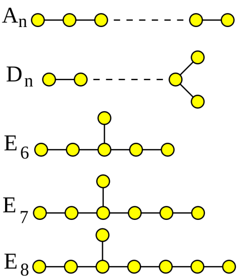
Subfactors in a nutshell

Or: ADE and all that

Daniel Tubbenhauer



May 2022

Jones' revolution

Throughout

Please convince yourself that I haven't messed up
while picking my pictures and text from my stolen material

Jones' revolution

The history of subfactors can roughly be divided into three parts:

- Subfactors in operator theory — Before Jones ∼1983--
- Invariants of subfactors — Jones' revolution ∼1983-1985
- Subfactors without subfactors!? — After Jones ∼1985++

ANNALS OF MATHEMATICS
Vol. 37, No. 1, January, 1936

ON RINGS OF OPERATORS

BY F. J. MURRAY* AND J. v. NEUMANN

(Received April 3, 1935)

Invent. math. 72, 1–25 (1983)

Index for Subfactors

V.F.R. Jones

Jones' revolution

Along the way Jones rediscovered the Temperley–Lieb (TL) algebra and found a Markov trace on it

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The history of subf... ts:

▶ S...

▶ I...

▶ S...

Along the way Jones rediscovered the Temperley–Lieb (TL) algebra and found a Markov trace on it

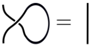
Why is that important? Well, because of the (Birman–)Jones polynomial:

Jones Polynomial

Jones met Birman in 1984:

Markov trace on the braid group ⇒ knot invariant

It was surprising that the Markov trace naturally comes from the trace of the II_1 factor,

$\tau(x\sigma_n) = \tau(x), \forall x \in TL_n, \longrightarrow$  , (Reidemeister move I)

therefore leading to a knot invariant, well-known as the Jones polynomial, by which Jones answered a series of old questions in knot theory in 1985.

 $= t + t^3 - t^4.$  $= t^{-1} + t^{-3} - t^{-4}.$

Reflection: $t \to t^{-1}$.

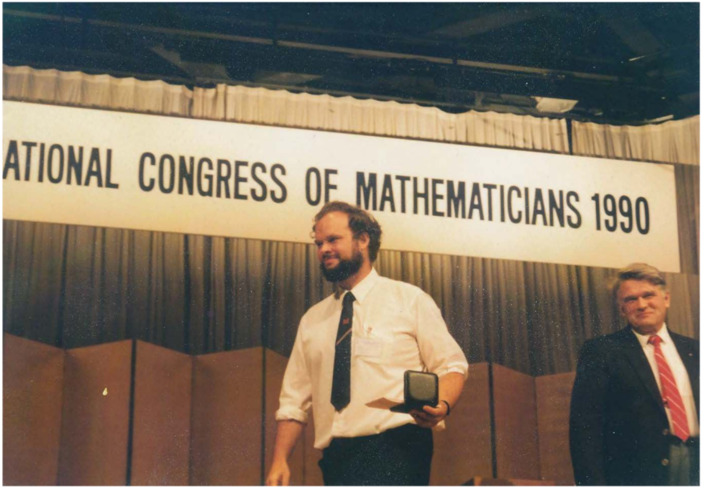
ON ... 3) ... tors

By F.

Jones' revolution

Th

Jones was awarded the Fields Medal at Kyoto in 1990 for these breakthroughs.



C

By

rs

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- Invari
- Subfa

Jones also implicitly coined the name "TL algebra" ∼1985:

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 1, January 1985

A POLYNOMIAL INVARIANT FOR KNOTS VIA VON NEUMANN ALGEBRAS[1]

BY VAUGHAN F. R. JONES[2]

For real t, D. Evans pointed out that an explicit representation of A_n on $\mathbf{C}^{2n+2}$ was discovered by H. Temperley and E. Lieb [**23**], who used it to show the equivalence of the Potts and ice-type models of statistical mechanics. A

(1983)

ON RI

factors

By F. J. M

(Received April 3, 1935)

V.F.R. Jones

Jones' revolution

The history

- Subfac
- Invaria
- Subfac

5 (1983)

V

ON RI

bfactors

By F. J. M

(R

Today's talk is based on:

my memory (horrible reference...)

Subfactors

in Memory of Vaughan Jones

Zhengwei Liu

Tsinghua University

Math-Science Literature Lecture Series
November 23, 2020, Harvard CMSA and Tsinghua YMSC

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 51, Number 2, April 2014, Pages 277–327
S 0273-0979(2013)01442-3
Article electronically published on December 24, 2013

THE CLASSIFICATION OF SUBFACTORS OF INDEX AT MOST 5

VAUGHAN F. R. JONES, SCOTT MORRISON, AND NOAH SNYDER

The above are easy to google (it is worth it!)

Antediluvian ∼1983--

Von Neumann's wartime Los Alamos ID badge photo

Why was everyone related to this part of math in Los Alamos?

- Factor = von Neumann algebra with trivial center
- Murray–von Neumann ∼1930++ classified factors by types: I, II_1, II_∞ and III
- II_1 are the "most exciting" ones They have a unique trace!
  We will stick with these momentarily (I drop the "of type II_1")
- A subfactor is an inclusion of factors $N \subset M$

Antediluvian ∼1983--

ОПЫТЪ СИСТЕМЫ ЭЛЕМЕНТОВЪ,

ОСНОВАННОЙ НА ИХЪ АТОМНОМЪ ВѢСѢ И ХИМИЧЕСКОМЪ СХОДСТВѢ.

			Ti=50	Zr=90	?=180.
			V=51	Nb=94	Ta=182.
			Cr=52	Mo=96	W=186.
			Mn=55	Rh=104,4	Pt=197,1.
			Fe=56	Ru=104,4	Ir=198.
			Ni=Co=59	Pd=106,6	Os=199.
H=1			Cu=63,4	Ag=108	Hg=200.
	Be=9,4	Mg=24	Zn=65,2	Cd=112	
	B=11	Al=27,3	?=68	Ur=116	Au=197?
	C=12	Si=28	?=70	Sn=118	
	N=14	P=31	As=75	Sb=122	Bi=210?
	O=16	S=32	Se=79,4	Te=128?	
	F=19	Cl=35,5	Br=80	I=127	
Li=7	Na=23	K=39	Rb=85,4	Cs=133	Tl=204.
		Ca=40	Sr=87,6	Ba=137	Pb=207.
		?=45	Ce=92		
		?Er=56	La=94		
		?Yt=60	Di=95		
		?In=75,6	Th=118?		

Д. Менделѣевъ

Chemistry	Group theory	Operator theory
Matter	Groups	von Neumann algebras
Elements	Simple groups	Factors
Simpler substances	Jordan–Hölder theorem	The theorem below
Periodic table	Classification of simple groups	Classification of factors

Theorem (von Neumann ∼1949)

Every von Neumann algebra on a separable Hilbert space has an essentially unique decomposition into direct integrals of factors

Antediluvian ∼1983--

Example (factor)

Bounded operators $B(H)$ on a Hilbert space H
If $\dim H < \infty$, then these are just matrices

			Fe=56	Ru=104,4	Ir=198.
			Ni=Co=59	Pd=106,6	Os=199.
H=1			Cu=63,4	Ag=108	Hg=200.
	Be=9,4	Mg=24	Zn=65,2	Cd=112	
	B=11	Al=27,3	?=68	Ur=116	Au=197?
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Example (factor)

Bounded operators $B(H)$ on a Hilbert space H
If dim $H < \infty$, then these are just matrices

I need the following terminology

Projections $E = EE = E^*$

$\leftrightsquigarrow$

orthogonal projection onto some closed subspace

Subspaces and thus projections are ordered by inclusion

Finite projections No $0 < F < E$ equivalent to E
For example, in fin-dim-land everything is finite

Minimal projections No $0 < F < E$

Theorem (von Neumann ∼1949)

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Antediluvian ∼1983--

Type I = there is a minimal nonzero projection

Example (type I factor)

Every type I factor is isomorphic to bounded operators $B(H)$ on a Hilbert space H

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F=19 Cl=35,5 Br=80 I=127

Type III = no nonzero finite projection

Non-example (type III factor)

Murray–von Neumann write ∼ 1936:

Problem 3. Which ones of the classes (I_n)–(III_∞) *do really exist?*
We will be able to contribute considerable material to clarify these questions, but the only one to which we can give a complete answer is Problem 5.

and do not solve Problem 3 since they did not find a type III factor

They exist! But giving an example gets me too far off track
They are also too hard for me anyway...

Theorem (von Neumann ∼1949)

Every von Neumann algebra on a separable Hilbert space has an essentially unique decomposition into direct integrals of factors

Antediluvian ∼1983

Type II = no minimal projections but there are nonzero finite projections

Type II_1 = id operator is finite; Type II_∞ = rest of type II

Example (type II factor)

Discrete group G with infinite nonidentity conjugacy classes, *e.g.* F_2
von Neumann group algebra $L(G)$ is a type II_1 factor
($L(G)$ = the commutant of the left regular representation on $\ell^2(G)$)

?=45 Ce=92
?Er=56 La=94

Reminder

$\ell^2(G)$ = formal sequences $\sum_{g\in G}\lambda_g g$ with $\sum_{g\in G}|\lambda_g|^2<\infty$
If $|G|<\infty$, then $L(G)=\mathbb{C}[G]$

Elements	Simple groups	Factors
Simpler substances	Jordan–Hölder theorem	The theorem below
Periodic table	Classification of simple groups	Classification of factors

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Elements	Simple groups	Factors

A famous open problem

$L(F_2)\cong L(F_k)$ for $k>2$

This is nontrivial: recall that *e.g.* $\mathbb{C}[D_4]\cong\mathbb{C}[O_8]$ but $D_4\not\cong O_8$

Theorem (v

Every von Neumann algebra on a separable Hilbert space has an essentially unique decomposition into direct integrals of factors

Antediluvian ∼1983--

I did not find a picture of Murray

Murray, Francis Joseph

MR Author ID:	**128490**
Earliest Indexed Publication:	1935
Total Publications:	**34**
Total Related Publications:	**2**
Total Citations:	**592**

⊞ Published as: Murray, F. J. (25)

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The word "algebra" is highlighted because there will be representations!

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Example

M is a factor, G a nice group, then $M^G \subset M$ is a subfactor

Example

G a nice group with infinite conjugacy classes
$H \subset G$ a nice subgroup with infinite conjugacy classes
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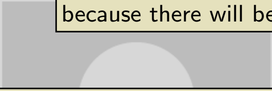

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- ... and III
- ...
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Vague slogan Subfactors ⇝ fixed points of a "quantum group" G action

The is also a version of Galois correspondence
and one can recover G from $M^G \subset M$
In this sense subfactor theory generalizes group theory

I did not find ... picture of Mur...

... s Joseph

128490

...n: 1935

34

2

Total Citations: 592

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Philosophy

(hard to track back – happened around Jones' revolution)

Focus on the abstract symmetry defined by a subfactor

+

de-emphasize the factors themselves

Really good **invariants** which we see today

and that **shift the focus** to combinatorics/categories

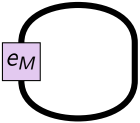
Index (number), principal graph, standard invariant ("tensor category")

Roughly:

$$\text{Subfactor} \underset{\text{Reconstruction } \Phi}{\overset{\text{Representation } \Psi}{\rightleftarrows}} \textit{Standard Invariant}$$

- Fa...
- M... and III
- II_1 ... W...
- A subfactor is an inclusion of factors $N \subset M$

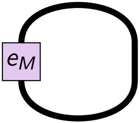
Jones' construction ∼1983-1985

$[M : N] \leftrightsquigarrow$  $\in \mathbb{R}_{\geq 0}$

- Subfactor $N \subset M$, M is a N-module by left multiplication
- Assume that M is finitely generated projective N-module
  The index $[M : N] \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ is the trace of the idempotent e_M for M
- Jones' index theorem ∼1983 The index is an invariant of $N \subset M$ and

$$[M : N] \in \{4\cos^2(\tfrac{\pi}{k+2}) | k \in \mathbb{N}\} \cup [4, \infty]$$

Jones' construction ∼1983-1985

$[M : N] \leftrightsquigarrow$  $\in \mathbb{R}_{\geq 0}$

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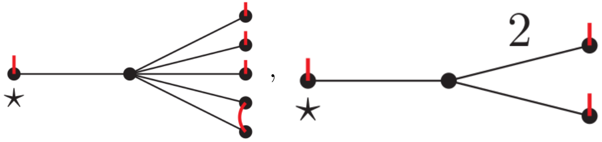
$$[M : N] \in \{4\cos^2(\tfrac{\pi}{k+2}) | k \in \mathbb{N}\} \cup [4, \infty]$$

Note the "quantization" below 4:
This was a weird/exciting result!
Jones: The most challenging part is constructing these subfactors

Jones' construction ∼1983-1985

$N \subset N \rtimes S_3$:



- $M_k = M \otimes_N M \otimes_N \dots$ (k copies of M), M_k is a N-N, N-M, M-N and M-M bimodule
- Enter, the principal graph(s) Γ!

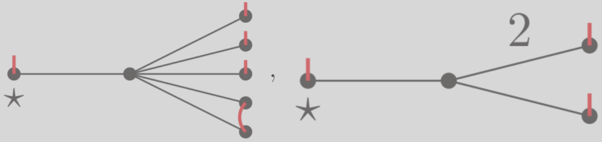
Definition 2.3. The *principal graph* of $N \subset M$ is the pointed bipartite graph whose vertices are the (isomorphism classes of) irreducible $N - N$ and $N - M$ bimodules contained in all the M_k, with $\dim(\mathrm{Hom}_{N-N}(V, W))$ edges between an $N - M$ bimodule V and an $N - N$ bimodule W. The distinguished vertex of Γ is the $N - N$ bimodule N itself, and we will use a $\star$ to denote it on the graph.

The distance from $\star$ to a vertex is called its depth. Note that the vertices at even depths are $N - N$ bimodules while the vertices at odd depths are $N - M$ bimodules.

Similarly, the "dual principal graph" is obtained by restricting irreducible $M - M$ bimodules to $M - N$ bimodules.

Jones' construction ∼1983-1985

$N \subset N \rtimes S_3$:



We typically indicate slightly more information when giving a pair of principal graphs. Namely, any bimodule has a dual, or contragredient, bimodule. The dual of an $A - B$ bimodule (where A and B are each one of M and N) is a $B - A$ bimodule, so duals of even vertices are even vertices on the same graph, while duals of odd vertices are odd vertices on the other graph. We record this duality data by using red tags to indicate duality on even vertices and by having odd vertices at each depth of each graph at the same relative height as its dual on the other graph.

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Jones' construction ~1983-1985

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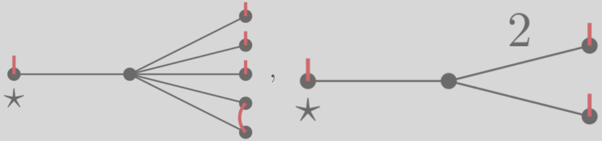


Example

Principal graph for $N \subset N \rtimes G$:
as an N-N bimodule $N \rtimes G$ is the direct sum, over G, of g-twisted trivial bimodules
Even vertices ⇝ the group elements

Example

Dual principal graph for $N \subset N \rtimes G$:
Even vertices ⇝ the simple group reps
with as many edges as dimension of the reps

whose vertices are the (isomorphism classes of) irreducible $N-N$ and $N-M$ bimodules c… edges between an $N-M$ bim… shed vertex of Γ is the $N-N$ b… on the graph.

The dista… hat the vertices at even depths … depths are $N-M$ bimodules.

Similarly, the "dual principal graph" is obtained by restricting irreducible $M-M$ bimodules to $M-N$ bimodules.

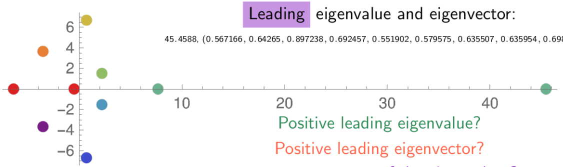
Jones' construction ∼1983-1985

$N \subset$

Perron–Frobenius theorem (Perron ∼1907, Frobenius ∼1912)

$$\begin{pmatrix} 3 & 0 & 5 & 1 & 8 & 7 & 0 & 1 & 4 & 7 \\ 4 & 8 & 0 & 6 & 3 & 4 & 2 & 6 & 8 & 3 \\ 8 & 6 & 6 & 7 & 6 & 0 & 9 & 4 & 8 & 5 \\ 3 & 7 & 7 & 1 & 5 & 6 & 4 & 1 & 7 & 4 \\ 4 & 0 & 3 & 4 & 4 & 8 & 8 & 1 & 4 & 2 \\ 0 & 3 & 7 & 3 & 2 & 4 & 2 & 2 & 3 & 8 \\ 6 & 3 & 6 & 1 & 5 & 6 & 1 & 6 & 4 & 4 \\ 2 & 4 & 0 & 2 & 8 & 8 & 1 & 4 & 8 & 6 \\ 6 & 7 & 6 & 3 & 4 & 2 & 9 & 6 & 5 & 0 \\ 0 & 6 & 9 & 9 & 8 & 3 & 9 & 9 & 1 & 9 \end{pmatrix}$$

What on earth is going on? Strange patterns with the eigenvalues and vectors:



Leading eigenvalue and eigenvector:

45.4588, (0.567166, 0.64265, 0.897238, 0.692457, 0.551902, 0.579575, 0.635507, 0.635954, 0.698596, 1)

Positive leading eigenvalue?

Positive leading eigenvector?

Irreducible matrices with entries from $\mathbb{R}_{\geq 0}$
have an eigenvalue $pf \in \mathbb{R}_{\geq 0}$ and
an associated eigenvector $\vec{pf} \in \mathbb{R}^n_{\geq 0}$
The growth rate of R^N is roughly given by pf^N

- $M_k =$ … bimodule
- Enter

D… e graph w… $N-M$ bi… ween an N… x of Γ is th… h.

…rtices at ev… $N-M$ bimodules.

Similarly, the "dual principal graph" is obtained by restricting irreducible $M-M$ bimodules to $M-N$ bimodules.

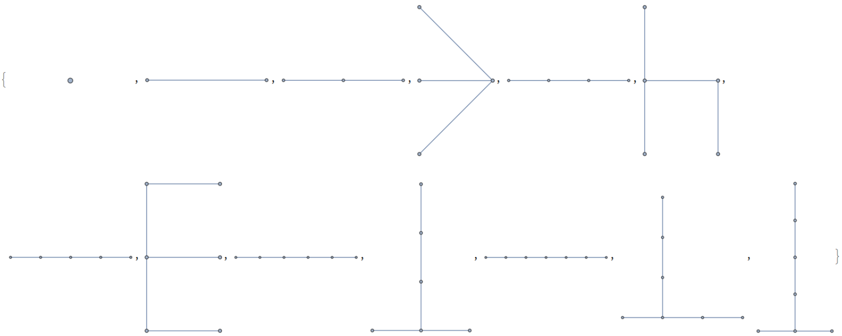
Sketch of Jones' quantization argument

$$[M : N] = pf(\Gamma)^2 \text{ (unless } [M : N] = \infty)$$

Then use Kronecker's theorem ∼1857

```
In[51]:= findgraphs4[n_] := Select[GraphData /@ Flatten[Table[GraphData["Connected", m], {m, 1, n}], 1], Last[Sort[Eigenvalues[AdjacencyMatrix[#]], Less]] < 2 &];
findgraphs4[7]
```

Out[52]=



The distanc… that the vertices at even depths ar… d depths are $N - M$ bimodules.

Similarly, th… ng irreducible $M - M$ bimodules to $M - N$ bimodules.

Upshot of Jones' construction

"Subfactors = graphs + combinatorics" in quotation marks

Jones' construction ∼1983-1985

Jones' type invariants were later also
introduced in other field, for example the study of tensor categories ∼2005

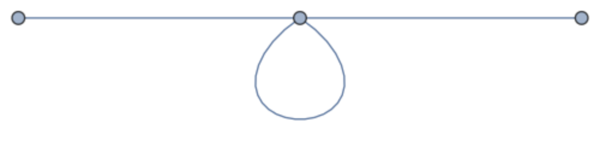
Annals of Mathematics, **162** (2005), 581–642

On fusion categories

By Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik

Example (principal graph of *Rep*(S_3))

The graph for the action of the standard rep of S_3 is:



PF miracle

Take $T = M(triv) + M(std)^2 + M(triv)$
$pf(T) = 6 = |S_3|$
pf eigenvector $\leftrightsquigarrow$ regular rep $\mathbb{C}[S_3]$

even depths are $N - N$ bimodules while the vertices at odd depths are $N - M$ bimodules.

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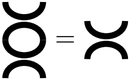
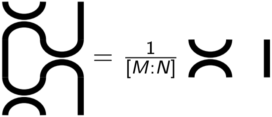
Jones' construction ∼1983-1985

§4. Possible Values of the Index

§4.1. Certain Algebras Generated by Projections

Jones' projectors satisfy the scaled TL relations for $\delta = [M:N] = 4\cos^2(\frac{\pi}{k+2})$

$e_k \leftrightsquigarrow \frac{1}{[M:N]}$ 

- $e_k e_k = e_k \leftrightsquigarrow$ 
- $e_k e_{k\pm1} e_k = \frac{1}{[M:N]} e_k \leftrightsquigarrow$ 
- $e_k e_l = e_l e_k$ for $|k-l| > 1 \leftrightsquigarrow$ far commutativity

Jones' construction ∼1983-1985

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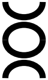
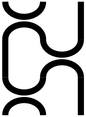
- $e_k e_k = e_k \leftrightsquigarrow$  $=$ 
- $e_k e_{k\pm 1} e_k = \frac{1}{[M:N]} e_k \leftrightsquigarrow$  $= \frac{1}{[M:N]}$ 
- $e_k e_l = e_l e_k$ for $|k-l| > 1 \leftrightsquigarrow$ far commutativity

I will be more precise later
(using a more modern language)
but for now, Jones shows that
$\{e_i | i \geq n\}$ generate a factor R_n
$R_2 \subset R_1$ is a subfactor of index $4\cos^2(\frac{\pi}{k+2})$

Jones' c

The Markov property!

They satisfy the relations

(I) $e_i^2 = e_i, e_i^* = e_i$,

(II) $e_i e_{i\pm1} e_i = t/(1+t)^2 e_i$, TL relations

(III) $e_i e_j = e_j e_i$ if $|i-j| \geq 2$.

Here t is a complex number. It has been shown by H. Wenzl [**24**] that an arbitrarily large family of such projections can only exist if t is either real and positive or $e^{\pm 2\pi i/k}$ for some $k = 3, 4, 5, \ldots$. When t is one of these numbers, there exists such an algebra for all n possessing a trace $\mathrm{tr}\colon A_n \to \mathbf{C}$ completely determined by the normalization $\mathrm{tr}(1) = 1$ and

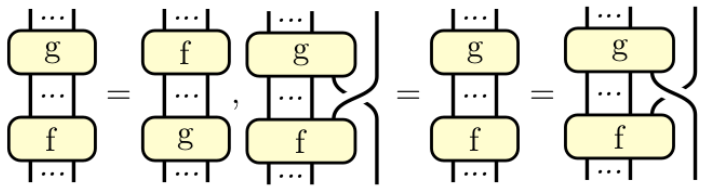
(IV) $\mathrm{tr}(ab) = \mathrm{tr}(ba)$,

(V) $\mathrm{tr}(we_{n+1}) = t/(1+t)^2\, \mathrm{tr}(w)$ if w is in A_n,

(VI) $\mathrm{tr}(a^* a) > 0$ if $a \neq 0$ Markov trace

(note $A_0 = \mathbf{C}$).

In braid pictures (crossing is given by Kauffman skein formula)



Jones' pr $^2(\frac{\pi}{k+2})$

$e_k \rightsquigarrow$ [M

▶ $e_k e_k$

▶ $e_k e_{k\pm1} e_k$

▶ $e_k e_l = e_l e$

In hindsight the crucial result Jones' proved about $\boldsymbol{TL}_{\mathbb{C}}(\delta)$ is the existence of a Markov trace $\Rightarrow$ Jones polynomial

Postdeluvian ∼1985++

$$N\boldsymbol{Bim} \underset{{}_M M_N}{\overset{{}_N M_M}{\rightleftarrows}} M\boldsymbol{Bim}$$

- We have collections of bimodules of the four flavors
- One flavor of bimodule forms a tensor category
- C^*-2-category with two objects (corresponding to M and N) together with a choice of generating 1-morphism (corresponding to ${}_N M_M$)+axioms =

Standard invariant

- The N-N bimodules and the M-M bimodules are the even parts of the standard invariant, and the others the odd part

Postd

Theorem (Popa ∼1994)

(Amenable) Subfactor —Representation Ψ→ (Amenable) *Standard Invariant*

(Amenable) Subfactor ←Reconstruction Φ— (Amenable) *Standard Invariant*

(Amenable = nice for the purpose of this talk)

$\Phi\Psi = id$ and $\Psi\Phi = id$

An amenable subfactor can be reconstructed from its standard invariant

- W
- O
- C^*-2-category with two objects (corresponding to M and N) together with a choice of generating 1-morphism (corresponding to ${}_N M_M$)+axioms =

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a

This is also called quantum Tannaka–Krein duality (TKD)

Roughly, TKD = reconstruction of compact groups from their representations

THE
TOHOKU MATHEMATICAL JOURNAL

Über den Dualitätssatz der nichtkommutativen topologischen Gruppen,

von

Tadao Tannaka in Sendai.

This was 1939

I didn't find a pdf copy of
M.G. Krein, A principle of duality for a bicompact group and a square block algebra
Dokl. Akad. Nauk. SSSR, N. S. 69, 725–728 (1949)

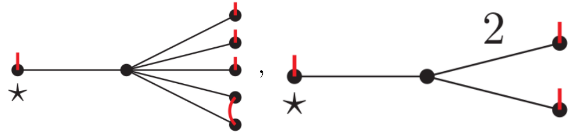
Postdeluvian ∼1985++

$$N\boldsymbol{Bim} \underset{{}_M M_N}{\overset{{}_N M_M}{\rightleftarrows}} M\boldsymbol{Bim}$$

- W
- Or
- C^* … with a ch…

Example

$N \subset N \rtimes S_3$:



Here the N-N tensor categories are $\boldsymbol{Vect}(S_3)$ and $\boldsymbol{Rep}(S_3)$

- The N-N bimodules and the M-M bimodules are the even parts of the standard invariant, and the others the odd part

Postdeluvian ∼1985++

$$\textit{Type } A\colon\quad \mathbb{1} \rightleftarrows \mathrm{L}_1 \rightleftarrows \mathrm{L}_2 \rightleftarrows \dots \rightleftarrows \mathrm{L}_k$$

$$\textit{Type } D\colon\quad \mathbb{1} \rightleftarrows \mathrm{L}_1 \rightleftarrows \mathrm{L}_2 \rightleftarrows \dots \rightleftarrows \mathrm{L}_k \begin{array}{l} \nearrow\!\!\swarrow\ \mathrm{L}_{k+1} \\ \searrow\!\!\nwarrow\ \mathrm{L}'_{k+1} \end{array}$$

Type E is not displayed

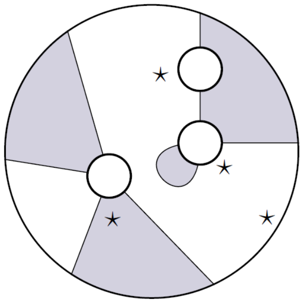
A reformulation of Jones' result is:

- Let L_1 be a generator of a fusion category with $pf(L_1) < 2$
- Then $pf(L_1) = 2\cos(\frac{\pi}{k+2})$ and the fusion graph of L_1 is of ADE type
- In type A the fusion category is $\boldsymbol{Rep}(U_q(\mathrm{sl}_2))^{ss}$ for $q = \exp(\frac{2\pi i}{k+2})$

Postdeluvian ∼1985++

Jones constructed $\textbf{\textit{Rep}}(U_q(\mathrm{sl}_2))^{ss}$ via the TL calculus

Main toolbox are planar algebras



Theorem 3.1. *The number of subfactor planar algebras realizing each of the ADE Dynkin diagrams is given by the following table.*

Principal graph	A_n	D_{2n+1}	D_{2n}	E_6	E_7	E_8
Realizations	1	0	1	2	0	2

This theorem is due to **Ocneanu and others** ∼**1988**

Typ ... $k+1$ ... $k+1$

A ref

- Then $pf(L_1) = 2\cos(\frac{\pi}{k+2})$ and the fusion graph of L_1 is of ADE type
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"Quantum McKay correspondence" **Theorem (Popa ~1994)**

The $pf^2 = 4$ case is given by affine Dynkin diagrams:

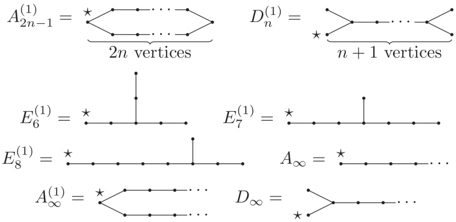


FIGURE 2. The affine Dynkin diagrams together with the only possible location for $\star$ up to symmetry.

Theorem 3.2. *The number of subfactor planar algebras realizing each of the starred affine Dynkin diagrams is given by the following table.*

Principal graph	$A_{2n}^{(1)}$	$A_{2n-1}^{(1)}$	$D_n^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$	A_∞	D_∞
Realizations	0	n	$n-2$	1	1	1	1	1

A_∞ is again the TL calculus

Type

L_{k+1}

L'_{k+1}

A refor

- Let
- Th
- In type A the fusion category is $\boldsymbol{Rep}(U_q(\mathrm{sl}_2))^{ss}$ for $q = \exp(\frac{2\pi i}{k+2})$

Jones's ADE subfactors are known to be related to
CFTs, knot theory, representation theory,
category theory, TQFT, integrable models, more...

Big question (open as far as I can tell):

Do other subfactors give "exotic" (non quantum group) versions of these?

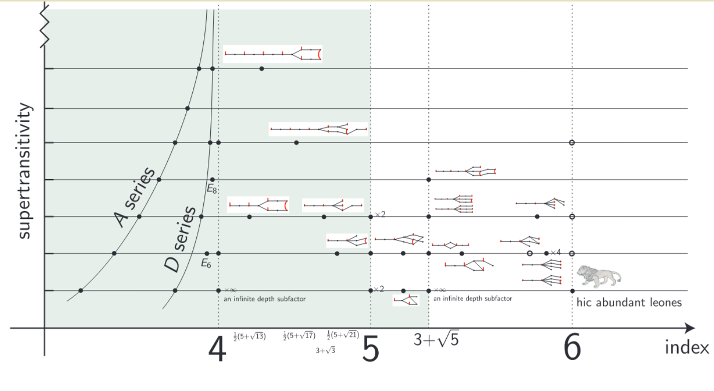
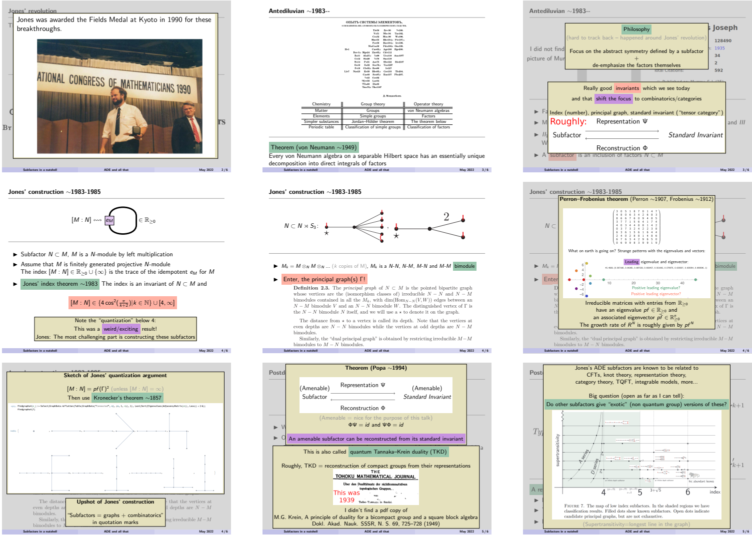
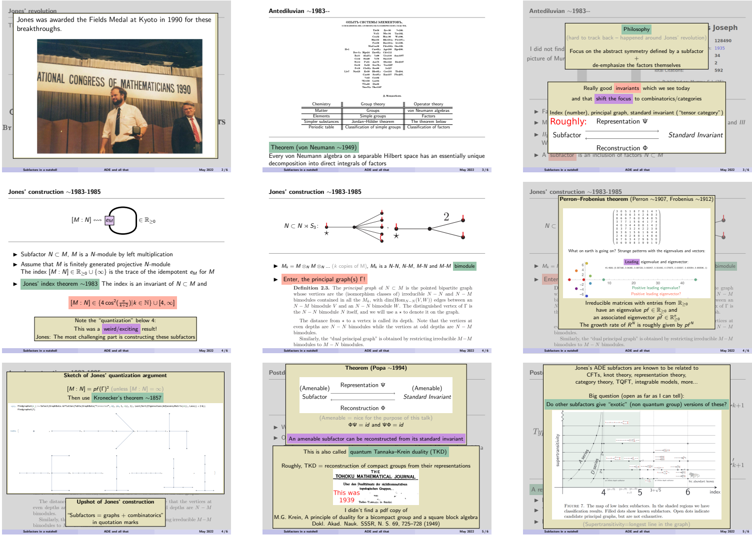


FIGURE 7. The map of low index subfactors. In the shaded regions we have classification results. Filled dots show known subfactors. Open dots indicate candidate principal graphs, but are not exhaustive.

(Supertransitivity=longest line in the graph)



There is still much to do...



Thanks for your attention!