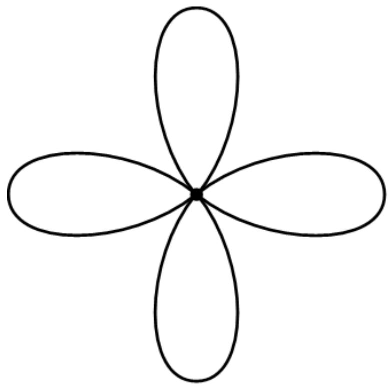
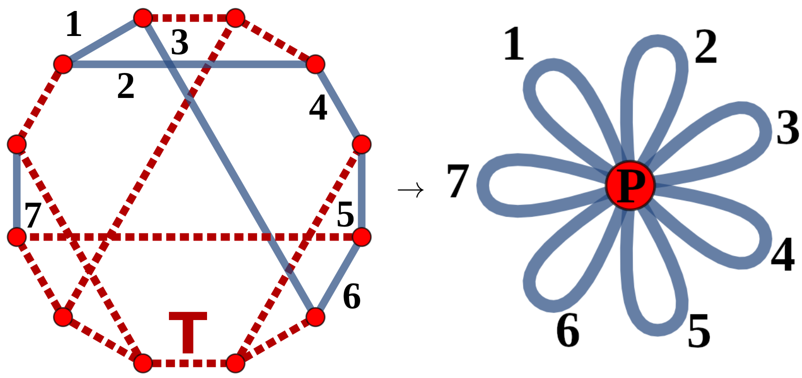
ASSIGNMENT 1 – SOLUTIONS: LECTURE ALGEBRAIC TOPOLOGY

Exercise 1. An n rose, or a bouquet of n circles, is $\bigvee_{i=1}^{n} S^1$, *e.g.* for $n = 4$:



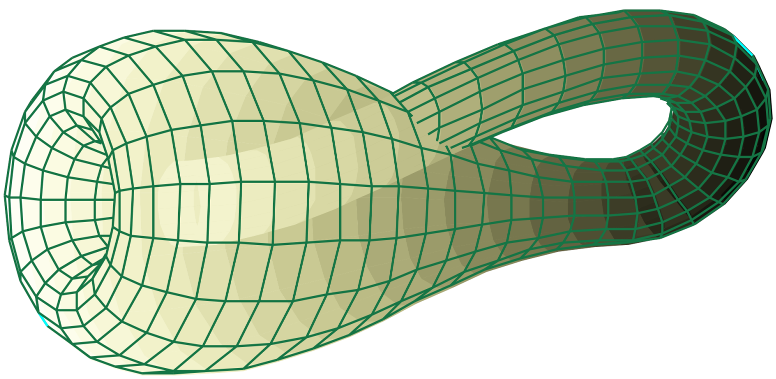
Show that any connected, finite graph is homotopy equivalent to an n rose for some n.

Solution (sketch) 1. Take a spanning tree T in G and contract it to a point P. (This works by the assumptions on G.)



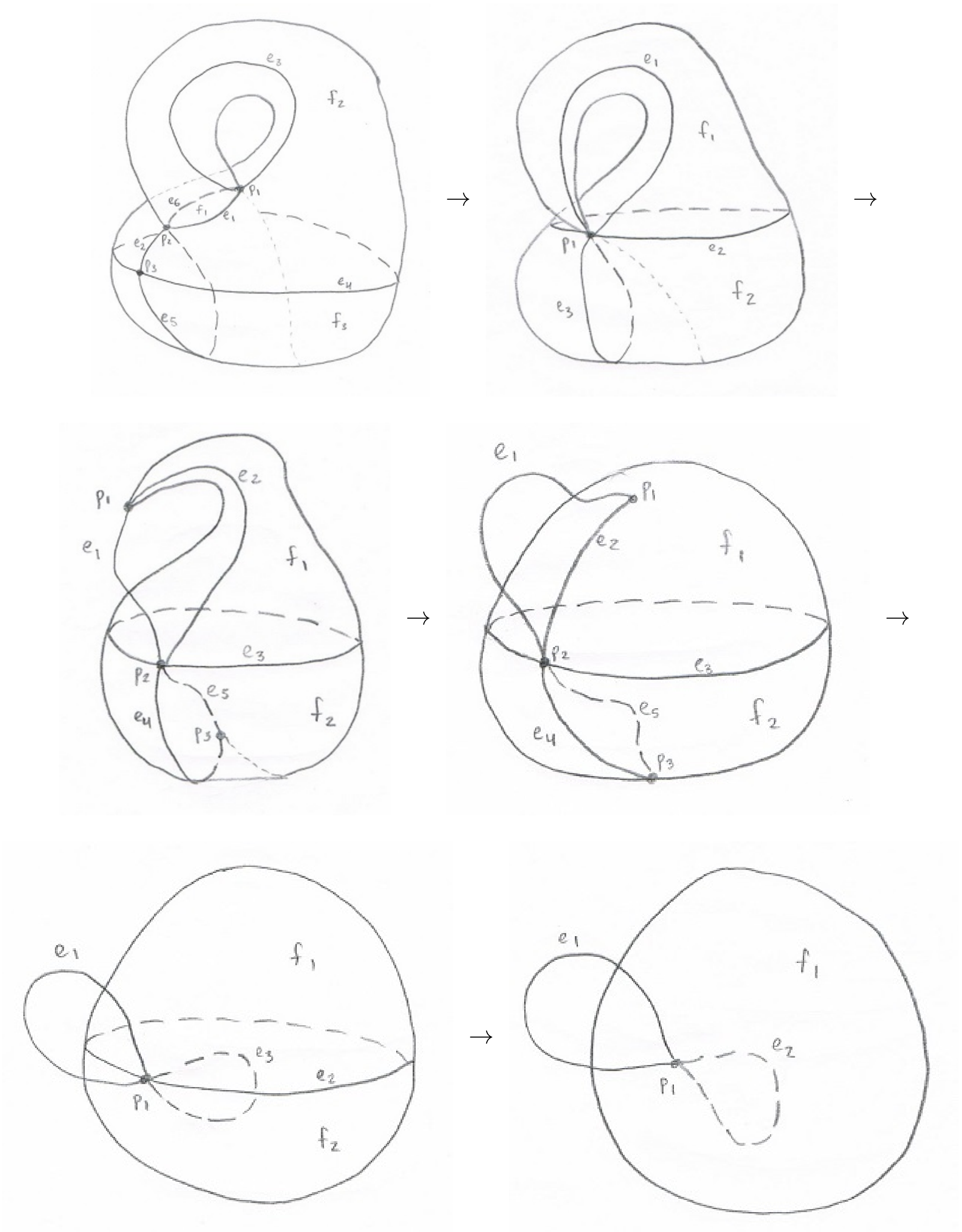
Since each edge not contained in T closes precisely one cycle in G, and T is identified to a point, the result is an n rose for n being the number of edges not contained in T. (The number n is thus the number of vertices of G minus 1.)

Exercise 2. Let X be the subset of $\mathbb{R}^3$ given by the most common immersion of the Klein bottle into $\mathbb{R}^3$ (we consider X as a subset of $\mathbb{R}^3$ and not as the Klein bottle itself):



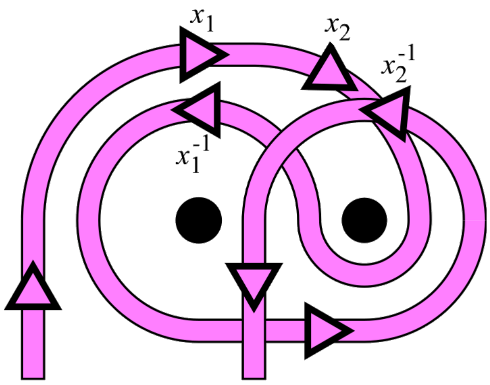
Show, *e.g.* by drawing the relevant pictures, that $X \simeq S^1 \vee S^1 \vee S^2$.

Solution (sketch) 2. Here is a sequence of pictures showing that $X \simeq S^1 \vee S^1 \vee S^2$:



Exercise 3. Compute $\pi_1(S^1 \vee S^1 \vee S^1)$ and solve the following variant of Spivak's hanging-pictures-puzzle: "Hang a picture on three nails so that removing any two nails fells the picture, but removing any one nail leaves the picture hanging.".

Hint: The solution to the original puzzle "Hang a picture on two nails so that removing any nail fells the picture." in algebraic notation is

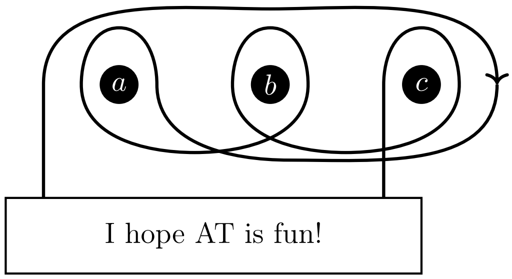
 $\rightsquigarrow$ 

$\rightsquigarrow x_2^{-1}x_1^{-1}x_2x_1.$

Solution (sketch) 3. A straightforward application of Seifert–van Kampen gives

$$\pi_1(S^1 \vee S^1 \vee S^1) \simeq \mathbb{Z} * \mathbb{Z} * \mathbb{Z} \simeq \langle a, b, c \rangle,$$

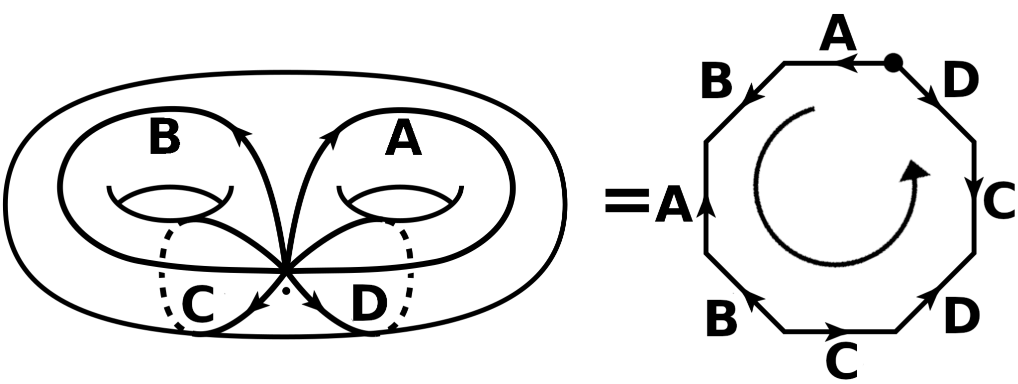
the free group in three generators a, b, c. Observe that $S^1 \vee S^1 \vee S^1$ is homotopy equivalent with the disc with three holes (say ordered left to right after choosing an embedding into the "wall" $\mathbb{R}^2$ as in the picture below), and we can identify the homotopy classes of the loops going around clockwise with a, b, c, in order from left to right. A solution for the puzzle is then easily verified to be

$c^{-1}b^{-1}a^{-1}cba \rightsquigarrow$ 

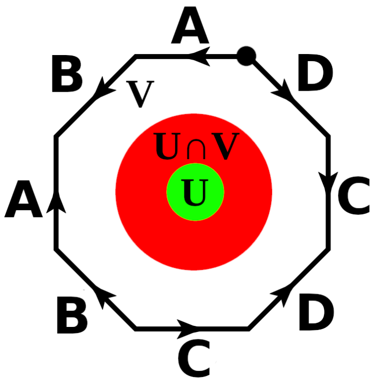
where the holes correspond to three nails.

Exercise 4. Let $M_{g,0}$ be the surface of genus g and with no boundary. Compute $\pi_1(M_{g,0})$ for $g > 0$.
Addendum:

- You can assume that $M_{g,0}$ is defined via its fundamental polygon obtained by identifying edges of a $4g$-gon as in the picture below.
- Hint:



Solution (sketch) 4. Note that $\pi_1(\bigvee_{i=1}^n S^1)$ is isomorphic ot the free group in n generators, which is an easy application of Seifert–van Kampen. Then "poking holes" gives



$$\pi_1(U \simeq \text{point}) \simeq 1$$
$$\pi_1(V \simeq \vee_{i=1}^4 S^1) \simeq \langle A, B, C, D\rangle$$
$$\pi_1(U \cap V \simeq S^1) \simeq \langle ABA^{-1}B^{-1}CDC^{-1}D^{-1}\rangle$$

where the generators A, B, C, D can be identified with the corresponding paths. (Note hereby that $ABA^{-1}B^{-1}CDC^{-1}D^{-1}$ is of infinite order, so generates a copy of $\mathbb{Z}$.) Seifert–van Kampen then gives

$$\pi_1(M_{2,0}) \simeq \langle A, B, C, D \mid ABA^{-1}B^{-1}CDC^{-1}D^{-1}\rangle.$$

This immediately generalizes to arbitrary $g > 0$ and we get

$$\pi_1(M_{g,0}) \simeq \langle A_1, B_1, \dots, A_g, B_g \mid [A_1, B_1] \dots [A_g, B_g]\rangle$$

where $[A, B] = ABA^{-1}B^{-1}$.