EXERCISES 8: LECTURE ALGEBRAIC TOPOLOGY

Exercise 1. Show that the following are equivalent under the assumption of the existence of a homology theory satisfying the dimension axiom.

1. $\mathbb{R}^m \cong \mathbb{R}^n$ (homeomorphism).
2. $S^{m-1} \simeq S^{n-1}$ (homotopy equivalent).
3. $m = n$.

Addendum:

- The dimension axiom was $H_\bullet(\text{point}) \cong 0$.
- Hint: Separating spheres means $H_\bullet(S^{m-1}) \cong H_\bullet(S^{n-1})$ if and only if $m = n$, and this is always true for homology theories satisfying the dimension axiom.
- The terminology of a homology theory satisfying the dimension axiom that separates spheres will be used in all exercises below.

Exercise 2. Proof the Brouwer fixed point theorem under the assumption of the existence of a homology theory satisfying the dimension axiom.
Addendum:

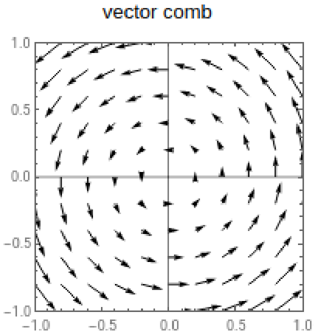
- The Brouwer fixed point theorem says that any map $f\colon D^n \to D^n$ has a fixed point.
- This very surprising result implies that no matter how much you stir a cup of coffee, some point of the liquid will return to its original position

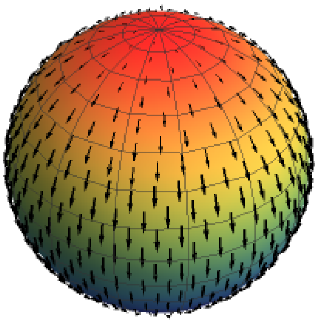
(This analogy was apparently proposed by Brouwer.)

Exercise 3. Proof the hairy ball theorem under the assumption of the existence of a homology theory satisfying the dimension axiom.
Addendum:

- A nonvanishing vector field on S^n is a map $\mathbb{R}^n \supset S^n \to \mathbb{R}$ such that $\langle x, v(x)\rangle = 0$ and $v(x) \neq 0$ for all $x \in S^n$. The hairy ball theorem say that these exists if and only if n is odd.

- Hint. One direction is a construction of a vanishing vector field. Here is one for $n = 2$:

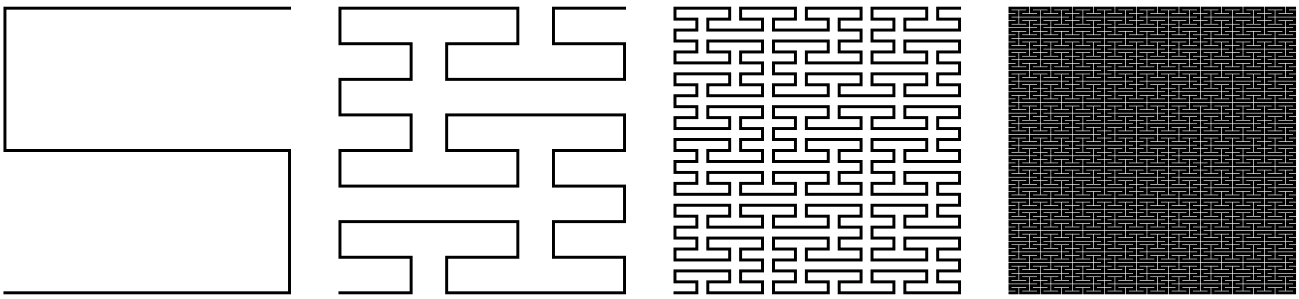




Exercise 4. Proof that an m-manifold can not homeomorphic to an n-manifold unless $m = n$ under the assumption of the existence of a homology theory satisfying the dimension axiom.

Addendum:

- See https://en.wikipedia.org/wiki/Topological_manifold for the key definitions, but we assume that m-manifolds are second-countable (which wikipedia does not assume).
- Hint: Second-countable implies that an n-manifold embeds into $\mathbb{R}^k$ for some $k \geq n$.
- This is not as trivial as it sounds. For example there are continues bijections $[0, 1] \to [0, 1]^2$, the so-called space filling curves. These are constructed via iteration, *e.g.*



- The exercises are optimal and not mandatory. Still, they are highly recommend.
- There will be 12 exercise sheets, all of which have four exercises.
- The sheets can be found on the homepage www.dtubbenhauer.com/lecture-algtop-2021.html.
- If not specified otherwise, spaces are topological space, maps are continuous *etc.*
- There might be typos on the exercise sheets, my bad, so be prepared.