y$

$U \times V \qquad U \in \mathcal{T}_X$

$\qquad\qquad\quad V \in \mathcal{T}_Y$

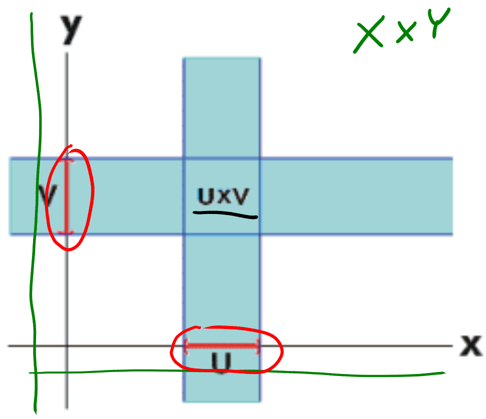
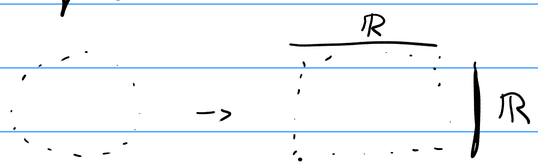
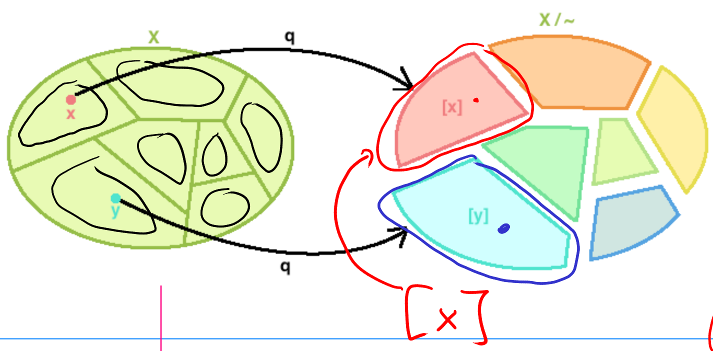


Fig. 8

Example: $\mathbb{R}^2 \simeq \mathbb{R} \times \mathbb{R}$





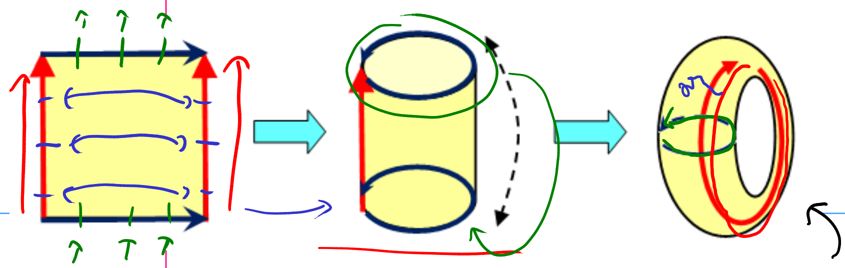
$(X, \mathcal{T})$, $\sim$ equi relation on X

$Y = X/\sim$

$q : X \longrightarrow Y = X/\sim$

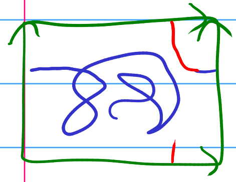
$\mathcal{T}_Y = \{ U \subset Y \mid q^{-1}(U) \in \mathcal{T}_X \}$

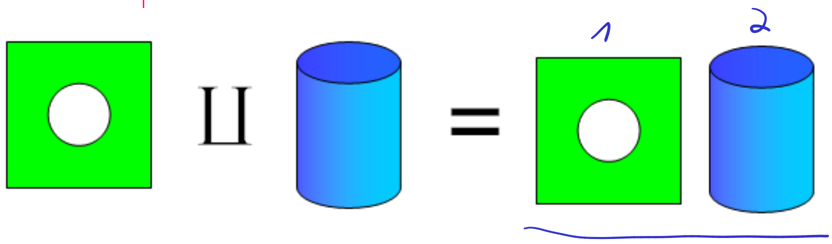
Quotient



$T \simeq \text{Square}/\sim$

fundamental polygon of T





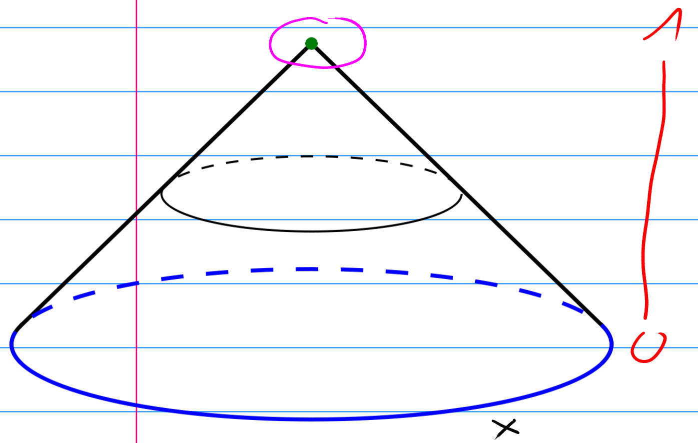
Union

$X \sqcup Y = X \dot\cup Y = X \amalg Y$

a set is open if it is of the form $U \dot\cup V$ $\quad U \in \tau_X$, $V \in \tau_Y$

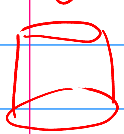
$\{0, 1, 2\} \cup \{0, 1\}$

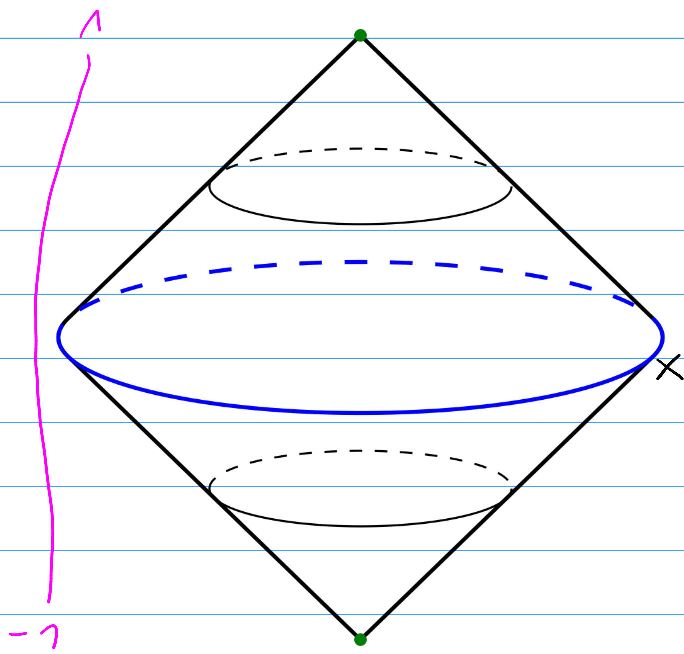
$= \{0_1, 1_1, 2_1, 0_2, 1_2\}$



cone

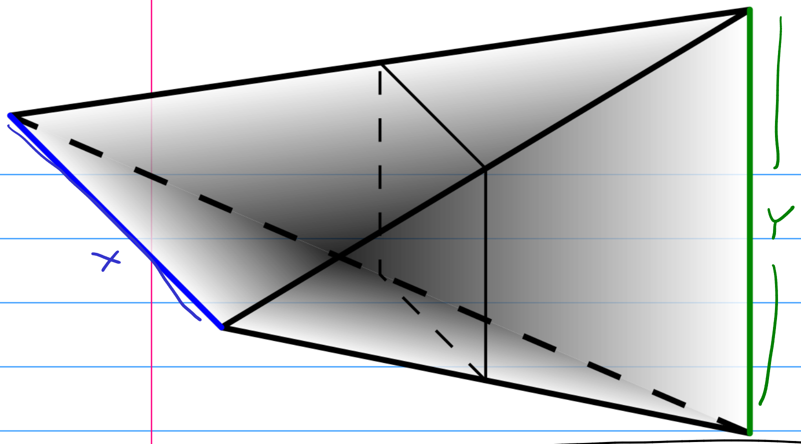
$X \times [0,1] / (X \times \{1\})$





suspension

$X \times [-1, 1] / (X \times \{-1\}, X \times \{1\})$

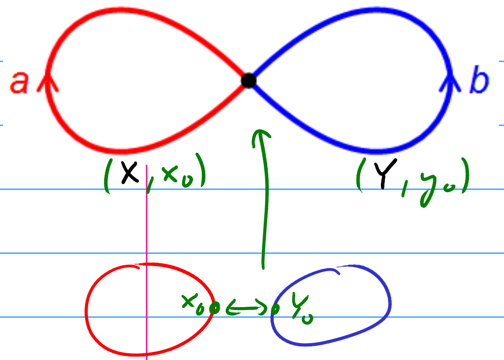


Join

$X * Y := X \times Y \times [0,1] / \sim$

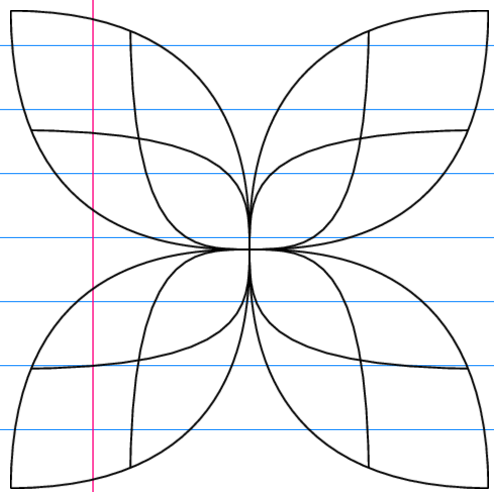
identifies $\{x\} \times Y \times \{0\}$ $\forall x$

and $X \times \{y\} \times \{1\}$



Wedge sum

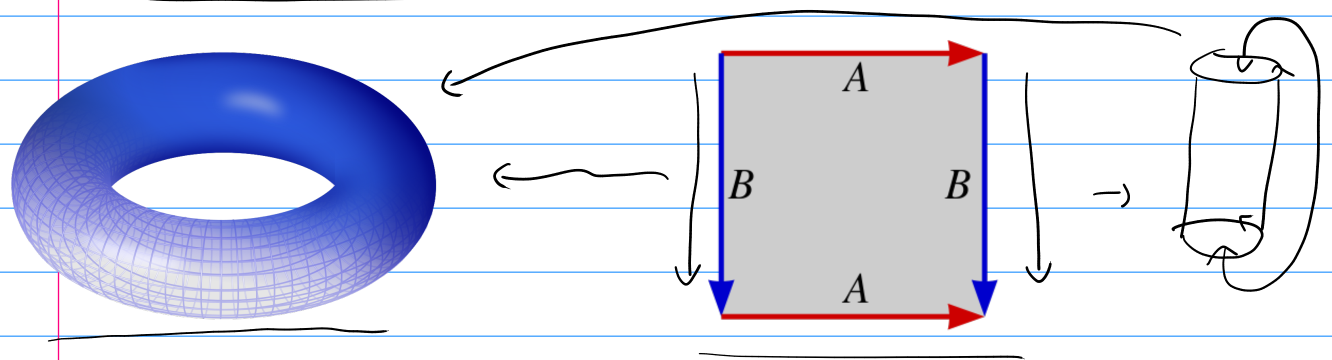
$X \vee Y = X \sqcup Y / x_0 = y_0$

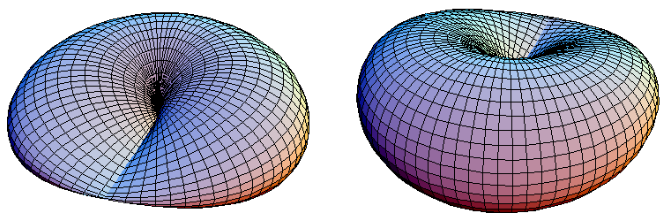


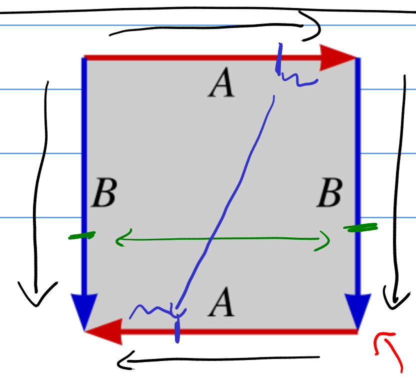
$X \wedge Y = (X \times Y) / (X \vee Y)$

$$X \wedge Y = (X \times Y)/(X \vee Y).$$

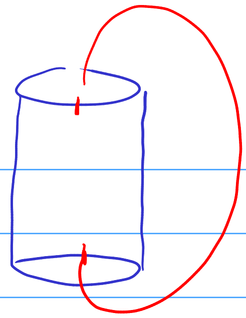
$\leftarrow [0,1] \wedge [0,1]$

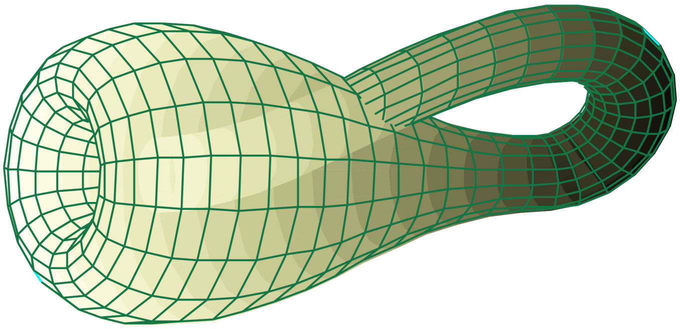




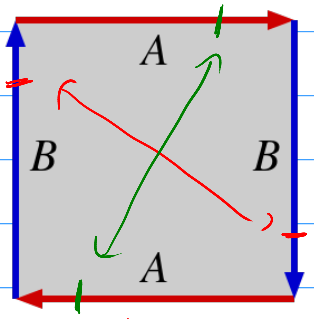


$\mathbb{R}P^2$ real projective plane

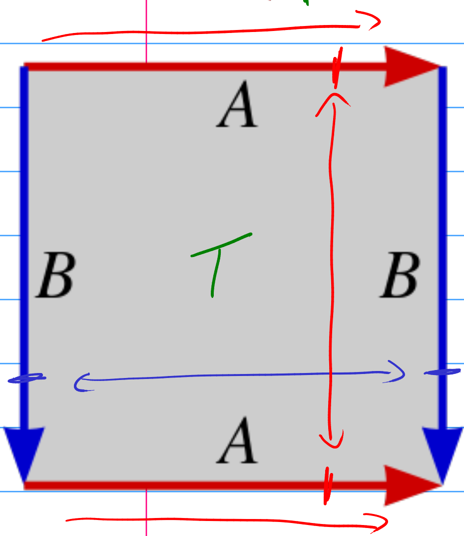




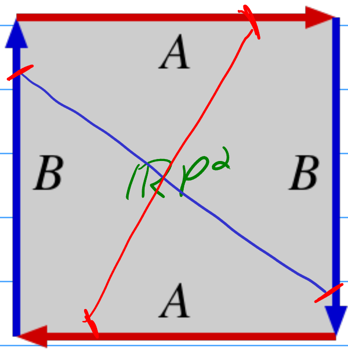
Klein bottle K



quotient $\rightsquigarrow$ $\mathbb{R}P^2$



$\neq$



$\neq$