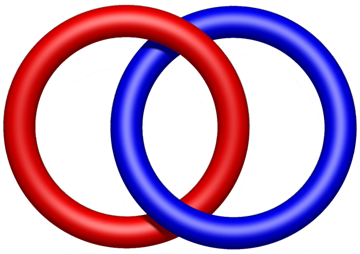
EXERCISES 11: LECTURE ALGEBRAIC TOPOLOGY

Exercise 1. A space X is called homologically self-dual if

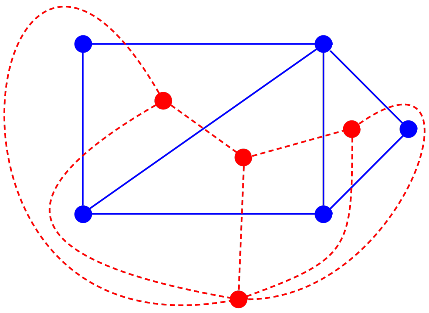
$$H_k(X) \cong H^k(X)$$

for all $0 \leq k \leq n$. Find a homologically self-dual space X. What does being homologically self-dual imply for the Hilbert–Poincaré polynomial $P(X)$?

Exercise 2. A Seifert surface is an orientable 2-manifold whose boundary is a given knot or link. Find the Seifert surface of the Hopf link H:

$H =$ 

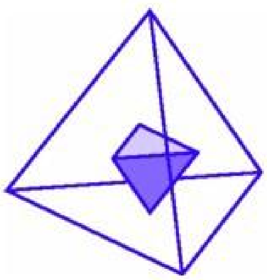
Exercise 3. The dual graph $G^\star$ of a plane (plane=embedded in the plane) graph G is a graph that has a vertex for each face of G, an edge for each pair of faces in G that are separated from each other by an edge, and a self-loop when the same face appears on both sides of an edge. For example, the following two plane graphs are dual:



Such a graph is self-dual if $G \cong G^\star$ as graphs. Find a self-dual graph, and explain why Poincaré duality implies that each such graph has $2\#V - 2$ edges, where $\#V$ is the number of vertices.

Addendum:

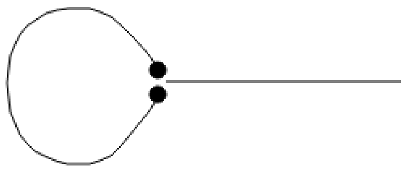
- Hint: Here is a tetrahedron in a tetrahedron:



- Hint: https://en.wikipedia.org/wiki/Dual_polyhedron#Self-dual_polyhedra

Exercise 4. Consider the following space X:

$$X = \mathbb{R}/\sim \text{ with } x \sim y \Leftrightarrow \big(x = -y \text{ and } |x| > 1\big)$$

$X \rightsquigarrow$ 

(The two black dots are the equivalence classes of $+1$ and -1.)

Show that X is non-orientable but has open neighborhoods homeomorphic to $\mathbb{R}$.

Addendum:

- This is an example of an "almost 1-manifold" that is non-orientable.
- Hint: https://math.stackexchange.com/questions/635980

- The exercises are optimal and not mandatory. Still, they are highly recommend.
- There will be 12 exercise sheets, all of which have four exercises.
- The sheets can be found on the homepage www.dtubbenhauer.com/lecture-algtop-2021.html.
- If not specified otherwise, spaces are topological space, maps are continuous *etc.*
- There might be typos on the exercise sheets, my bad, so be prepared.