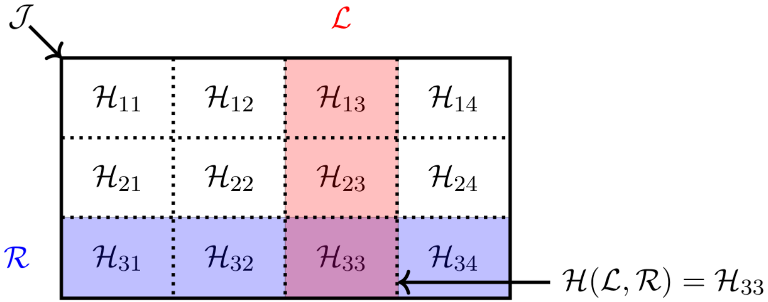
Representation theory of monoids and monoidal categories

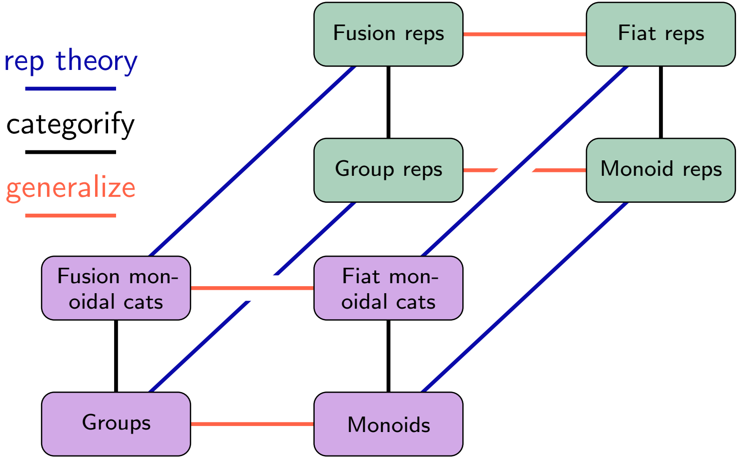
Or: Cells and actions

Daniel Tubbenhauer



Joint with Marco Mackaay, Volodymyr Mazorchuk, Vanessa Miemietz and Xiaoting Zhang

Where are we?



- **Green, Clifford, Munn, Ponizovskiĭ ∼1940++ many others**
  Representation theory of (finite) monoids

- Goal Find some categorical analog

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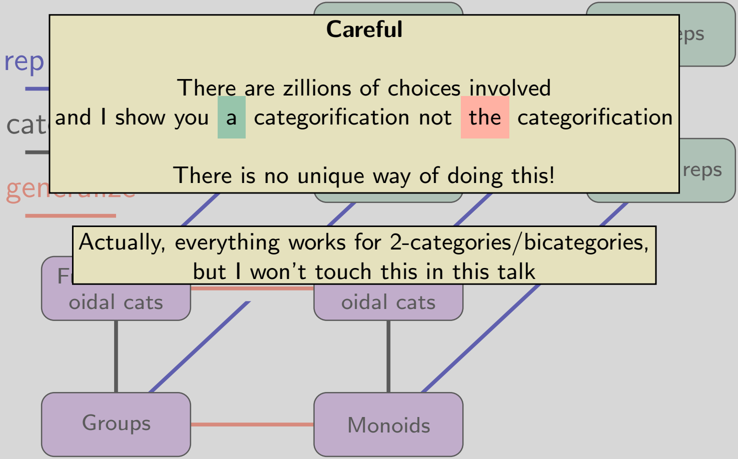

Careful

There are zillions of choices involved
and I show you a categorification not the categorification

There is no unique way of doing this!

Actually, everything works for 2-categories/bicategories,
but I won't touch this in this talk

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Today

I explain monoid and fiat rep theory

fiat monoidal categories $\xrightarrow{\text{categorify}}$ certain fin dim algebras $\supset$ monoid algebras

fiat reps $\xrightarrow{\text{categorify}}$ reps of certain fin dim algebras $\supset$ monoid reps

- **Goal** Find some categorical analog

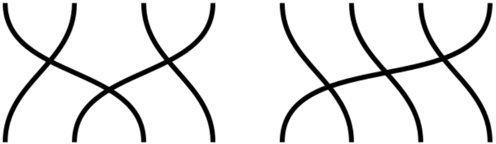
Where are we?

Examples of monoids

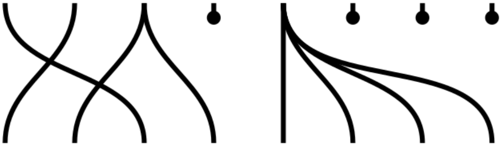
Groups

Multiplicative closed sets of matrices (these need not to be unital, but anyway)

Symmetric groups $\mathrm{Aut}(\{1,...,n\})$

$(24138567) \leftrightsquigarrow$ 

Transformation monoids $\mathrm{End}(\{1,...,n\})$

$(23135555) \leftrightsquigarrow$ 

- Representation theory of (finite) monoids
- Goal Find some categorical analog

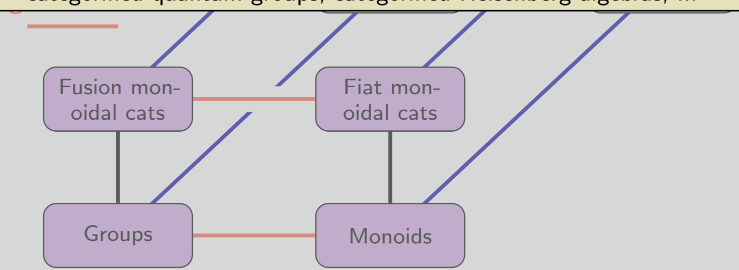
Examples of monoidal categories

G-graded vector spaces $\mathscr{V}\mathrm{ect}_G$, module categories $\mathscr{R}\mathrm{ep}(G)$, same for monoids

$\mathscr{R}\mathrm{ep}$(Hopf algebra), tensor or fusion or modular categories,

Soergel bimodules ("the Hecke category"),

categorified quantum groups, categorified Heisenberg algebras, ...



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Examples of reps of these

Categorical modules, functorial actions,

(co)algebra objects, conformal embeddings of affine Lie algebras, the LLT algorithm,

cyclotomic Hecke/KLR algebras, categorified (anti-)spherical module, ...

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Groups Monoids

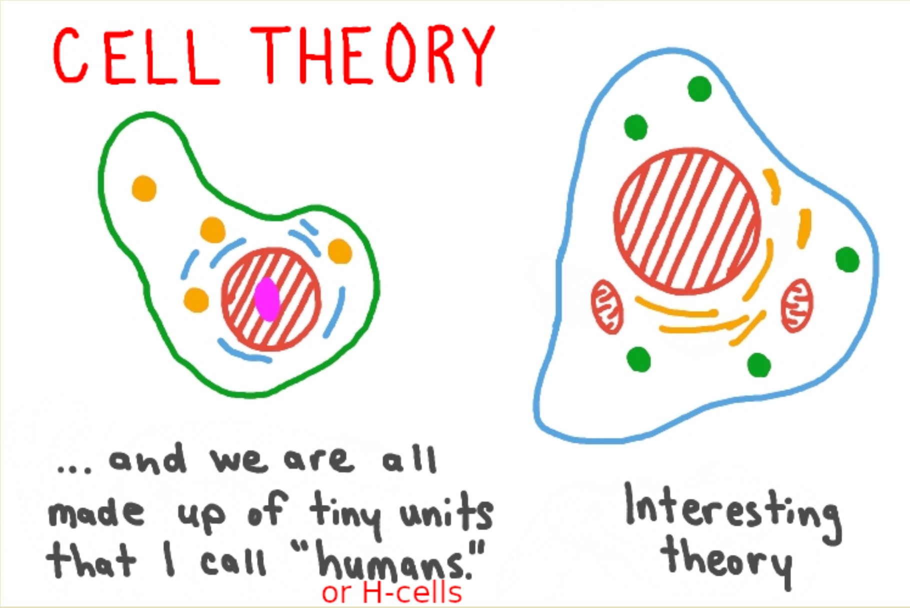
Applications of categorical representations

- Representation theory (classical and modular), link homologies, combinatorics,

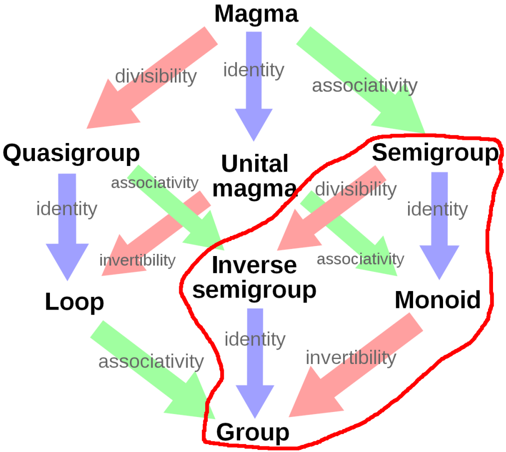
  TQFTs, quantum physics, geometry, ...

- Goal Find some categorical analog

The main player today: cell theory!



The theory of monoids (Green ∼1950++)



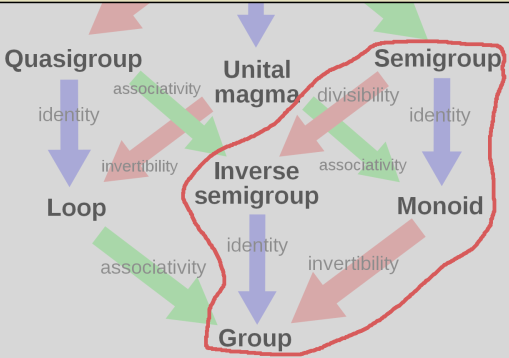
- ▶ Associativity ⇒ reasonable theory of matrix reps
- ▶ Southeast corner ⇒ reasonable theory of matrix reps

The

Adjoining identities is "free" and there is no essential difference between semigroups and monoids, or inverses semigroups and groups

The main difference is semigroups/monoids *vs.* inverses semigroups/groups

Today I will stick with the more familiar monoids and groups



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The point of monoid theory is to keep track of information loss

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Monoids appear naturally in categorification

Group-like structures

	Totality$^{\alpha}$	Associativity	Identity	Invertibility	Commutativity
Semigroupoid	Unneeded	Required	Unneeded	Unneeded	Unneeded
Small category	Unneeded	Required	Required	Unneeded	Unneeded
Groupoid	Unneeded	Required	Required	Required	Unneeded
Magma	Required	Unneeded	Unneeded	Unneeded	Unneeded
Quasigroup	Required	Unneeded	Unneeded	Required	Unneeded
Unital magma	Required	Unneeded	Required	Unneeded	Unneeded
Semigroup	Required	Required	Unneeded	Unneeded	Unneeded
Loop	Required	Unneeded	Required	Required	Unneeded
Inverse semigroup	Required	Required	Unneeded	Required	Unneeded
Monoid	Required	Required	Required	Unneeded	Unneeded
Commutative monoid	Required	Required	Required	Unneeded	Required
Group	Required	Required	Required	Required	Unneeded
Abelian group	Required	Required	Required	Required	Required

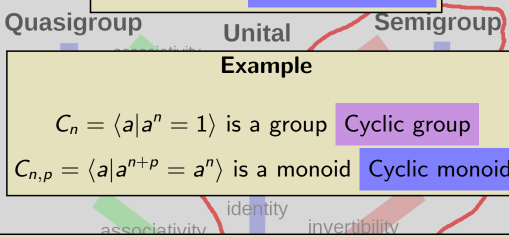
- Associativity =
- Southeast corr

The theory of monoids (Green ∼1950++)

Example

$\mathbb{Z}$ is a group **Integers**

$\mathbb{N}$ is a monoid **Natural numbers**



Example

$C_n = \langle a | a^n = 1 \rangle$ is a group **Cyclic group**

$C_{n,p} = \langle a | a^{n+p} = a^n \rangle$ is a monoid **Cyclic monoid**

Example

$S_n = \mathrm{Aut}(\{1, ..., n\})$ is a group **Symmetric group**

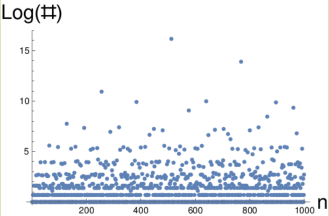
$T_n = \mathrm{End}(\{1, ..., n\})$ is a monoid **Transformation monoid**

- Associativity $\Rightarrow$ reasonable theory of matrix reps
- Southeast corner $\Rightarrow$ reasonable theory of matrix reps

Finite groups are kind of random...

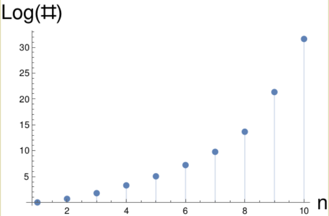
A000001 Number of groups of order n.
(Formerly M0098 N0035)

0, 1, 1, 1, 2, 1, 2, 1, 5, 2, 2, 1, 5, 1, 2, 1, 14, 1, 5, 1, 5, 2, 2, 1, 15, 2, 2, 5, 4, 1, 4, 1, 51, 1, 2, 1, 14, 1, 2, 2, 14, 1, 6, 1, 4, 2, 2, 1, 52, 2, 5, 1, 5, 1, 15, 2, 13, 2, 2, 1, 13, 1, 2, 4, 267, 1, 4, 1, 5, 1, 4, 1, 50, 1, 2, 3, 4, 1, 6, 1, 52, 15, 2, 1, 15, 1, 2, 1, 12, 1, 10, 1,



A058133 Number of monoids (semigroups with identity) of order n, considered to be equivalent when they are isomorphic or anti-isomorphic (by reversal of the operator).

0, 1, 2, 6, 27, 156, 1373, 17730, 858977, 1844075697, 52991253973742 (list; graph; refs; listen; history;



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The theory of monoids (Green ∼1950++)

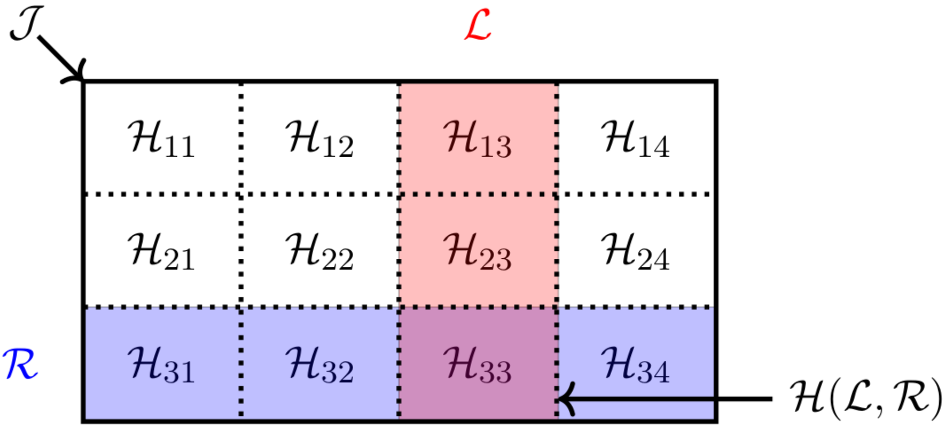
The cell orders and equivalences:

$$x \leq_L y \Leftrightarrow \exists z\colon y = zx,$$
$$x \leq_R y \Leftrightarrow \exists z'\colon y = xz',$$
$$x \leq_{LR} y \Leftrightarrow \exists z, z'\colon y = zxz',$$
$$x \sim_L y \Leftrightarrow (x \leq_L y) \wedge (y \leq_L x),$$
$$x \sim_R y \Leftrightarrow (x \leq_R y) \wedge (y \leq_R x),$$
$$x \sim_{LR} y \Leftrightarrow (x \leq_{LR} y) \wedge (y \leq_{LR} x).$$

Left, right and two-sided cells (a.k.a. L, R and J-cells): equivalence classes

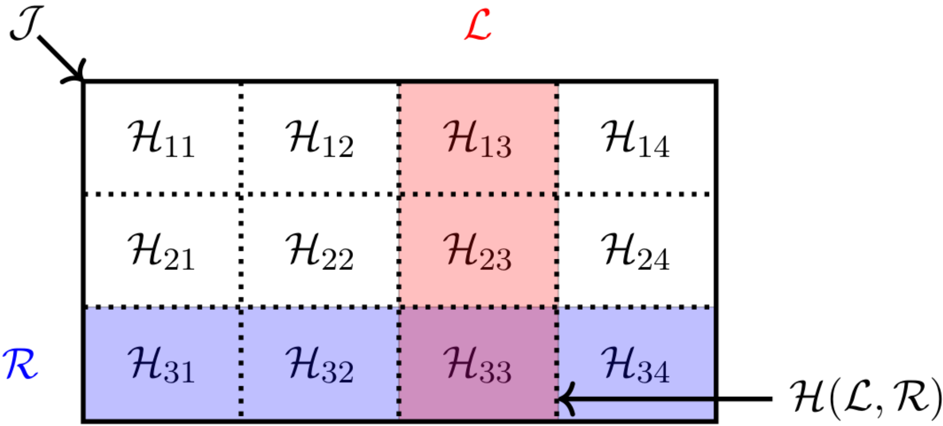
- H-cells = intersections of left and right cells
- Slogan Cells measure information loss

The theory of monoids (Green ∼1950++)



- *H*-cells = intersections of left and right cells
- Slogan Cells partition monoids into matrix-type-pieces

The theory of monoids (Green ∼1950++)

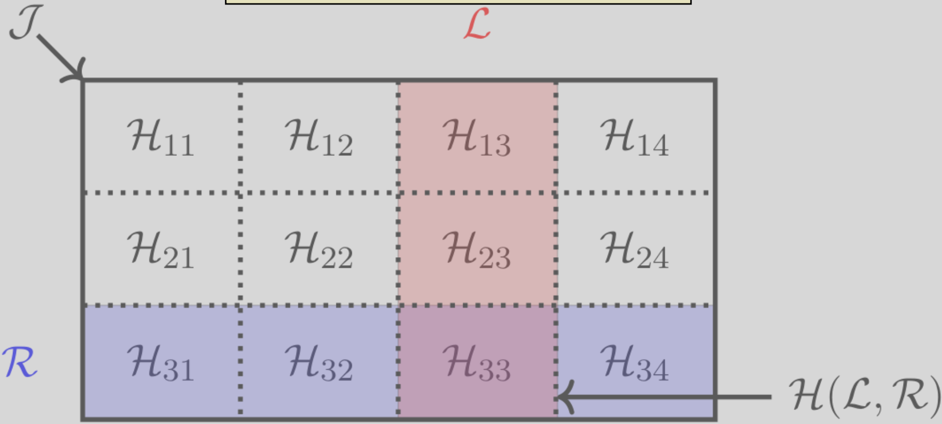


- Each $\mathcal{H}$ contains no or 1 idempotent e; every e is contained in some $\mathcal{H}(e)$
- Each $\mathcal{H}(e)$ is a maximal subgroup No internal information loss

The theory of mon

Example (group-like)

All invertible elements form the smallest cell



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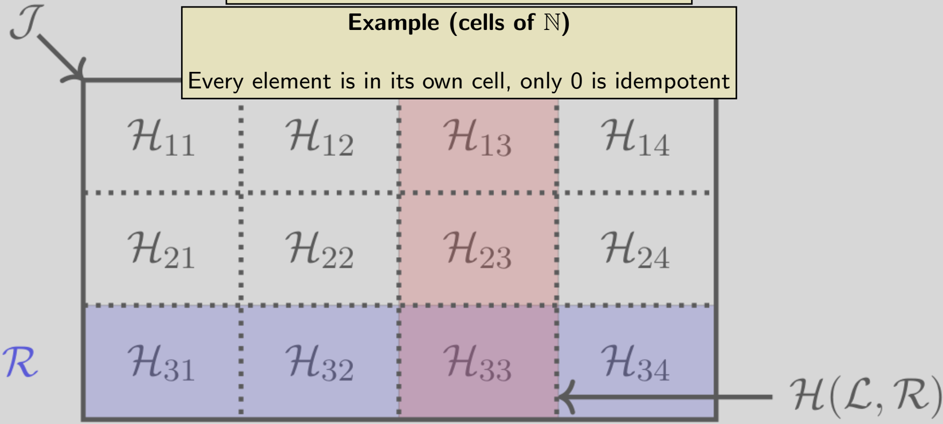
Every element is in its own cell, only 0 is idempotent



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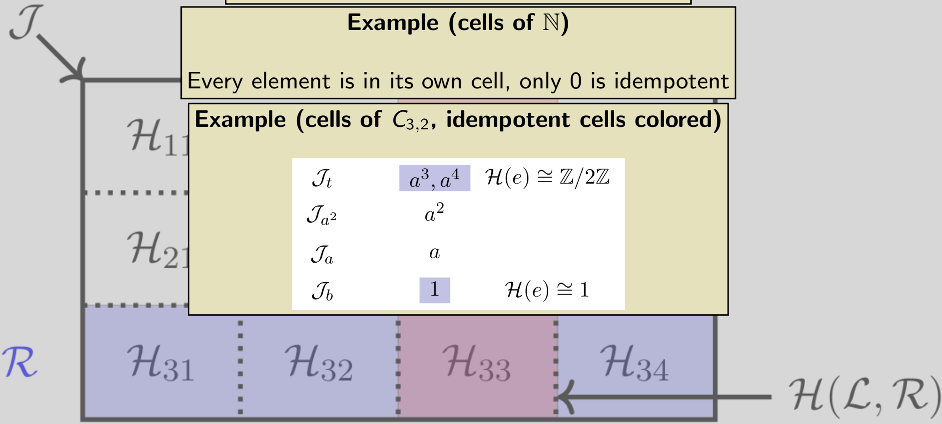

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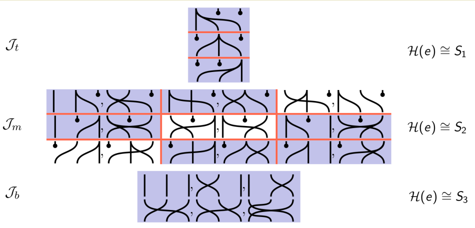
Example (cells of T_3, idempotent cells colored)

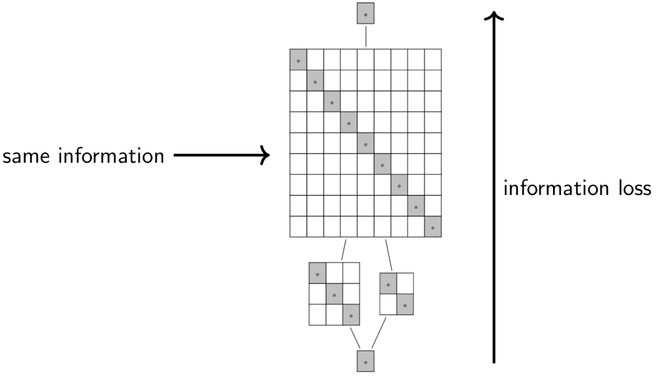


$\mathcal{J}$ $\mathcal{H}_{11}$ $\mathcal{H}_{21}$ $\mathcal{R}$ $\mathcal{H}_{31}$ $\mathcal{H}(\mathcal{L}, \mathcal{R})$

- Each $\mathcal{H}$ cont … in some $\mathcal{H}(e)$
- Each $\mathcal{H}(e)$ is … oss

The theory of monoids (Green ∼1950++)

Computing these "egg box diagrams" is one of the main tasks of monoid theory



GAP can do these calculations for you (package semigroups)

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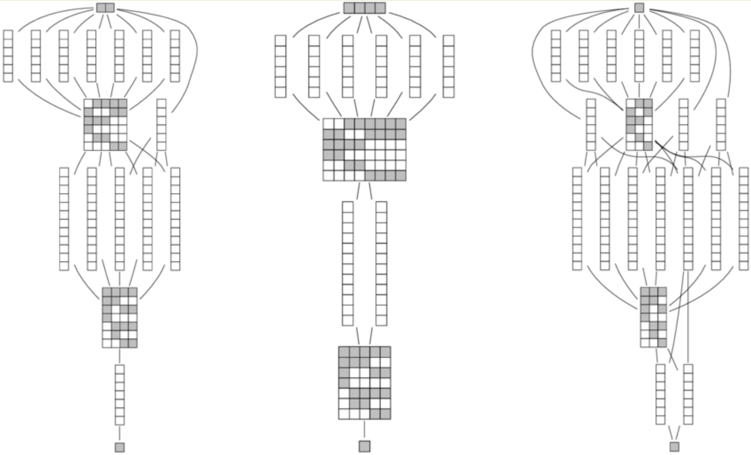
The th…

Examples (no specific monoids)



Grey boxes are idempotent H-cells

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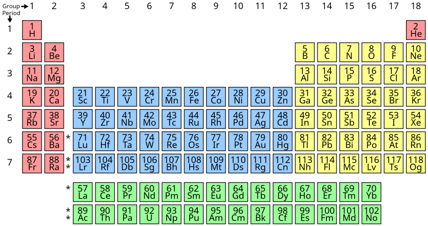
The simple reps of monoids

$\phi\colon S\to \mathrm{GL}(V)$ S-representation on a $\mathbb{K}$-vector space V, S is some monoid

- A $\mathbb{K}$-linear subspace $W\subset V$ is S-invariant if $S\,.\,W\subset W$ Substructure
- $V\neq 0$ is called simple if $0, V$ are the only S-invariant subspaces Elements
- Careful with different names in the literature: S-invariant $\leftrightsquigarrow$ subrepresentation, simple $\leftrightsquigarrow$ irreducible
- A crucial goal of representation theory

Find the periodic table of simple S-representations



S_3	(1)	(12)	(123)
χ_{triv}	1	1	1
χ_{sgn}	1	-1	1
χ_{stand}	2	0	-1

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Chemistry	Group theory	Rep theory
Matter	Groups	Reps
Elements	Simple groups	Simple reps
Simpler substances	Jordan–Hölder theorem	Jordan–Hölder theorem
Periodic table	Classification of simple groups	Classification of simple reps

- A crucial goal of representation theory

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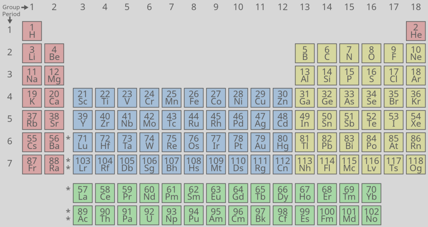

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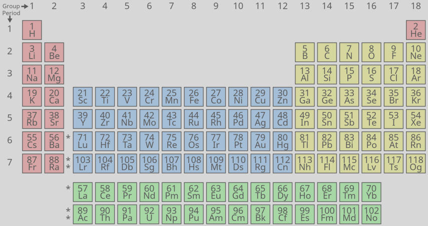
Frobenius ∼1895++and others

For groups and $\mathbb{K} = \mathbb{C}$ this theory is really satisfying



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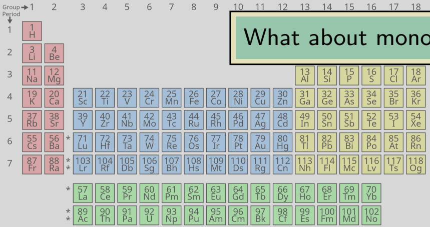

What about monoids?

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The simple reps of monoids

Clifford, Munn, Ponizovskiĭ ∼1940++ ***H*-reduction**

There is a one-to-one correspondence

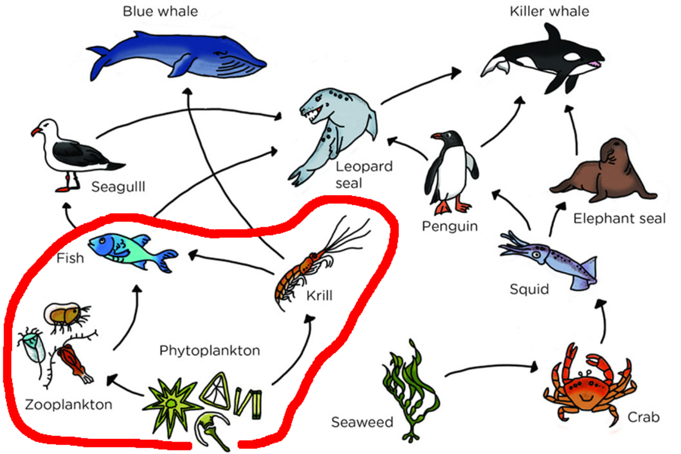
$$\left\{\begin{matrix}\text{simples with}\\ \text{apex } \mathcal{J}(e)\end{matrix}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{matrix}\text{simples of (any)}\\ \mathcal{H}(e) \subset \mathcal{J}(e)\end{matrix}\right\}$$

Reps of monoids are controlled by their maximal subgroups

- Each simple has a unique maximal $\mathcal{J}(e)$ whose $\mathcal{H}(e)$ does not kill it Apex
- In other words (smod means the category of simples, take $\mathbb{K} = \overline{\mathbb{K}}$):

$$S\text{-}\mathrm{smod}_{\mathcal{J}(e)} \simeq \mathcal{H}(e)\text{-}\mathrm{smod}$$

Example (anti apex predator)



"Apex = fish" means that the red bubble does not annihilate your rep and the rest does

The simple reps of monoids

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Example (groups)

Groups have only one cell – the group itself

H-reduction is trivial for groups

Reps of ... ubgroups

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The simple reps

Clifford, Munn,

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$$\left\{ \begin{matrix} \text{simple} & \ldots & (\text{any}) \\ \text{apex} & \ldots & \mathcal{J}(e) \end{matrix} \right\}$$

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Three simple reps over $\mathbb{C}$:
one for 1 and two for $\mathbb{Z}/2\mathbb{Z}$

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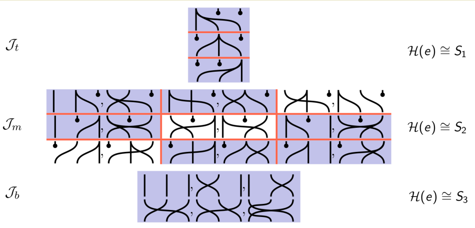

Six simple reps over $\mathbb{C}$:
three for S_3, two for S_2 and one for S_1

Reps ... groups

- Each simple ... not kill it Apex
- In other word ... $\mathbb{K} = \overline{\mathbb{K}}$):

Clifford, Munn, Ponizovskiĭ ∼1940++ *H*-reduction

There is a one-to

$\left\{ \begin{array}{l} \text{simple} \\ \text{apex} \end{array} \right.$ … (any) $\mathcal{J}(e)$ $\}$

Reps … roups

Example (cells of $C_{3,2}$, idempotent cells colored)

$\mathcal{J}_t$	a^3, a^4	$\mathcal{H}(e) \cong \mathbb{Z}/2\mathbb{Z}$
$\mathcal{J}_{a^2}$	a^2	
$\mathcal{J}_a$	a	
$\mathcal{J}_b$	1	$\mathcal{H}(e) \cong 1$

Trivial rep of 1 induces to $C_{3,2}$ and has apex $\mathcal{J}_b$

$\mathcal{J}_a, \mathcal{J}_{a^2}, \mathcal{J}_t$ act by zero

Trivial rep of $\mathbb{Z}/2\mathbb{Z}$ induces to $C_{3,2}$ and has apex $\mathcal{J}_t$

Nothing acts by zero

- Each simple … ot kill it Apex
- In other wor … $= \overline{\mathbb{K}}$):

$$S\text{-smod}_{\mathcal{J}(e)} \simeq \mathcal{H}(e)\text{-smod}$$

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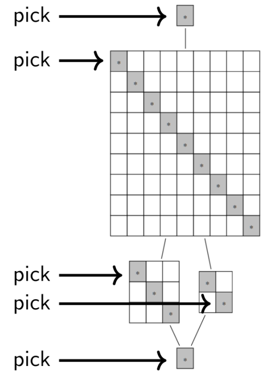
Example (no specific monoid)



Five apexes: bottom cell, big cell, 2x2 cell, 3x3 cell, top cell

Simples for the 2x2 cell are acted on as zero by elements from 3x3 cell, top cell

H-reduction It is sufficient to pick one $\mathcal{H}(e)$ per block

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Clifford, Mu

There is a or

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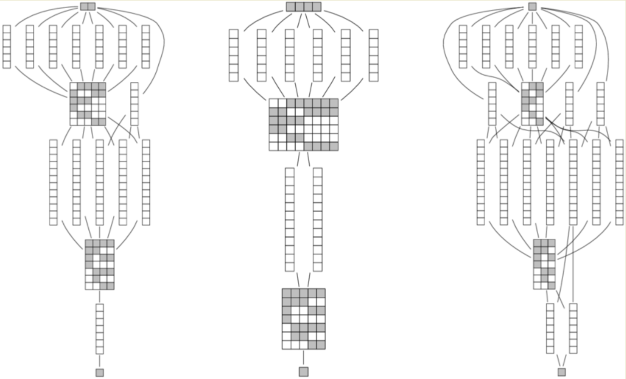
Examples (no specific monoids)

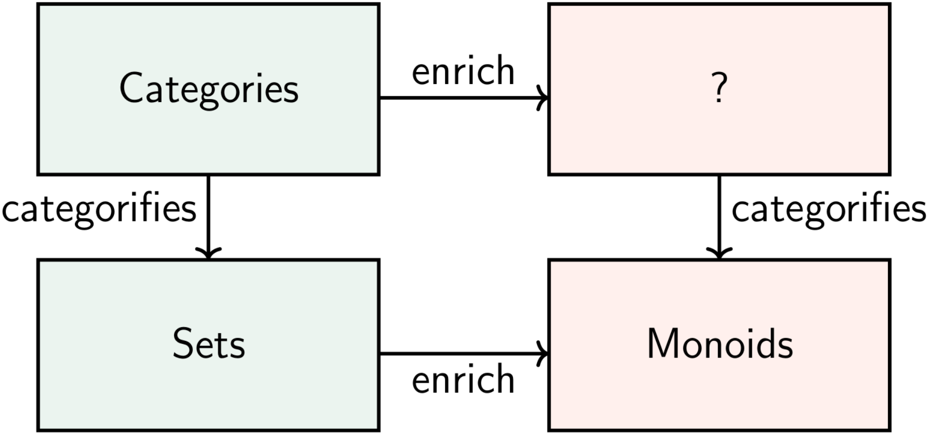


All of these have four apexes

- Each si ... ll it Apex
- In other ... $\mathbb{K}$):

$$S\text{-smod}_{\mathcal{J}(e)} \simeq \mathcal{H}(e)\text{-smod}$$

Categorification of monoid reps



- Usual answer ? = monoidal cats
- I need more structure than plain monoidal cats Specific categorification!

Categorification of monoid reps

- Let $\mathscr{C} = \mathscr{R}\mathrm{ep}(G)$ (G a finite group)
- $\mathscr{C}$ is monoidal and nice. For any $\mathtt{M}, \mathtt{N} \in \mathscr{C}$, we have $\mathtt{M} \otimes \mathtt{N} \in \mathscr{C}$:

$$g(m \otimes n) = gm \otimes gn$$

for all $g \in G, m \in \mathtt{M}, n \in \mathtt{N}$. There is a trivial representation $\mathbb{1}$

- Finitary = linear + additive + idempotent split + finitely many indecomposables + fin dim hom spaces Cat of a fin dim algebra
- Fiat = finitary + involution + adjunctions + monoidal
- Fusion = fiat + semisimple
- Reps are on finitary cats

Finitary + fiat are additive analogs of tensor cats

Tensor cats as in **Etingof–Gelaki–Nikshych–Ostrik** $\sim$**2015**

Categorification of monoid reps

Examples instead of formal defs

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- $\mathscr{C}$ is monoidal and nice. For any $\mathtt{M}, \mathtt{N} \in \mathscr{C}$, we have $\mathtt{M} \otimes \mathtt{N} \in \mathscr{C}$:

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  for all $g \in G, m \in \mathtt{M}, n \in \mathtt{N}$. There is a trivial representation $\mathbb{1}$

- The regular cat representation $\mathscr{M} : \mathscr{C} \to \mathscr{E}\mathrm{nd}(\mathscr{C})$:

$$\begin{array}{ccc} \mathtt{M} & \longrightarrow & \mathtt{M} \otimes \_ \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes \_} \\ \mathtt{N} & \longrightarrow & \mathtt{N} \otimes \_ \end{array}$$

- The decategorification is the regular representation

Categorification of monoid reps

- Let $K \subset G$ be a subgroup
- $\mathcal{R}\mathrm{ep}(K)$ is a cat representation of $\mathscr{R}\mathrm{ep}(G)$, with action

$$\mathcal{R}es^G_K \otimes \_ \colon \mathscr{R}\mathrm{ep}(G) \to \mathscr{E}\mathrm{nd}(\mathcal{R}\mathrm{ep}(K)),$$

  which is indeed a cat action because $\mathcal{R}es^G_K$ is a $\otimes$-functor
- The decategorifications are $\mathbb{N}$-representations

Categorification of monoid reps

- Let $\psi \in H^2(K, \mathbb{C}^*)$ (ground field is now $\mathbb{C}$)
- Let $\mathcal{V}(K, \psi)$ be the category of projective K-modules with Schur multiplier ψ, *i.e.* vector spaces $\mathtt{V}$ with $\rho\colon K \to \mathcal{E}\mathrm{nd}(\mathtt{V})$ such that

$$\rho(g)\rho(h) = \psi(g, h)\rho(gh), \text{ for all } g, h \in K$$

- Note that $\mathcal{V}(K, 1) = \mathcal{R}\mathrm{ep}(K)$ and

$$\otimes\colon \mathcal{V}(K, \phi) \boxtimes \mathcal{V}(K, \psi) \to \mathcal{V}(K, \phi\psi)$$

- $\mathcal{V}(K, \psi)$ is also a cat representation of $\mathscr{C} = \mathscr{R}\mathrm{ep}(G)$:

$$\mathscr{R}\mathrm{ep}(G) \;\boxtimes\; \mathcal{V}(K, \psi) \xrightarrow{\mathcal{R}es_K^G \boxtimes \mathrm{Id}} \mathcal{R}\mathrm{ep}(K) \;\boxtimes\; \mathcal{V}(K, \psi) \xrightarrow{\otimes} \mathcal{V}(K, \psi)$$

- The decategorifications are $\mathbb{N}$-representations

Categorification of monoid reps

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- Le ... iplier ψ, *i.e*
- N
-
-

Classical

An S module is called simple (the "elements")

if it has no S-stable ideals

We have the Jordan–Hölder theorem: every module is built from simples

Goal Find the periodic table of simples

Categorical

A $\mathscr{C}$ module is called simple (the "elements")

if it has no $\mathscr{C}$-stable monoidal ideals

We have the weak Jordan–Hölder theorem: every module is built from simples

Goal Find the periodic table of simples

Categorification of monoid reps

- Let $\psi \in H^2(K, \mathbb{C}^*)$ (ground field is now $\mathbb{C}$)
- Let $\mathcal{V}(K,\psi)$ be the category of projective K-modules with Schur multiplier ψ, *i.e.* vector spaces $\mathtt{V}$ with $\rho\colon K \to \mathcal{E}\mathrm{nd}(\mathtt{V})$ such that

Folk theorem?

Completeness All simples of $\mathscr{R}\mathrm{ep}(G,\mathbb{C})$ are of the form $\mathcal{V}(K,\psi)$

Non-redundancy We have $\mathcal{V}(K,\psi) \cong \mathcal{V}(K',\psi')$
$\Leftrightarrow$
the subgroups are conjugate and $\psi' = \psi^g$, where $\psi^g(k,l) = \psi(gkg^{-1}, glg^{-1})$

-
-

$$\mathscr{R}\mathrm{ep}(G) \boxtimes \mathcal{V}(K,\psi) \xrightarrow{\mathcal{R}es_K^G \boxtimes \mathrm{Id}} \mathcal{R}\mathrm{ep}(K) \boxtimes \mathcal{V}(K,\psi) \xrightarrow{\otimes} \mathcal{V}(K,\psi)$$

- The decategorifications are $\mathbb{N}$-representations

Clifford, Munn, Ponizovskiĭ categorically

The cell orders and equivalences $(\mathrm{X},\mathrm{Y},\mathrm{Z}$ indecomposable, $\Subset_{\oplus}$ = direct summand):

$$\mathrm{X} \leq_L \mathrm{Y} \Leftrightarrow \exists \mathrm{Z}\colon \mathrm{Y} \Subset_{\oplus} \mathrm{Z}\mathrm{X}$$
$$\mathrm{X} \leq_R \mathrm{Y} \Leftrightarrow \exists \mathrm{Z}'\colon \mathrm{Y} \Subset_{\oplus} \mathrm{X}\mathrm{Z}'$$
$$\mathrm{X} \leq_{LR} \mathrm{Y} \Leftrightarrow \exists \mathrm{Z},\mathrm{Z}'\colon \mathrm{Y} \Subset_{\oplus} \mathrm{Z}\mathrm{X}\mathrm{Z}'$$
$$\mathrm{X} \sim_L \mathrm{Y} \Leftrightarrow (\mathrm{X} \leq_L \mathrm{Y}) \wedge (\mathrm{Y} \leq_L \mathrm{X})$$
$$\mathrm{X} \sim_R \mathrm{Y} \Leftrightarrow (\mathrm{X} \leq_R \mathrm{Y}) \wedge (\mathrm{Y} \leq_R \mathrm{X})$$
$$\mathrm{X} \sim_{LR} \mathrm{Y} \Leftrightarrow (\mathrm{X} \leq_{LR} \mathrm{Y}) \wedge (\mathrm{Y} \leq_{LR} \mathrm{X})$$

Left, right and two-sided cells (a.k.a. *L*, *R* and *J*-cells): equivalence classes

- *H*-cells $\mathscr{S}_{\mathcal{H}} = \mathrm{Add}(\mathrm{X} \in \mathcal{H}, \mathbb{1})$ mod higher terms
- Slogan Cells measure information loss

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Left, right and two-sided cells (a.k.a. *L*, *R* and *J*-cells): equivalence classes

- *H*-cells $\mathscr{S}$
- Slogan Ce

Compare to monoids:

Indecomposables instead of elements, $\Subset_{\oplus}$ instead of $=$

Otherwise the same!

Clifford, Munn, Ponizov

Example (cells of $\mathscr{R}\mathrm{ep}(G,\mathbb{C})$)

Only one cell since $\mathbb{1} \Subset \mathrm{XX}^*$

The cell orders and equiva e, $\Subset$ = direct summand):

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$\mathcal{J}_t$	Z_3	$[\mathscr{S}_{\mathcal{H}}] \cong 3\mathbb{Z}$
$\mathcal{J}_b$	$\mathrm{Z}_1,\mathrm{Z}_2$	$[\mathscr{S}_{\mathcal{H}}] \cong \mathbb{Z}/2\mathbb{Z}$

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Clifford, Munn, Ponizov[…]

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Example (cells of the Hecke category of type B_2, pseudo idempotent cells colored)

$\mathcal{J}_{w_0}$	B_{1212}		$\mathcal{S}_{\mathcal{H}}$ "$\simeq$" $\mathcal{V}\mathrm{ect}$
$\mathcal{J}_{\mathrm{middle}}$	$\mathrm{B}_1,\mathrm{B}_{121}$	B_{21}	$\mathcal{S}_{\mathcal{H}}$ "$\simeq$" $\mathcal{V}\mathrm{ect}_{\mathbb{Z}/2\mathbb{Z}}$
	B_{12}	$\mathrm{B}_2,\mathrm{B}_{212}$	
$\mathcal{J}_{\emptyset}$	$\mathrm{B}_{\emptyset}$		$\mathcal{S}_{\mathcal{H}} \simeq \mathcal{V}\mathrm{ect}$

Categorical ***H*****-reduction**
There is a one-to-one correspondence

$$\left\{\begin{matrix}\text{simples with}\\ \text{apex } \mathcal{J}\end{matrix}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{matrix}\text{simples of (any)}\\ \mathscr{S}_{\mathcal{H}} \subset \mathscr{S}_{\mathcal{J}}\end{matrix}\right\}$$

Almost *verbatim* as for monoids

- Each simple has a unique maximal $\mathcal{J}$ whose $\mathscr{S}_{\mathcal{H}}$ does not kill it Apex
- In other words (smod means the category of simples):

$$\mathscr{S}\text{-smod}_{\mathcal{J}(e)} \simeq \mathscr{S}_{\mathcal{H}}\text{-smod}$$

Example (cells of $\mathscr{R}\mathrm{ep}(G,\mathbb{C})$)

No reduction

Clifford, Munn, Ponizovs

Categorical H-reduction

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Clifford, Munn, Ponizov

Example (cells of $\mathscr{R}\mathrm{ep}(G,\mathbb{C})$)

No reduction

Catego

There is

Example (cells of $\mathscr{R}\mathrm{ep}(\mathbb{Z}/3\mathbb{Z},\overline{\mathbb{F}_3})$, pseudo idempotent cells colored)

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Two apexes, three simples (2+1)

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Clifford, Munn, Ponizov...

Catego...

There is...

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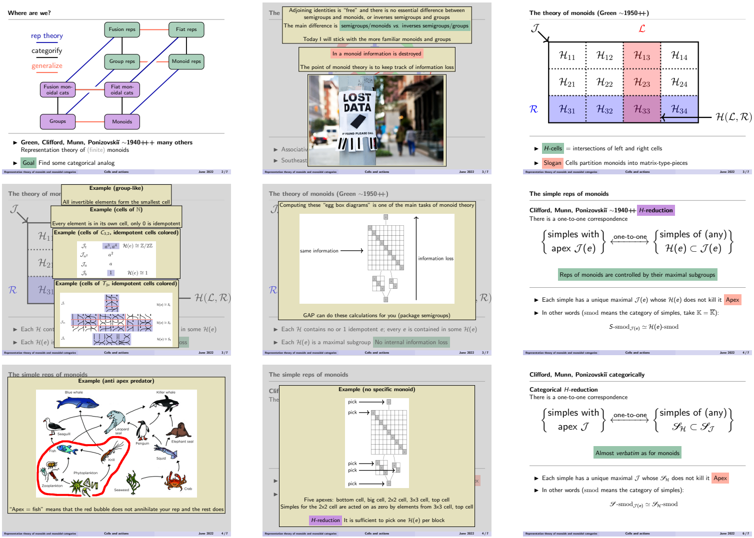
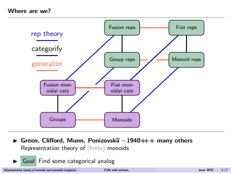
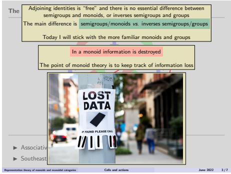
	B_{12}	B_2, B_{212}	
$\mathcal{J}_{\emptyset}$	$B_{\emptyset}$		$\mathscr{S}_{\mathcal{H}}\simeq\mathscr{V}\mathrm{ect}$

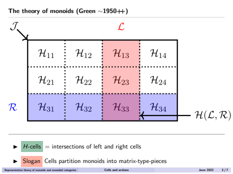
Three apexes, four simples (1+2+1)

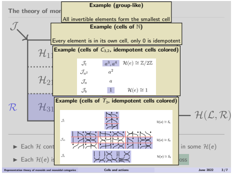


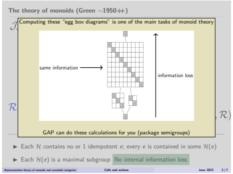
There is still much to do...

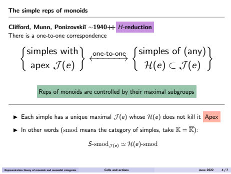


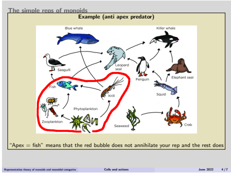


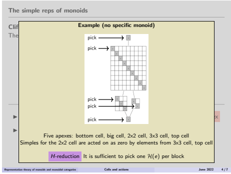


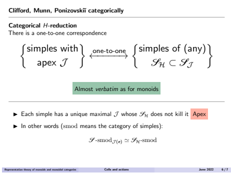












Thanks for your attention!