

All I know about Artin–Tits groups

Or: Why type A is so much easier...

Daniel Tubbenhauer

The first "published" braid diagram.

Bestimmung der Crossing

a	1	1	2+1	3+1	2+1	1+1
b	2	2	1	1	1	1
c	3	4	3	4	4	3+1
d	4	3+1	3+1	2+1	3+1	4+1

Es kommt dann den zue Begriff der Kreuzung,
 so als Begriff im Artin-artigen Satz
 man nicht alle Teile eines Knotens

Wahrscheinlich sind die halben Linien
 einer Linie die andere auf dem letzten Strang
 sein gegeben.
 In diesen Beispiel
 ab-ab-ab-ab

Man kann wie in jeder Linie zu jeder wie oft + mit - verbindet

(Page 283 from Gauß' handwritten notes, volume seven, ≤ 1830).

Joint with David Rose

April 2019

Let Γ be a Coxeter graph.

Artin \sim 1925, **Tits** \sim 1961++. The Artin–Tits group and its Coxeter group quotient are given by generators-relations:

$$\begin{aligned} \text{AT}(\Gamma) &= \langle \ell_i \mid \underbrace{\cdots \ell_i \ell_j \ell_i}_{m_{ij} \text{ factors}} = \underbrace{\cdots \ell_j \ell_i \ell_j}_{m_{ij} \text{ factors}} \rangle \\ &\downarrow \\ \text{W}(\Gamma) &= \langle \sigma_i \mid \sigma_i^2 = 1, \underbrace{\cdots \sigma_i \sigma_j \sigma_i}_{m_{ij} \text{ factors}} = \underbrace{\cdots \sigma_j \sigma_i \sigma_j}_{m_{ij} \text{ factors}} \rangle \end{aligned}$$

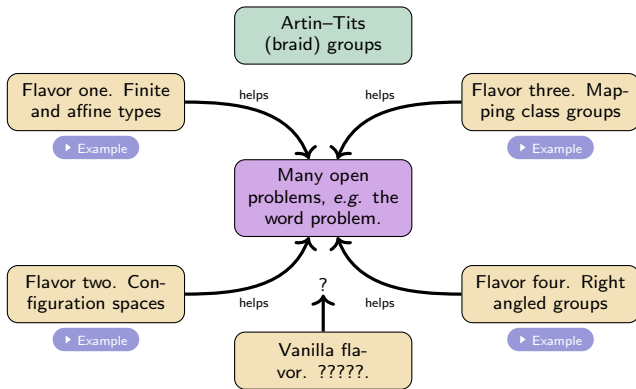
Artin–Tits groups generalize classical braid groups, Coxeter groups
polyhedron groups.

▶ generalize

My failure. What I would like to understand, but I do not.

Artin–Tits groups come in four main flavors.

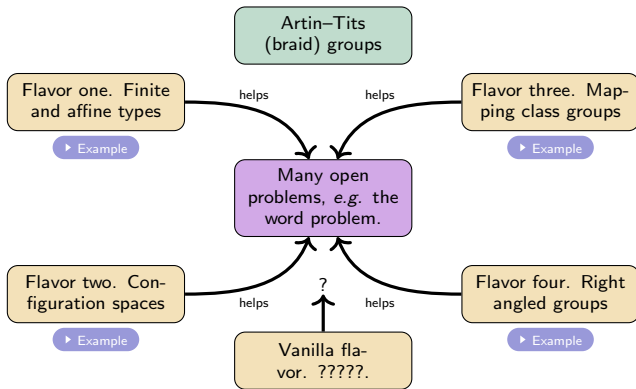
Question: Why are these special? What happens in general type?



My failure. What I would like to understand, but I do not.

Artin–Tits groups come in four main flavors.

Question: Why are these special? What happens in general type?



Maybe some categorical considerations help
In particular, what can Artin–Tits groups tell you about flavor two?

Let Γ be a Coxeter graph. The following commuting diagram exists in any type:

$$\begin{array}{ccc}
 \text{AT}(\Gamma) & \xlongequal{\quad} & \text{AT}(\Gamma) \\
 \llbracket - \rrbracket \circlearrowleft & & \circlearrowleft \llbracket - \rrbracket \\
 \mathcal{K}^b(\mathcal{S}^{\mathfrak{q}}(\Gamma)) & \xrightarrow{\text{decat.}} & \mathcal{H}^{\mathfrak{q}}(\Gamma) \\
 \downarrow & & \downarrow \\
 \mathcal{K}^b(\mathcal{Z}^{\mathfrak{q}}(\Gamma)) & \xrightarrow{\text{decat.}} & \mathcal{B}^{\mathfrak{q}}(\Gamma)
 \end{array}$$

Question. How does this help to study Artin–Tits groups?

Here (killing idempotents for the last row):

- ▶ Hecke algebra $\mathcal{H}^{\mathfrak{q}}(\Gamma)$, homotopy category of Soergel bimodules $\mathcal{K}^b(\mathcal{S}^{\mathfrak{q}}(\Gamma))$.
- ▶ Hecke action $\llbracket - \rrbracket$, Rouquier complex $\llbracket - \rrbracket$.
- ▶ Burau representation $\mathcal{B}^{\mathfrak{q}}(\Gamma)$, homotopy category of representations of zigzag algebras $\mathcal{K}^b(\mathcal{Z}^{\mathfrak{q}}(\Gamma))$.

Let Γ be a Coxeter graph. The following commuting diagram exists in any type:

$$\text{AT}(\Gamma) \xlongequal{\quad\quad\quad} \text{AT}(\Gamma)$$

Faithfulness?

The Hecke action is known to be faithful in very few cases, e.g. for Γ of rank 1, 2.
But there is “no way” to prove this in general.

Example (seems to work). Hecke distinguishes the braids where Burau failed:

```
sage: R.<q> = LaurentPolynomialRing(ZZ)
sage: H = IwahoriHeckeAlgebra('A5', q, -q^-1)
sage: T=H.T(1);
sage: ps11 = T[4] * T[5]^(-1) * T[2]^(-1) * T[1]
sage: ps12 = T[4]^(-1) * T[5]^(2) * T[2] * T[1]^(-2)
sage: Ps11 = T[1]^(-1) * T[2] * T[5] * T[4]^(-1)
sage: Ps12 = T[1]^(2) * T[2]^(-1) * T[5]^(-2) * T[4]
sage: w1 = Ps11 * T[3] * ps11
sage: w2 = Ps12 * T[3] * ps12
sage: W1 = Ps11 * T[3]^(-1) * ps11
sage: W2 = Ps12 * T[3]^(-1) * ps12
sage: w1 * w2 * W1 * W2
```

evaluate

WARNING: Output truncated!
[full_output.txt](#)

$-(q^{-21} \cdot 10^4 q^{-19} + 50 q^{-17} \cdot 168 q^{-15} + 428 q^{-13} \cdot 882 q^{-11} + 1531 q^{-9} \cdot 2303 q^{-7} + 3067 q^{-5} \cdot 3676 q^{-3} + 4012 q^{-1} \cdot 4012 q + 3676 q^3 \cdot 3067 q^5 + 2303 q^7 \cdot 1531 q^9 + 882 q^{11} \cdot 428 q^{13} + 168 q^{15} \cdot 50 q^{17} + 10 q^{19} \cdot q^{21}) \cdot T[1, 2, 3, 4, 5, 1, 2, 3, 4, 1, 2, 3, 1, 2, 1] +$

Let Γ be a Coxeter graph. The following commuting diagram exists in any type:

Faithfulness?

Rouquier's action is known to be faithful in quite a few cases:
 finite type (Khovanov–Seidel, Brav–Thomas),
 affine type A (Gadbled–Thiel–Wagner), affine type C (handlebody).
 There might be hope to prove this in general.

Example (the whole point). Zigzag already distinguishes braids:

```

sage: R.<t,q> = LaurentPolynomialRing(ZZ);
sage: psi1 = z4 * z5^(-1) * z2^(-1) * z1
sage: psi2 = z4^(-1) * z5^(2) * z2 * z1^(-2)
sage: w1 = psi1^(-1) * z3 * psi1
sage: w2 = psi2^(-1) * z3 * psi2
sage: (w1 * w2 * w1^(-1) * w2^(-1)).substitute(t=-1), (w1 * w2 * w1^(-1) * w2^(-1)).substitute(t=1,q=-2)

```

	evaluate
[1 0 0 0]	[-6900766331/4782969 119949646700/4782969 -27606410000/1594323 -1446875300/59049 10123227400/177147]
[0 1 0 0]	[-6008522000/1594323 104398156073/1594323 -24028111250/531441 -1259219000/19683 8810639500/59049]
[0 0 1 0]	[3077274850/1594323 -53464229650/1594323 12305843941/531441 644883850/19683 -4512158300/59049]
[0 0 0 1]	[2639191750/4782969 -45868537000/4782969 10557771250/1594323 553296799/59049 -3871127000/177147]
[0 0 0 1]	[-6175410800/4782969 107290158950/4782969 -24693841250/1594323 -1294131800/59049 9055019047/177147]

Question

Here (kill)

- ▶ Hecke
- ▶ Hecke

$\mathcal{Z}^q(\Gamma)$.

- ▶ Burau representation $\mathcal{B}^q(\Gamma)$, homotopy category of representations of zigzag algebras $\mathcal{K}^b(\mathcal{Z}^q(\Gamma))$.

Let Γ be a Coxeter graph. The following commuting diagram exists in any type:

$$\Delta T(\Gamma) \xlongequal{\quad} \Delta T(\Gamma)$$

Theorem (handlebody faithfulness).

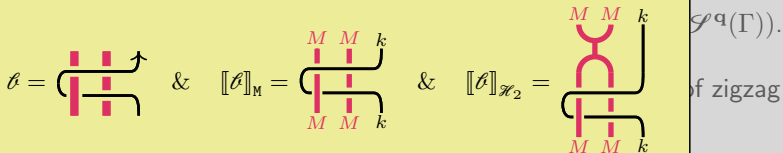
For all g, n , Rouquier's action $[[-]]$ gives rise to a family of faithful actions

$$\begin{array}{ccc} \mathcal{B}r(g, n) \curvearrowright \mathcal{K}^b(\mathcal{S}^q(\Gamma)), \ell \mapsto [[\ell]]_M & & \\ \downarrow & \xrightarrow{\text{decat.}} & \downarrow \\ \mathcal{K}^b(\mathcal{L}^q(\Gamma)) & & \mathcal{B}^q(\Gamma) \end{array}$$

Theorem (handlebody HOMFLYPT homology).

This action extends to a HOMFLYPT invariant for handlebody links.

Mnemonic:



▶ Please stop!

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$\mathcal{S}^q(\Gamma)$.

of zigzag

Rouquier ~ 2004 . The 2-braid group $\mathcal{AT}(\Gamma)$ is $\text{im}(\llbracket - \rrbracket) \subset \mathcal{K}^b(\mathcal{S}_s^{\mathfrak{q}}(\Gamma))$.

If you have a configuration space picture for $\text{AT}(\Gamma)$ one can define the category of braid cobordisms $\mathcal{B}_{\text{cob}}(\Gamma)$ in four space.

Fact (well-known?). For Γ of type A, $B = C$ or affine type C we have

$$\mathcal{AT}(\Gamma) = \text{inv}(\mathcal{B}_{\text{cob}}(\Gamma)).$$

Corollary (strictness). We have a categorical action

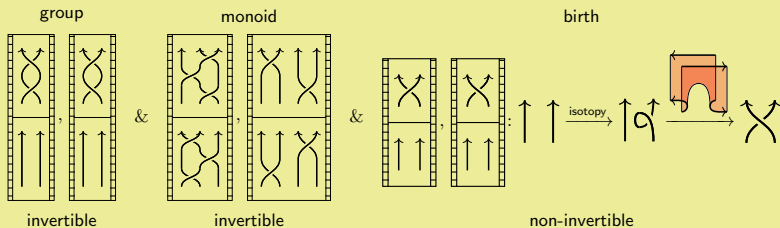
$$\text{inv}(\mathcal{B}_{\text{cob}}(g, n)) \curvearrowright \mathcal{K}^b(\mathcal{S}^{\mathfrak{q}}(\Gamma)), \mathcal{a} \mapsto \llbracket \mathcal{a} \rrbracket, \mathcal{a}_{\text{cob}} \mapsto \llbracket \mathcal{a}_{\text{cob}} \rrbracket.$$

Question (functoriality). Can we lift $\llbracket - \rrbracket$ to a categorical action

$$\mathcal{B}_{\text{cob}}(g, n) \curvearrowright \mathcal{K}^b(\mathcal{S}^{\mathfrak{q}}(\Gamma))?$$

Example (type A).

Braid cobordisms are movies of braids. E.g. some generators are



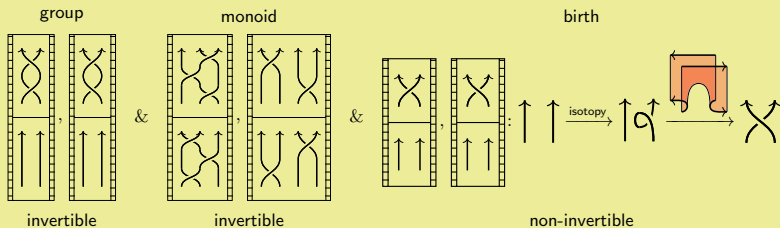
Invertible ones encode isotopies, non-invertible ones “more interesting” topology.

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Invertible ones encode isotopies, non-invertible ones "more interesting" topology.

Question (functoriality). Can we lift $\llbracket - \rrbracket$ to a categorical action

Theorem (well-known?).

The Rouquier complex is functorial in types
A, B = C and affine C.

Rouquier \sim 2004. The 2-braid group $\mathcal{AT}(\Gamma)$ is $\text{im}(\llbracket - \rrbracket) \subset \mathcal{K}^b(\mathcal{S}_s^{\mathfrak{q}}(\Gamma))$.

If you have a configuration space picture for $\mathcal{AT}(\Gamma)$ one can define the category of braid

Theorem (handlebody functoriality).

Fact For all g, n , Rouquier's action $\llbracket - \rrbracket$ gives rise to a family of functorial actions

$$\mathcal{B}_{\text{cob}}(g, n) \curvearrowright \mathcal{K}^b(\mathcal{S}^{\mathfrak{q}}(\Gamma)), \ell \mapsto \llbracket \ell \rrbracket_{\mathbb{M}}, \ell_{\text{cob}} \mapsto \llbracket \ell_{\text{cob}} \rrbracket_{\mathbb{M}}.$$

Coro ($\mathcal{B}_{\text{cob}}(g, n)$ is the 2-category of handlebody braid cobordisms.)

$$\text{inv}(\mathcal{B}_{\text{cob}}(g, n)) \curvearrowright \mathcal{K}^b(\mathcal{S}^{\mathfrak{q}}(\Gamma)), \ell \mapsto \llbracket \ell \rrbracket, \ell_{\text{cob}} \mapsto \llbracket \ell_{\text{cob}} \rrbracket.$$

Question (functoriality). Can we lift $\llbracket - \rrbracket$ to a categorical action

$$\mathcal{B}_{\text{cob}}(g, n) \curvearrowright \mathcal{K}^b(\mathcal{S}^{\mathfrak{q}}(\Gamma))?$$

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Question (functoriality)

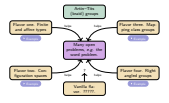
Final observation.

In all (non-trivial) cases I know
"faithful \Leftrightarrow functorial".

Is there a general statement?

cal action

My Takens. What I would like to understand, but I do not.
 Artin-Tits groups come in four main flavors.
 Question: Why are these special? What happens in general type?



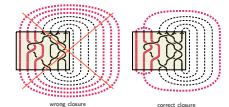
Maybe some categorical considerations help? In particular, what can Artin-Tits groups tell you about flavor tags?

Proof?
 Essentially: Reduce the problem to the mapping class $\mathcal{M}(\Sigma)$ group of a surface Σ , which acts on $\pi_1(\Sigma, \text{boundary})$ via Dehn twist.
 Then $\langle \sigma_i^{\pm 1} \rangle \cong \text{AT}(\Sigma) \cong \mathcal{M}(\Sigma) / \langle \pi_1(\Sigma, \text{boundary}) \text{ acts faithfully} \rangle$
 Example: The surface Σ is built from Γ by gluing annuli: Ann_i

$1 \rightarrow \langle \sigma_i \rangle \rightarrow \mathcal{M}(\Sigma) \rightarrow \text{AT}(\Sigma) \rightarrow 1$

Dehn twist along the **vertical** curve

The Alexander closure on $\text{ab}(\mathfrak{g}, \infty)$ is given by merging core strands at infinity

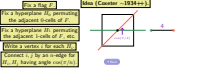


This is different from the classical Alexander closure.



Figure: The Coxeter graph of finite type (from the original Coxeter's original paper, 1934)

Examples.
 This gives a generator-relation presentation
 Type $A_n \rightarrow$ n hyperplanes in n -space
 Type $B_n \rightarrow$ n half-spaces in n -space
 Type $H_3 \rightarrow$ dodecahedron (icosahedron \rightarrow exceptional Coxeter group
 For $\mathfrak{h}_3(\mathbb{R})$ we have a 3-gon



Let $\text{Br}(\mathfrak{g}, n)$ be the group defined as follows.

Generators. Braid and twist generators

$$\sigma_i = \begin{array}{c} | \quad | \\ | \quad | \\ \hline | \quad | \\ | \quad | \end{array} \quad \tau_i = \begin{array}{c} | \quad | \\ | \quad | \\ \hline | \quad | \\ | \quad | \end{array}$$

Relations. Reidemeister braid relations, type C relations and special relations, e.g.

Involves three players and invariance

$$r_1 r_2 r_3 r_2 = r_2 r_1 r_2 r_3 \quad (r_1 r_2 r_3^{-1}) r_2 = r_2 (r_1 r_2 r_3^{-1})$$

The Alexander closure on $\text{ab}(\mathfrak{g}, \infty)$ at infinity

Theorem (Laudrop-Zukerman – 1993).
 For any link L in the genus g handlebody H_g , there is a braid in $\text{ab}(\mathfrak{g}, \infty)$ whose (correct) closure is isotopic to L

Fact.
 \mathfrak{P}_g is given by a complement in the 3-sphere S^3 by an open tubular neighborhood of the embedded graph obtained by gluing $g+1$ subnormal ‘‘type’’ edges to two vertices.

This is the 3-ball $\mathfrak{P}_g \cong \mathbb{D}^3$ a braid \mathfrak{B}_g

Lawrence – 1989, Kramer – 2000, Bigelow – 2000 (Cohn-Wales – 2000, Digne – 2000). Let Γ be of finite type. There exists a faithful action of $\text{AT}(\Gamma)$ on a finite-dimensional vector space.

Uphor: One can ask a computer program questions about braid!

```

sage: B = BraidGroup(3)
sage: T = B([1, 2])
sage: B.isfaithful(T)
False
sage: B.isfaithful(B)
True

```

Figure: SAGE in action: The Braid $[1, 2]$ action is not faithful, the LKB is.

Theorem (Hiring-Odenberg-Laudrop-Zukerman – 2002, Verdiani – 1998)

The map

is an isomorphism of groups $\text{Br}(\mathfrak{g}, n) \cong \text{ab}(\mathfrak{g}, n)$

$\cos(\pi/4)$ twice on a line:

Currently known (to the best of my knowledge).			
Genus	type A	type C	
$g=0$	$\text{ab}(\mathfrak{g}) \cong \text{AT}(A_{n-1})$		
$g=1$	$\text{ab}(1, n) \cong \mathbb{Z} \times \text{AT}(A_{n-1}) \cong \text{AT}(A_{n-1})$	$\text{ab}(1, n) \cong \text{AT}(C_n)$	
$g=2$		$\text{ab}(2, n) \cong \text{AT}(C_n)$	
$g \geq 3$			
Genus	type D	type B	
$g=0$			
$g=1$	$\text{ab}(1, n)_{\text{type D}} \cong \text{AT}(D_n)$	$\text{ab}(1, n)_{\text{type B}} \cong \text{AT}(B_n)$	
$g=2$	$\text{ab}(2, n)_{\text{type D}} \cong \text{AT}(D_n)$	$\text{ab}(2, n)_{\text{type B}} \cong \text{AT}(B_n)$	
$g \geq 3$			

(For orbifolds ‘‘genus’’ is just an analogy.)

There is still much to do...

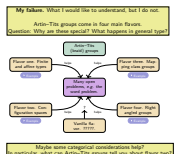


Figure: The Coxeter graph of finite type (from Bourbaki's Lie groups and Lie algebras)

Examples. This gives a generator-relation presentation.

Type A_n — n hyperplanes H_i generating the adjacent i -cells of \mathbb{R}^n .

Type B_n — n hyperplanes H_i generating the adjacent i -cells of \mathbb{R}^n , etc.

Type H_3 — dodecahedron (icosahedron) — exceptional Coxeter group.

For $h_3(3)$ we have a 3-gon:

For a flag \mathcal{F} :

Fix a hyperplane H_i generating the adjacent i -cells of \mathbb{R}^n .

Fix a hyperplane H_j generating the adjacent j -cells of \mathbb{R}^n , etc.

Write a vertex v for each \mathcal{F} .

Connect v, v' by an n -edge for $\mathcal{F}, \mathcal{F}'$ having single $\text{con}(v, v')$.

Lawrence — 1989, Kramer — 2000, Bigelow — 2000 (Cohn-Wales — 2000, Digne — 2000). Let Γ be finite type. There exists a faithful action of $AT(\Gamma)$ on a finite-dimensional vector space.

Upshot: One can ask a computer program questions about braid!

```

sage: G = CoxeterGroup([3, 3, 3])
sage: G
Coxeter group of type [3, 3, 3]
sage: G.order()
216
sage: G.rank()
3
sage: G.is_finite_type()
True
sage: G.is_coxeter_group()
True
sage: G.is_artin_tits_group()
True
sage: G.is_garside_group()
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sage: G.is_coxeter_group_of_rank_80()
False
sage: G.is_coxeter_group_of_rank_81()
False
sage: G.is_coxeter_group_of_rank_82()
False
sage: G.is_coxeter_group_of_rank_83()
False
sage: G.is_coxeter_group_of_rank_84()
False
sage: G.is_coxeter_group_of_rank_85()
False
sage: G.is_coxeter_group_of_rank_86()
False
sage: G.is_coxeter_group_of_rank_87()
False
sage: G.is_coxeter_group_of_rank_88()
False
sage: G.is_coxeter_group_of_rank_89()
False
sage: G.is_coxeter_group_of_rank_90()
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sage: G.is_coxeter_group_of_rank_91()
False
sage: G.is_coxeter_group_of_rank_92()
False
sage: G.is_coxeter_group_of_rank_93()
False
sage: G.is_coxeter_group_of_rank_94()
False
sage: G.is_coxeter_group_of_rank_95()
False
sage: G.is_coxeter_group_of_rank_96()
False
sage: G.is_coxeter_group_of_rank_97()
False
sage: G.is_coxeter_group_of_rank_98()
False
sage: G.is_coxeter_group_of_rank_99()
False
sage: G.is_coxeter_group_of_rank_100()
False

```

Figure: SAGE in action: The Bura $\Gamma(1)$ action is not faithful, the LXX is.

Proof?

Essentially: Reduce the problem to the mapping class $\mathcal{M}(\Sigma)$ group of a surface Σ , which acts on $\pi_1(\Sigma, \text{boundary})$ via Dehn twist.

Then $\langle \mathcal{M}^n \rangle \cong AT(\Gamma) \subset \mathcal{M}(\Sigma) \subset \pi_1(\Sigma, \text{boundary})$ acts faithfully.

Example: The surface Σ is built from Γ by gluing annuli: Ann_i .

Dehn twist along the annuli center.

Let $Is(\mathfrak{g}, n)$ be the group defined as follows.

Generators. Braid and twist generators

$$s_i = \begin{matrix} \uparrow & \downarrow \\ \uparrow & \downarrow \\ \uparrow & \downarrow \\ \uparrow & \downarrow \\ \uparrow & \downarrow \end{matrix} \quad \& \quad t_i = \begin{matrix} \uparrow & \downarrow \\ \uparrow & \downarrow \\ \uparrow & \downarrow \\ \uparrow & \downarrow \\ \uparrow & \downarrow \end{matrix}$$

Relations. Reidemeister braid relations, type C relations and special relations, e.g.

$\delta_1 \delta_2 \delta_3 \delta_2 = \delta_2 \delta_1 \delta_3 \delta_2$

$(\delta_1 \delta_2 \delta_3^2) \delta_2 = \delta_2 (\delta_1 \delta_2 \delta_3^2)$

Involves three players and invariance

Theorem (Häring-Oldenburg-Laudropoulos — 2002, Verhölten — 1998)

The map

is an isomorphism of groups $Is(\mathfrak{g}, n) \cong \mathcal{M}(\Sigma, n)$.

The Alexander closure on $is(\mathfrak{g}, \infty)$ is given by merging core strands at infinity

wrong closure

correct closure

This is different from the classical Alexander closure.

Theorem (Laudropoulos — 1993).

For any link L in the genus g handlebody \mathcal{H}_g , there is a braid in $is(\mathfrak{g}, \infty)$ whose (correct) closure is isotopic to L .

Fact.

\mathcal{H}_g is given by a complement in the 3-sphere S^3 by an open tubular neighborhood of the embedded graph obtained by gluing $g+1$ subnormal 'type' edges to two vertices.

This is

the 3-ball $\mathcal{H}_1 \cong \mathcal{D}^3$ a torus \mathcal{H}_2 \mathcal{H}_3

$\text{con}(\pi/4)$ twice on a line:

Currently known (to the best of my knowledge).		
Genus	type A	type C
$g=0$	$is(\mathfrak{g}) \cong AT(A_{n-1})$	
$g=1$	$is(\mathbb{1}, n) \cong \mathbb{Z} \times AT(A_{n-1}) \cong AT(A_{n-1})$	$is(\mathbb{1}, n) \cong AT(C_{n-1})$
$g=2$		$is(\mathbb{2}, n) \cong AT(C_{n-1})$
$g \geq 3$		
And some $\mathbb{Z}/2\mathbb{Z}$ -orbifolds $(\mathbb{Z}/2\mathbb{Z})$ -invariant:		
Genus	type D	type B
$g=0$		
$g=1$	$is(\mathbb{1}, n)_{\mathbb{Z}/2\mathbb{Z}} \cong AT(D_{n-1})$	$is(\mathbb{1}, n)_{\mathbb{Z}/2\mathbb{Z}} \cong AT(B_{n-1})$
$g=2$	$is(\mathbb{2}, n)_{\mathbb{Z}/2\mathbb{Z}} \cong AT(D_{n-1})$	$is(\mathbb{2}, n)_{\mathbb{Z}/2\mathbb{Z}} \cong AT(B_{n-1})$
$g \geq 3$		
(For orbifolds "genus" is just an analogy.)		

Thanks for your attention!

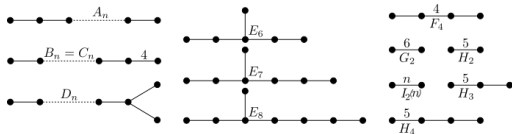


Figure: The Coxeter graphs of finite type. (Picture from https://en.wikipedia.org/wiki/Coxeter_group.)

Examples.

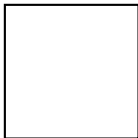
Type $A_3 \iff$ tetrahedron \iff symmetric group S_4 .

Type $B_3 \iff$ cube/octahedron \iff Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \times S_3$.

Type $H_3 \iff$ dodecahedron/icosahedron \iff exceptional Coxeter group.

For $I_2(4)$ we have a 4-gon:

Idea (Coxeter ~1934++).



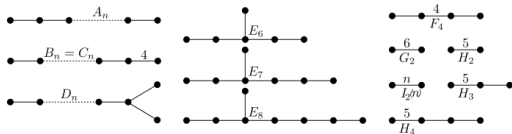


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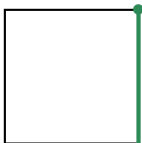
Type $B_3 \iff$ cube/octaneuron \iff wreath group $(\mathbb{Z}/2\mathbb{Z}) \ltimes S_3$.

Type $H_3 \iff$ dodecahedron/icosahedron \iff exceptional Coxeter group.

For $I_2(4)$ we have a 4-gon:

Fix a flag F .

Idea (Coxeter $\sim 1934++$).



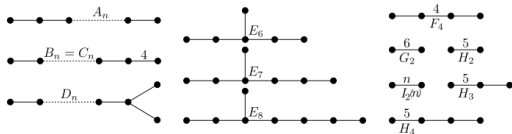


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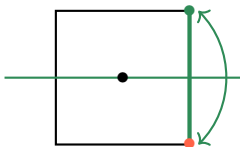
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For $I_2(4)$ we have a 4-gon:

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Fix a hyperplane H_0 permuting the adjacent 0-cells of F .



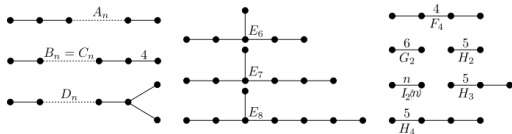


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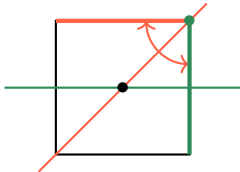
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Fix a hyperplane H_0 permuting the adjacent 0-cells of F .

Fix a hyperplane H_1 permuting the adjacent 1-cells of F , etc.



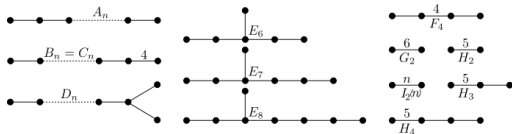


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For $I_2(4)$ we have a 4-gon:

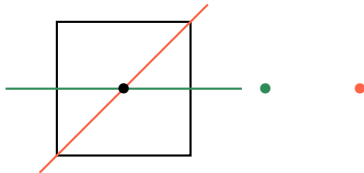
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Fix a hyperplane H_1 permuting the adjacent 1-cells of F , etc.

Write a vertex i for each H_i .



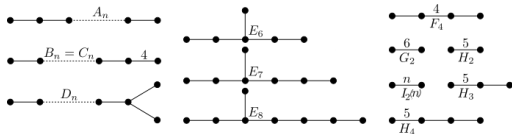


Figure: The Coxeter graphs of finite type. (Picture from https://en.wikipedia.org/wiki/Coxeter_group.)

Examples.

This gives a generator-relation presentation.

Type $A_3 \iff$ tetrahedron \iff symmetric group S_4 .

Type $B_3 \iff$ And the braid relation measures the angle between hyperplanes.

Type $H_3 \iff$ dodecahedron/icosahedron \iff exceptional Coxeter group.

For $I_2(4)$ we have a 4-gon:

Fix a flag F .

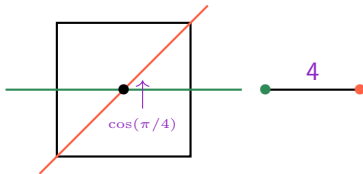
Idea (Coxeter \sim 1934++).

Fix a hyperplane H_0 permuting the adjacent 0-cells of F .

Fix a hyperplane H_1 permuting the adjacent 1-cells of F , etc.

Write a vertex i for each H_i .

Connect i, j by an n -edge for H_i, H_j having angle $\cos(\pi/n)$.



Lawrence ~ 1989 , Krammer ~ 2000 , Bigelow ~ 2000 (Cohen–Wales ~ 2000 , Digne ~ 2000). Let Γ be of finite type. There exists a faithful action of $AT(\Gamma)$ on a finite-dimensional vector space.

Upshot: One can ask a computer program questions about braids!

```
sage: B.<b1,b2,b3,b4,b5> = BraidGroup(6)
sage: psi1 = b4 * b5^(-1) * b2^(-1) * b1
sage: psi2 = b4^(-1) * b5^(2) * b2 * b1^(-2)
sage: w1 = psi1^(-1) * b3 * psi1
sage: w2 = psi2^(-1) * b3 * psi2
sage: print((w1 * w2 * w1^(-1) * w2^(-1)).TL_matrix(4))
sage: print(((w1 * w2 * w1^(-1) * w2^(-1)).LKB_matrix()).substitute(x=-1,y=1))

evaluate
[ 1 0 0 0 0]
[ 0 1 0 0 0]
[ 0 0 1 0 0]
[ 0 0 0 1 0]
[ 0 0 0 0 1]
[ -15 -80 -80 0 -16 -64 -64 16 0 0 80 64 80 64 -16]
[ 32 129 128 32 64 96 96 0 32 0 -96 -64 -96 -64 32]
[ -32 -128 -127 -32 -64 -96 -96 0 -32 0 96 64 96 64 -32]
[ 16 80 80 1 16 64 64 -16 0 0 -80 -64 -80 -64 16]
[ 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0]
[ -64 -192 -192 -32 -96 -127 -128 32 -32 0 160 96 160 96 -64]
[ 64 192 192 32 96 128 129 -32 32 0 -160 -96 -160 -96 64]
[ 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0]
[ -16 -80 -80 0 -16 -64 -64 16 1 0 80 64 80 64 -16]
[ 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0]
[ -64 -192 -192 -32 -96 -128 -128 32 -32 0 161 96 160 96 -64]
[ 32 128 128 32 64 96 96 0 32 0 -96 -63 -96 -64 32]
[ 64 192 192 32 96 128 128 -32 32 0 -160 -96 -159 -96 64]
[ -32 -128 -128 -32 -64 -96 -96 0 -32 0 96 64 96 65 -32]
[ 16 80 80 0 16 64 64 -16 0 0 -80 -64 -80 -64 17]
```

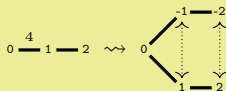
Figure: SAGE in action: The Burau (TL) action is not faithful, the LKB is.

Lawrence ~ 1989 , Krammer ~ 2000 , Bigelow ~ 2000 (Cohen–Wales ~ 2000 , Digne ~ 2000). Let Γ be of finite type. There exists a faithful action of $AT(\Gamma)$ on a finite-dimensional vector space.

Proof?

Uses root combinatorics of ADE diagrams and the fact that each $AT(\Gamma)$ of finite type can be embedded in types ADE.

Example. Type B “unfolds” into type A:

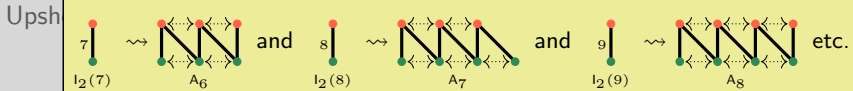


But there is also a different way, discussed later.

Figure: SAGE in action: The Burau (TL) action is not faithful, the LKB is.

Lawrence ~ 1989 , Krammer ~ 2000 , Bigelow ~ 2000 (Cohen–Wales ~ 2000 , Digne ~ 2000). Let Γ be of finite type. There exists a faithful action of $AT(\Gamma)$

on a ... **Example.** In the dihedral case these (un)foldings correspond to bicolorings:



Fact.

This gives $AT(I_2(n)) \hookrightarrow AT(\Gamma)$

\Leftrightarrow

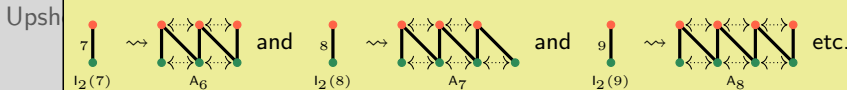
$\Gamma = ADE$ for $n = \text{Coxeter number}$.

```
[ 0 0 0 1 0 ]
[ 0 0 0 0 1 ]
[ -15 -80 -80 0 -16 -64 -64 16 0 0 80 64 80 64 -16 ]
[ 32 129 128 32 64 96 96 0 32 0 -96 -64 -96 -64 32 ]
[ -32 -128 -127 -32 -64 -96 -96 0 -32 0 96 64 96 64 -32 ]
[ 16 80 80 1 16 64 64 -16 0 0 -80 -64 -80 -64 16 ]
[ 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 ]
[ -64 -192 -192 -32 -96 -127 -128 32 -32 0 160 96 160 96 -64 ]
[ 64 192 192 32 96 128 129 -32 32 0 -160 -96 -160 -96 64 ]
[ 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 ]
[ -16 -80 -80 0 -16 -64 -64 16 1 0 80 64 80 64 -16 ]
[ 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 ]
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```
[ 0 0 0 1 0 ]
[ 0 0 0 0 1 ]
[ -15 -80 -80 0 -16 -64 -64 16 0 0 80 64 80 64 -16 ]
[ 32 129 128 32 64 96 96 0 32 0 -96 -64 -96 -64 32 ]
```

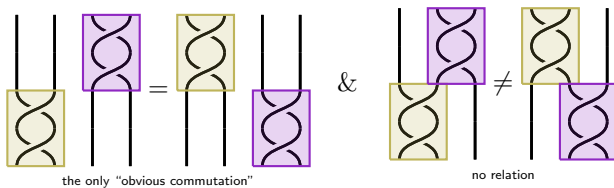
Example (SAGE; $n = 9$). LKB says it is true:

```
sage: B.<b1,b2,b3,b4,b5,b6,b7,b8> = BraidGroup(9)
sage: x = b1 * b3 * b5 * b7
sage: y = b2 * b4 * b6 * b8
sage: w = x * y * x * y * x * y * x * y * x * y * x
sage: v = y * x * y * x * y * x * y * x * y * x * y
sage: w == v
True
```

Figure: SAGE in action: The Burau (TL) action is not faithful, the LKB is.

Crisp–Paris ~ 2000 (Tits conjecture). For all $m > 1$, the subgroup $\langle \mathcal{C}_i^m \rangle \subset AT(\Gamma)$ is free (up to “obvious commutation”).

In finite type this is a consequence of LKB; in type A it is clear:



This should have told me something: I will come back to this later.

Proof?

Cr

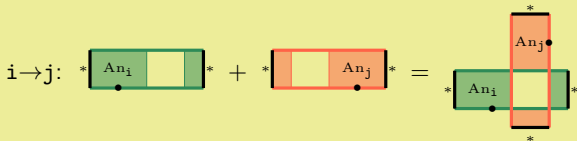
$\langle \ell_i^m \rangle$

Essentially: Relate the problem to the mapping class $\mathcal{M}(\Sigma)$ group of a surface Σ , which acts on $\pi_1(\Sigma, \text{boundary})$ via Dehn twist.

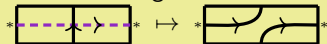
In

Then $\langle \ell_i^m \rangle \hookrightarrow \text{AT}(\Gamma) \rightarrow \mathcal{M}(\Sigma) \curvearrowright \pi_1(\Sigma, \text{boundary})$ acts faithfully.

Example. The surface Σ is built from Γ by gluing annuli A_{n_i} :



Dehn twist along the orchid curve:

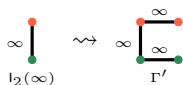


Th

Recall. Right-angled means $m_{ij} \in \{2, \infty\}$.

Fact (well-known?). Let Γ be of right-angled type. There exists a faithful action of $AT(\Gamma)$ on a finite-dimensional \mathbb{R} -vector space.

Example. $\Gamma = I_2(\infty)$, the infinite dihedral group.



Define a map

$$AT(\Gamma) \rightarrow W(\Gamma'), s \mapsto ss, t \mapsto tt.$$

Crazy fact: This is an embedding, and actually

$$W(\Gamma') \cong AT(\Gamma) \rtimes (\mathbb{Z}/2\mathbb{Z})^2.$$

Thus, via Tits' reflection representation, it follows that $AT(\Gamma)$ is linear.

Recall. Right-angled means $m_{ij} \in \{2, \infty\}$.

Fact (well-known?). Let Γ be of right-angled type. There exists a faithful action of $AT(\Gamma)$ on a finite-dimensional \mathbb{R} -vector space.

Example. $\Gamma = I_2^n$

Proof?

This works in general:

For each right-angled Γ there exists a Γ' such that
 $W(\Gamma') \cong AT(\Gamma) \rtimes (\mathbb{Z}/2\mathbb{Z})^i$.

Corollary.

Tits' reflection representation gives a faithful action on a finite-dimensional \mathbb{R} -vector space.

Define a map

$$AT(\Gamma) \rightarrow W(\Gamma'), s \mapsto ss, t \mapsto tt.$$

Crazy fact: This

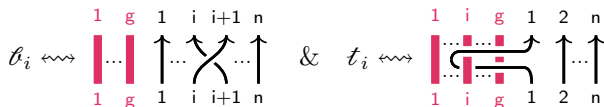
This is the only case where I know that the Artin–Tits group embeds into a Coxeter group.

$$W(\Gamma') \cong AT(\Gamma) \rtimes (\mathbb{Z}/2\mathbb{Z})^2.$$

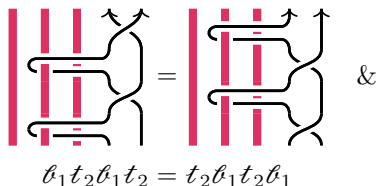
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Let $\text{Br}(g, n)$ be the group defined as follows.

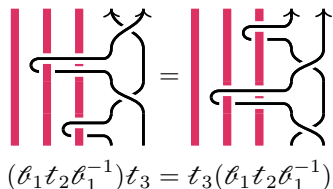
Generators. Braid and twist generators



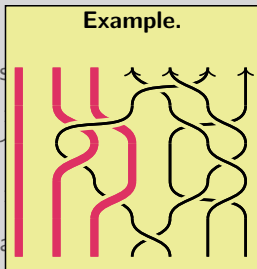
Relations. Reidemeister braid relations, type C relations and special relations, e.g.



Involves three players and inverses!

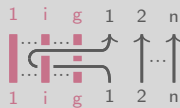


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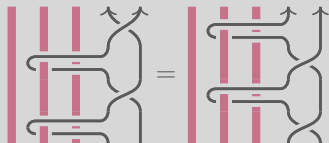
$\ell_i \leftrightarrow$



Relations. Reidemeister bra

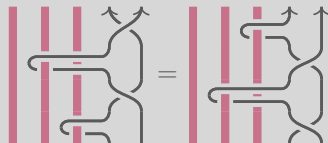
relations and special relations, e.g.

Involves three players and inverses!



$$\ell_1 t_2 \ell_1 t_2 = t_2 \ell_1 t_2 \ell_1$$

&

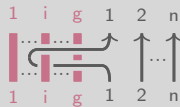
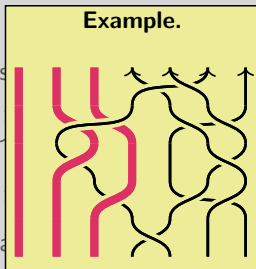


$$(\ell_1 t_2 \ell_1^{-1}) t_3 = t_3 (\ell_1 t_2 \ell_1^{-1})$$

Let $\text{Br}(g, n)$ be the group defined as follows.

Generators. Braid and twist

$\sigma_i \leftrightarrow$



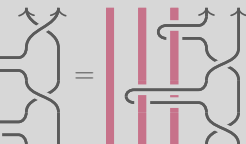
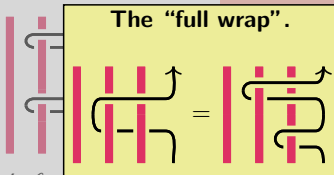
Relations. Reidemeister bra

relations and special relations, e.g.

Involves three players and inverses!



=



$$\sigma_1 t_2 \sigma_1 t_2 = t_2 \sigma_1 t_2 \sigma_1$$

$$(\sigma_1 t_2 \sigma_1^{-1}) t_3 = t_3 (\sigma_1 t_2 \sigma_1^{-1})$$

Let $\text{Br}(g, n)$ be the group defined as follows.

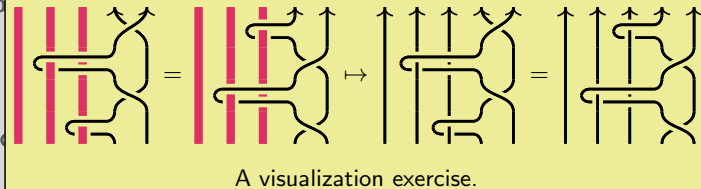
Generators. Braid and twist generators

1 g 1 i i+1 n 1 i g 1 2 n

Fact (type A embedding).

$\text{Br}(g, n)$ is a subgroup of the usual braid group $\mathcal{B}r(g+n)$.

Relation



relations, e.g.

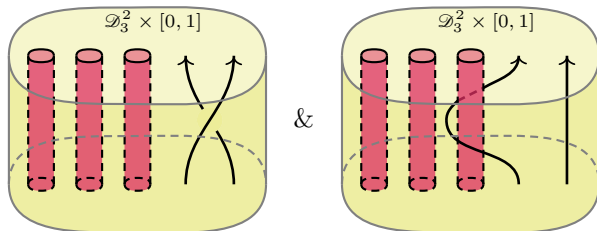
crosses!

$$\ell_1 t_2 \ell_1 t_2 = t_2 \ell_1 t_2 \ell_1$$

$$(\ell_1 t_2 \ell_1^{-1}) t_3 = t_3 (\ell_1 t_2 \ell_1^{-1})$$

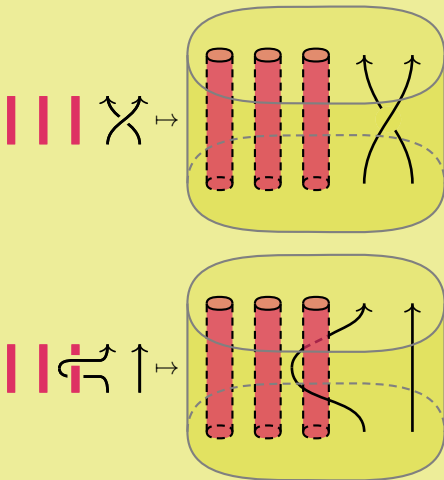
The group $\mathcal{B}r(g, n)$ of braid in a g -times punctures disk $\mathcal{D}_g^2 \times [0, 1]$:

Two types of braidings, the usual ones and “winding around cores”, e.g.



Theorem (Häring-Oldenburg–Lambropoulou ~2002, Vershinin ~1998).

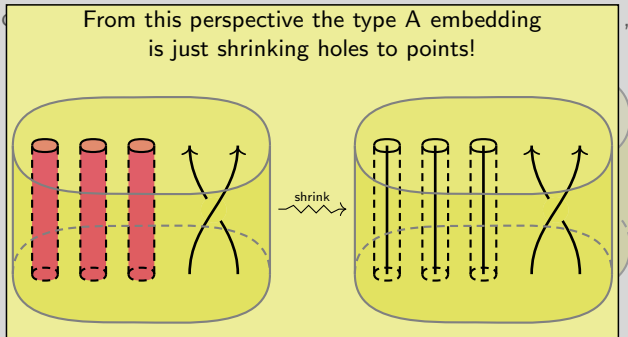
The map



is an isomorphism of groups $\text{Br}(g, n) \rightarrow \mathcal{B}\text{r}(g, n)$.

The group $\mathcal{B}r(g, n)$ of braid in a g -times punctures disk $\mathcal{D}_g^2 \times [0, 1]$:

Two types of embeddings, e.g.

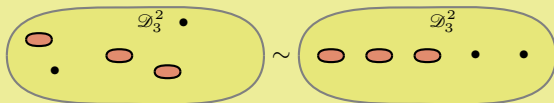


The group $\mathcal{B}r(g, n)$ of braid in a g -times punctures disk $\mathcal{D}_g^2 \times [0, 1]$:

Two types of braidings, the usual ones and “winding around cores” e.g.

Note.

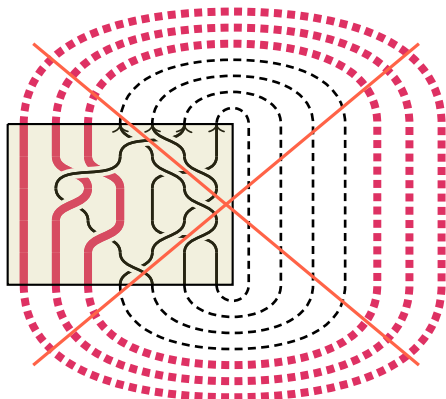
For the proof it is crucial that \mathcal{D}_g^2 and the boundary points of the braids \bullet are only defined up to isotopy, e.g.



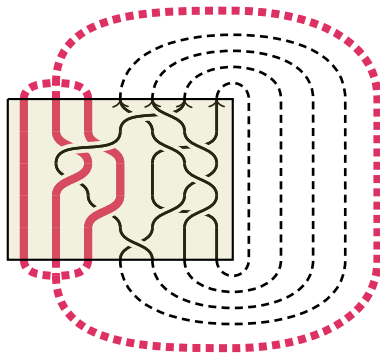
\Rightarrow one can always “conjugate cores to the left”.

This is useful to define $\mathcal{B}r(g, \infty)$.

The Alexander closure on $\mathcal{BR}(g, \infty)$ is given by merging core strands at infinity.



wrong closure



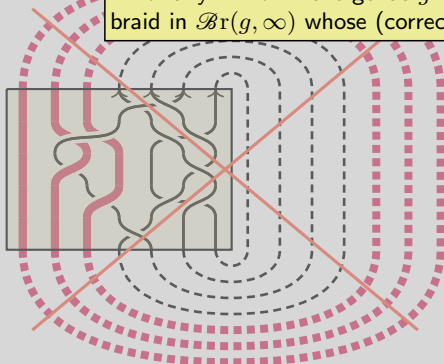
correct closure

This is different from the classical Alexander closure.

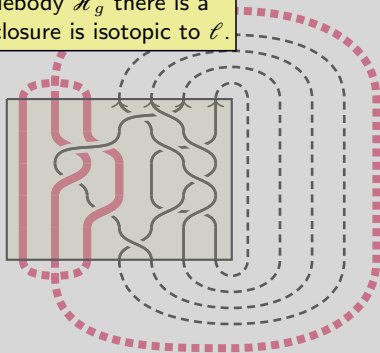
The Alexander closure on $\mathcal{B}_R(g, \infty)$ is given by merging core strands at infinity.

Theorem (Lambropoulou ~1993).

For any link ℓ in the genus g handlebody \mathcal{H}_g there is a braid in $\mathcal{B}_R(g, \infty)$ whose (correct!) closure is isotopic to ℓ .



wrong closure



correct closure

This is different from the classical Alexander closure.

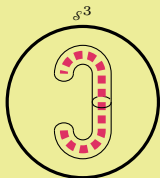
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Theorem (Lambropoulou ~1993).

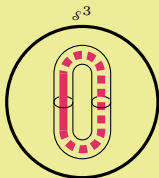
For any link ℓ in the genus g handlebody \mathcal{H}_g there is a braid in $\mathcal{B}_R(g, \infty)$ whose (correct!) closure is isotopic to ℓ .

Fact.

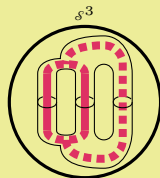
\mathcal{H}_g is given by a complement in the 3-sphere \mathcal{S}^3 by an open tubular neighborhood of the embedded graph obtained by gluing $g + 1$ unknotted "core" edges to two vertices.



the 3-ball $\mathcal{H}_0 = \mathcal{D}^3$



a torus \mathcal{H}_1



\mathcal{H}_2

This is

$\cos(\pi/3)$ on a line:

$$\text{type } A_{n-1}: 1 \text{ --- } 2 \text{ --- } \dots \text{ --- } n-2 \text{ --- } n-1$$

The classical case. Consider the map

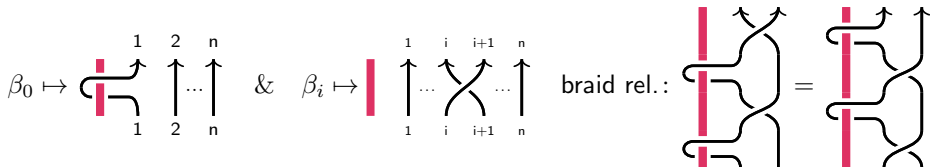
$$\beta_i \mapsto \begin{array}{cccc} 1 & i & i+1 & n \\ \uparrow & \swarrow & \nearrow & \uparrow \\ \dots & & & \dots \\ 1 & i & i+1 & n \end{array} \quad \text{braid rel.:} \quad \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \diagup \quad \diagdown \quad \diagup \\ \uparrow \quad \uparrow \quad \uparrow \end{array} = \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagup \quad \diagdown \quad \diagup \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array}$$

Artin ~1925. This gives an isomorphism of groups $AT(A_{n-1}) \xrightarrow{\cong} \mathcal{B}r(0, n)$.

$\cos(\pi/4)$ on a line:

$$\text{type } C_n: 0 \stackrel{4}{=} 1 - 2 - \dots - n-1 - n$$

The semi-classical case. Consider the map

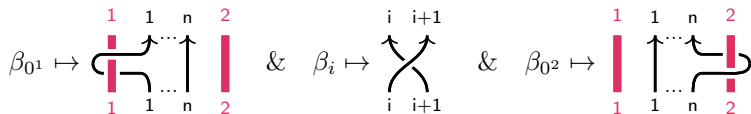


Brieskorn \sim 1973. This gives an isomorphism of groups $\text{AT}(C_n) \xrightarrow{\cong} \mathcal{B}\text{r}(1, n)$.

$\cos(\pi/4)$ twice on a line:

$$\text{type } \tilde{C}_n: 0^1 \overset{4}{=} 1 \text{ --- } 2 \text{ --- } \dots \text{ --- } n-1 \text{ --- } n \overset{4}{=} 0^2$$

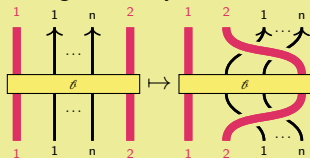
Affine adds genus. Consider the map



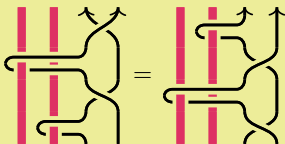
Allcock ~1999. This gives an isomorphism of groups $\text{AT}(\tilde{C}_n) \xrightarrow{\cong} \mathcal{B}r(2, n)$.

This case is strange – it only arises under conjugation:

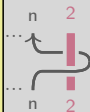
$\cos(\pi/4)$ twice



By a miracle, one can avoid the special relation



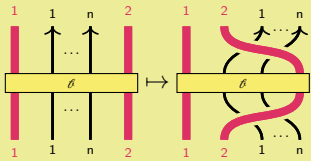
This relation involves three players and inverses. Bad!



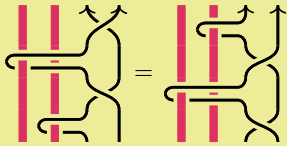
Allcock \sim 1999. This gives an isomorphism of groups $AT(\tilde{C}_n) \xrightarrow{\cong} \mathcal{B}r(2, n)$.

$\cos(\pi/4)$ twice

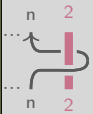
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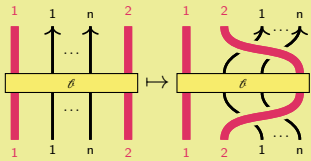
Affine adds ge

$\beta_{0^1} \mapsto$

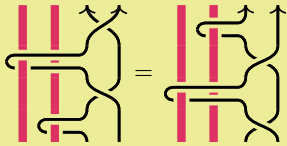
Currently, not much seems to be known, but I think the same story works.

Allcock ~1999. This gives an isomorphism of groups $AT(\tilde{C}_n) \xrightarrow{\cong} Br(2, n)$.

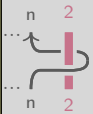
This case is strange – it only arises under conjugation:



By a miracle, one can avoid the special relation

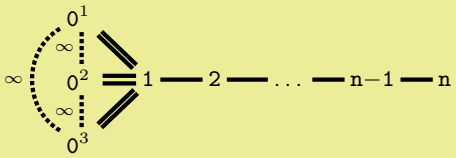


This relation involves three players and inverses. Bad!



Currently, not much seems to be known, but I think the same story works.

However, this is where it seems to end, e.g. genus $g = 3$ wants to be



In some sense this can not work; remember Tits conjecture.

$\cos(\pi/4)$ twice

Affine adds ge

$\beta_{01} \mapsto$

Allcock

1000 This gives an isomorphism of groups $\mathcal{AT}(\tilde{C}_n) \cong \mathcal{AT}(g, n)$.

$\cos(\pi/4)$ twice on a line:

Currently known (to the best of my knowledge).

Aff

Genus	type A	type C
$g = 0$	$\mathcal{B}r(n) \cong AT(A_{n-1})$	
$g = 1$	$\mathcal{B}r(1, n) \cong \mathbb{Z} \ltimes AT(\tilde{A}_{n-1}) \cong AT(\hat{A}_{n-1})$	$\mathcal{B}r(1, n) \cong AT(C_n)$
$g = 2$		$\mathcal{B}r(2, n) \cong AT(\tilde{C}_n)$
$g \geq 3$		

And some $\mathbb{Z}/2\mathbb{Z}$ -orbifolds ($\mathbb{Z}/\infty\mathbb{Z}$ = puncture):

All

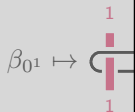
Genus	type D	type B
$g = 0$		
$g = 1$	$\mathcal{B}r(1, n)_{\mathbb{Z}/2\mathbb{Z}} \cong AT(D_n)$	$\mathcal{B}r(1, n)_{\mathbb{Z}/\infty\mathbb{Z}} \cong AT(B_n)$
$g = 2$	$\mathcal{B}r(2, n)_{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}} \cong AT(\tilde{D}_n)$	$\mathcal{B}r(2, n)_{\mathbb{Z}/\infty\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}} \cong AT(\tilde{B}_n)$
$g \geq 3$		

(For orbifolds "genus" is just an analogy.)

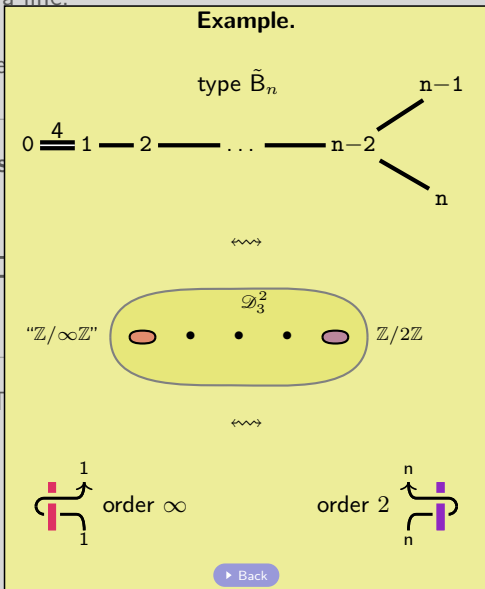
$\cos(\pi/4)$ twice on a line:

type

Affine adds genus



Allcock ~1999. T



$$\xrightarrow{\cong} \mathcal{B}r(2, n).$$