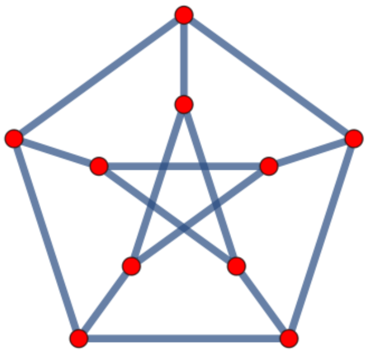
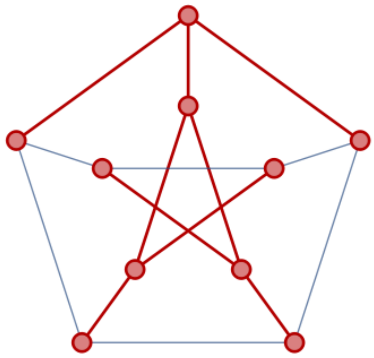
ASSIGNMENT 1 – SOLUTIONS: LECTURE ALGEBRAIC TOPOLOGY

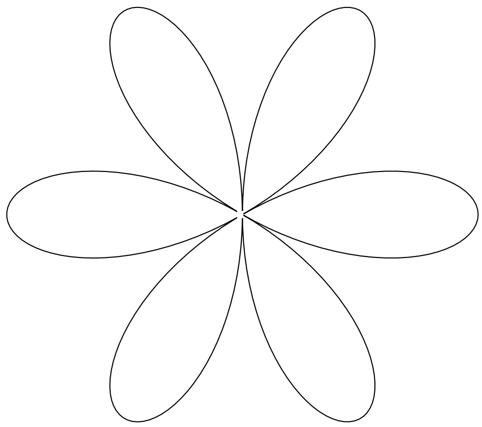
Exercise 1. Compute the homology $H_*(G)$ of the Petersen graph G:

$G =$ 

Can you guess what the homology of a general graph is?

Hint: The following two pictures should be helpful.





Solution (sketch) 1. Every connected finite graph G is homotopy equivalent to some n-rose. Here $n = E - V + 1$ is the number of edges not contained in a spanning tree of G, where V and E denote the number of vertices respectively edges of G. This immediately gives

$$H_*(G) \cong \mathbb{Z} \oplus t\mathbb{Z}^{\oplus(E-V+1)}$$

by observing that all gluing maps for n-roses are zero.

In particular, when G is the Petersen graph we get

$$H_*(G) \cong \mathbb{Z} \oplus t\mathbb{Z}^{\oplus 6}.$$

For a non-connected finite graph $G = \coprod_i G_i$ we use additivity of H_* and get

$$H_*(G) \cong \bigoplus_i \left(\mathbb{Z} \oplus t\mathbb{Z}^{\oplus(E_i - V_i + 1)}\right) \cong \mathbb{Z}^{\oplus i} \oplus t\mathbb{Z}^{\oplus(E-V+1)}.$$

Here the G_i are the connected component of G whose number of vertices and edges are denoted by V_i respectively E_i.

For an infinite graph set-theoretical issues arise. This case is omitted (and was not asked for).

Exercise 2. Classify the Platonic solids by using that they are cell complexes for the sphere S^2 and that $\chi(S^2) = 2$.

Addendum:

- Note that Platonic solids have a definition and are not arbitrary polyhedra: they are convex regular polyhedron in $\mathbb{R}^3$.

▶ Hint: We know the answer, so let us make a table where m, n are defined by $mV = 2E = nF$:

	m	n	V	E	F
Tetrahedron	3	3	4	6	4
Cube	3	4	8	12	6
Octahedron	4	3	6	12	8
Dodecahedron	3	5	20	30	12
Icosahedron	5	3	12	30	20

Observe that $\frac{1}{2} < \frac{1}{m} + \frac{1}{n}$ holds.

Solution (sketch) 2. Any Platonic solid P gives a cell structure of S^2. In particular, using the notions V, E and F for vertices, edges and faces, we get

$$\chi(P) = V - E + F = 2.$$

By definition of a Platonic solid every face of P has the same number of edges while every edge is adjacent to two faces. Thus, we can write $nF = 2E$ for some $n > 2$. (Yes, the Greek's excluded zerogons, monogons and digons. If you try to draw the as regular polygons you realize that you need curved edges - duh ;-)) The same is true for vertices, having the same number of surrounding edges by definition of a Platonic solid. So we get $mV = 2E$ for some $m > 2$. In other words, $F = 2/nE$ and $V = 2/mE$ giving

$$2 = \chi(P) = V - E + F = 2/mE - E + 2/nE = (1/m - 1/2 + 1/n)2E \Rightarrow 1/E = 1/m - 1/2 + 1/n.$$

In particular, since $1/E > 0$ we get

$$1/m + 1/n > 1/2.$$

This equation is precisely satisfied as in the solution table

	m	n	V	E	F
Tetrahedron	3	3	4	6	4
Cube	3	4	8	12	6
Octahedron	4	3	6	12	8
Dodecahedron	3	5	20	30	12
Icosahedron	5	3	12	30	20

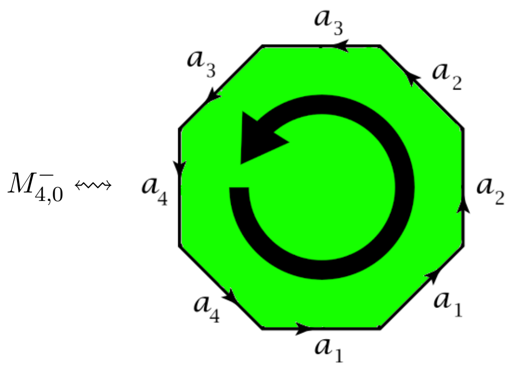
To see this observe that $m = 6$ already gives

$$1/n > 1/3$$

so maximally $n = 2$, which is excluded. Since bigger values of m let $1/2 - 1/m$ go even closer to $1/2$ we conclude that $2 < m < 6$. Similarly for n. It is then easy to see that $m = 4 = n$ or $m = 4, n = 5$ or $m = 5, n = 4$ do not satisfy $1/m + 1/n > 1/2$, So the table says we are done: every solution to $1/m + 1/n > 1/2$ is as in the table and, conversely, any such solution corresponds to the indicated polyhedron so is not a numerical solution.

Exercise 3. For $g \geq 1$ let $M_{g,0}^{-}$ denote the closed non-orientable surface of genus g defines via its fundamental polygon, *i.e.* a $2g$-sided polygon with attaching word $a_1^2 \dots a_g^2$. For example, for $g = 4$

we have:



Compute the homology $H_*(M^-_{g,0})$ and the Hilbert–Poincare polynomial $P(M^-_{g,0})$.

Hint: Note that $M^-_{1,0} \cong \mathbb{R}P^2$ and $M^-_{2,0}$ is the Klein bottle, and recall how to calculate their homologies. (Beware that the above are not the standard presentations of these two surfaces: a surface can be defined by different fundamental polygons.)

Solution (sketch) 3. The cellular chain complex is given by

$$C_*\colon \mathbb{Z} \xrightarrow[ker=0,rank=1]{\begin{pmatrix}2 & \dots & 2 & 2\end{pmatrix}^T} \mathbb{Z}^g \xrightarrow[ker=g,rank=0]{\begin{pmatrix}0 & \dots & 0 & 0\end{pmatrix}} \mathbb{Z}.$$

(Kernels and ranks indicated for the computation of the Hilbert–Poincaré polynomial.) After Gaussian elimination we get

$$C_*\colon \mathbb{Z} \xrightarrow{\begin{pmatrix}0 & \dots & 0 & 2\end{pmatrix}^T} \mathbb{Z}^g \xrightarrow{\begin{pmatrix}0 & \dots & 0 & 0\end{pmatrix}} \mathbb{Z}.$$

from which we directly read-off the homology:

$$H_*(M_{g,0}) \cong \mathbb{Z} \oplus t(\mathbb{Z}^{\oplus(g-1)} \oplus \mathbb{Z}/2\mathbb{Z}).$$

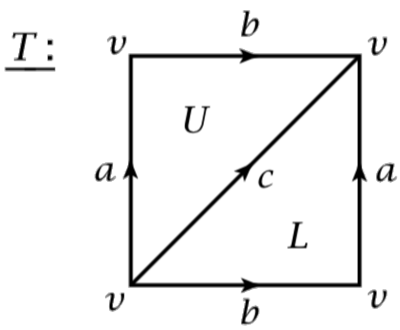
Extending scalars to $\mathbb{Q}$ and computing kernel-rank (indicated above) gives

$$P(M_{g,0}) = 1 + (g-1)t.$$

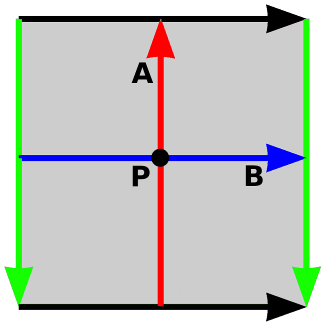
Exercise 4. Compute the cohomology ring $H^\bullet(T)$ of the torus T from the definitions (*i.e.* not going to the intersection ring).

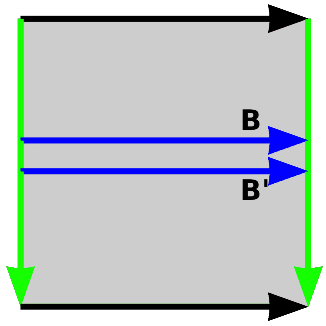
Addendum:

- ▶ You can assume that T is defined via the following simplicial structure:



▶ Hint: The main calculations in the intersection ring are

$[A] \cap [B] = [P] \rightsquigarrow$ 

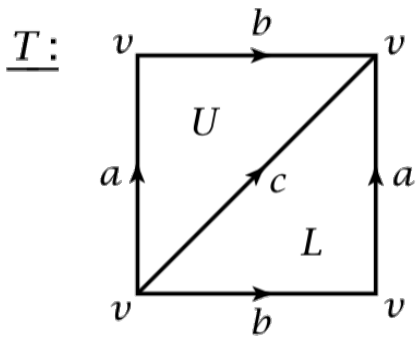
$[B] \cap [B] = 0 \rightsquigarrow$ 

The main point is to find expressions of $[A]$ and $[B]$ in $C^*(T)$. It is then not hard to verify that the intersection calculation is reflected in singular cohomology.

Solution (sketch) 4. Recall that

$$H_*(T) \cong H^*(T) \cong \mathbb{Z} \oplus t\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}.$$

By using the simplicial structure



it is easy to see that we can choose the following generators of homology, where * denotes the duals:

$$H_0(T) \cong \mathbb{Z}\{v\}, \quad H_1(T) \cong \mathbb{Z}\{a+c, b+c\}, \quad H_2(T) \cong \mathbb{Z}\{U+L\},$$
$$H^0(T) \cong \mathbb{Z}\{v^*\}, \quad H^1(T) \cong \mathbb{Z}\{a^*+c^*, b^*+c^*\}, \quad H^2(T) \cong \mathbb{Z}\{U^*+L^*\}.$$

The multiplication table (meaning $x \smile y$ for x indexed by the columns) is

	v^*	a^*+c^*	b^*+c^*	U^*+L^*
v^*	v^*	a^*+c^*	b^*+c^*	U^*+L^*
a^*+c^*	a^*+c^*	0	U^*+L^*	0
b^*+c^*	b^*+c^*	$-U^*-L^*$	0	0
U^*+L^*	U^*+L^*	0	0	0

.

Most of these are immediate.

First, v^* is the unit: By degree reasons the unit is a generator of $H^0(T)$, so it is either v^* or $-v^*$. We have $(v^* \smile v^*)(v) = v^*(v) = 1$, so v^* is the unit.

Moreover, U^* multiplies with everything except v^* to zero for degree reasons again, while $(a^*+c^*)^2 = (b^*+c^*)^2 = 0$, and $(a^*+c^*) \smile (b^*+c^*) = -(b^*+c^*) \smile (a^*+c^*)$ by graded anticommutativity.

It remains to argue that

$$\big((a^*+c^*) \smile (b^*+c^*)\big)(U+L) = -1.$$

This is immediate from the definition of the singular cochains. To see this let us denote the vertex of the above simplicial complex by 0 to 3 starting northeast and going clockwise. (This is for notation purposes – they are all the same vertex.) Then $U = [3,0,2]$ and $L = [1,0,2]$, while $a = [1,0] = [2,3]$, $b = [3,0] = [2,1]$ and $c = [2,0]$. So

$$\big((a^*+c^*) \smile (b^*+c^*)\big)(U) = (a^*+c^*)([3,0])(b^*+c^*)([0,2]) = 0.$$

Moreover,

$$\big((a^* + c^*) \smile (b^* + c^*)\big)(L) = (a^* + c^*)([1,0])(b^* + c^*)([0,2]) = a^*([1,0])c^*([0,2]) = -1,$$

and we are done. (Note that $[0,2] = -c$.)

The multiplication table gives us immediately the (expected) result

$$H^\bullet(T) \cong \mathbb{Z}\langle X, Y\rangle/(X^2 = Y^2, XY = -YX).$$