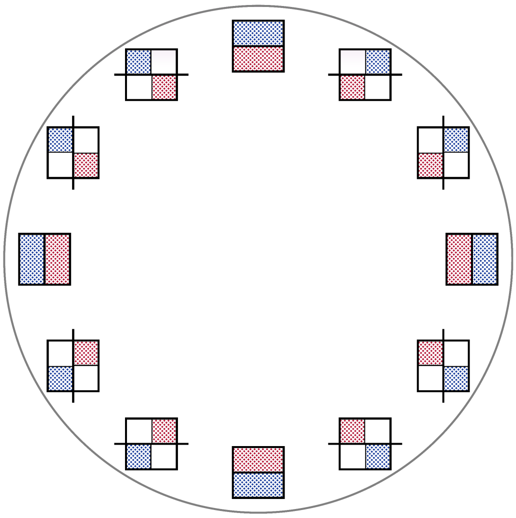
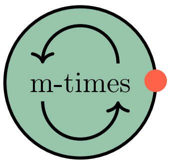
EXERCISES 12: LECTURE ALGEBRAIC TOPOLOGY

Exercise 1. Show that $\pi_n(X, x_0)$ is a commutative group for $n \geq 2$.

Hint: Have a look at the clock



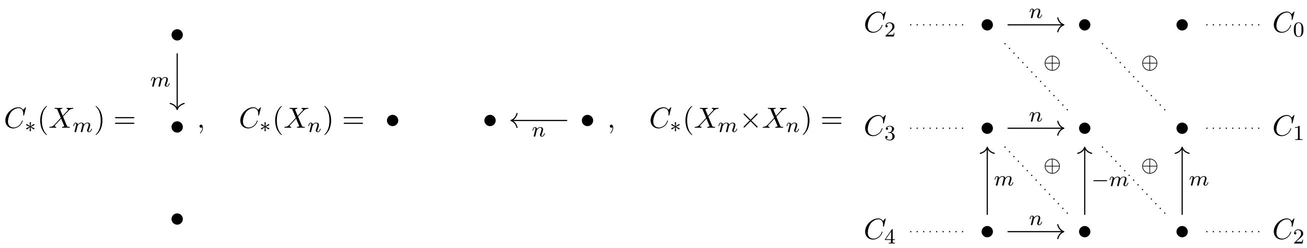
Exercise 2. For $m \in \mathbb{Z}_{\geq 1}$ let X_m be the cell complex obtained by attaching a disc via $S^1 \to S^1$ winding m-times:

$X_m =$ 

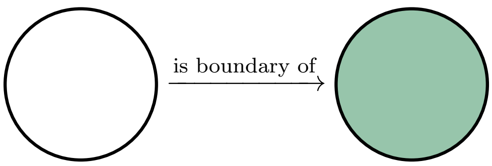
Compute the homology $H_*(X_m \times X_n)$ and the Hilbert–Poincare polynomial $P(X_m \times X_n)$ of the space $X_m \times X_n$. How do these compare to $H_*(X_m)$ and $H_*(X_n)$, respectively, to $P(X_m)$ and $P(X_n)$.

Addendum:

- The answer will only depend on m and n.
- Hint: Here is a picture of the tensor product of the cell complexes:



Exercise 3. A closed n-manifold is called null-cobordant if it is the boundary of a compact $(n+1)$-manifold. For example, S^1 is null-cobordant:

$S^1 \cong$  $\cong D^2$.

Decide whether $\mathbb{R}P^2$ is null-cobordant or not.

Hint: The Euler characteristic does the job https://math.stackexchange.com/questions/1385708

Exercise 4. Show that $\pi_n(S^n) \cong \mathbb{Z}$.

Addendum:

- The higher homotopy groups of spheres are notoriously hard to compute, and only partial results are known, *e.g.*

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	${\mathbb{Z}_2}^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$

- One problem is that the $\pi_k(S^n)$ appear to be pretty random, in particular, as one goes to the right along a row. However, there are a few patterns, *e.g.*:
  - The groups below the jagged black line are constant along the diagonals (as indicated by the red, green and blue coloring).
  - Most of the groups are finite. The only infinite groups are either on the main diagonal or immediately above the jagged line (highlighted in yellow).
- The exercise therefore asks to verify the leftmost non-trivial blue diagonal.

- The exercises are optimal and not mandatory. Still, they are highly recommend.
- There will be 12 exercise sheets, all of which have four exercises.
- The sheets can be found on the homepage www.dtubbenhauer.com/lecture-algtop-2021.html.
- If not specified otherwise, spaces are topological space, maps are continuous *etc.*
- There might be typos on the exercise sheets, my bad, so be prepared.