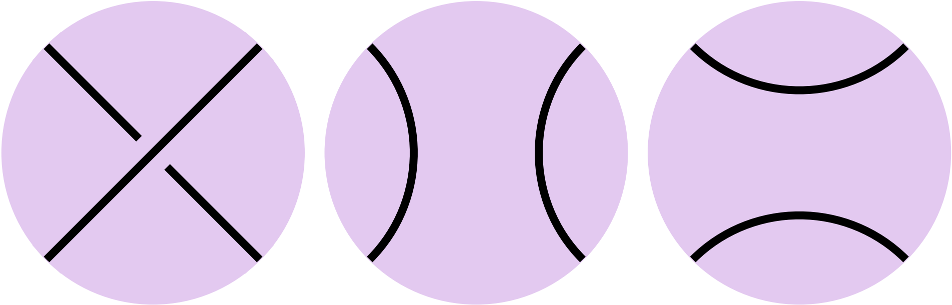
Temperley–Lieb times four

Or: Invariant theory, magnetism, subfactors and skein

Daniel Tubbenhauer



March 2022

The Temperley–Lieb (TL) calculus is everywhere

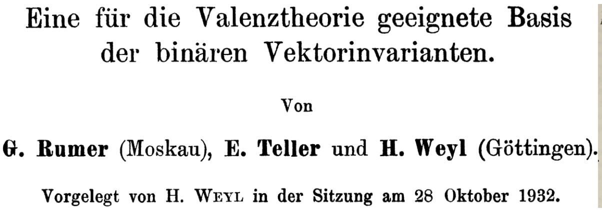
Throughout

Please convince yourself that I haven't messed up while picking my quotations from my stolen material

The Temperley–Lieb (TL) calculus is everywhere

The TL calculus was discovered several times, *e.g.*:

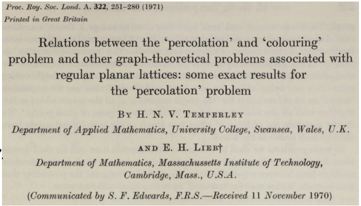
- Via valence bond theory Rumer–Teller–Weyl (RTW) ∼1932
- Via the Potts model Temperley–Lieb ∼1971
- Via subfactors Jones ∼1983
- Via skein theory Kauffman ∼1987


Eine für die Valenztheorie geeignete Basis der binären Vektorinvarianten.

Von

G. Rumer (Moskau), **E. Teller** und **H. Weyl** (Göttingen).

Vorgelegt von H. Weyl in der Sitzung am 28 Oktober 1932.


Proc. Roy. Soc. Lond. A. **322**, 251–280 (1971)
Printed in Great Britain

Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the 'percolation' problem

By H. N. V. Temperley

Department of Applied Mathematics, University College, Swansea, Wales, U.K.

and E. H. Lieb†

Department of Mathematics, Massachussetts Institute of Technology, Cambridge, Mass., U.S.A.

(*Communicated by S. F. Edwards, F.R.S.—Received* 11 *November* 1970)

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- Via valence bond theory Rumer–Teller–Weyl (RTW) ∼1932

Warning

I consider the two 1932 papers below as one

They are quite similar and appeared in the same issue of
Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen Mathematisch-Physikalische Klasse
(not continue from 1933 onward)

Eine

Von

G. Rumer (Moskau), **E. Teller** und **H. Weyl** (Göttingen).

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The Temperle

The TL calculu

- Via valenc
- Via the P
- Via subfac
- Via skein

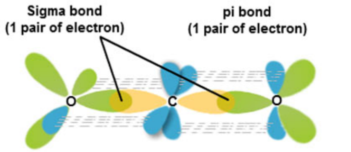
What we will see today

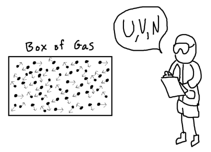
(2) Quantum chemistry

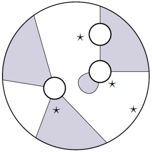
(3) Statistical mechanics

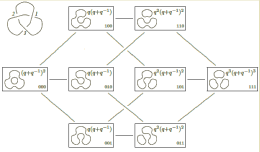
(4) Operator theory

(1) Quantum topology









Eine für die
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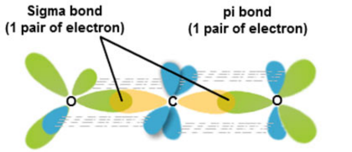

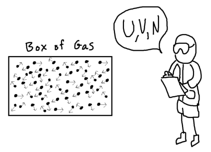

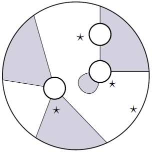

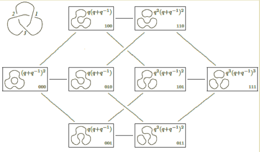

(1) is the newest incarnation of the TL calculus but easiest to explain, so I start with (1)

The Temperley–Lieb (TL) calculus is everywhere

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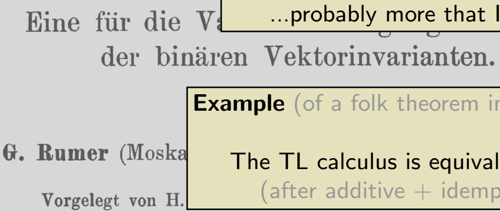
- Via valence bond theory Rumer–Teller–Weyl (RTW) ∼1932
- Via the Potts
- Via subfactor
- Via skein theo

Not discussed today, but honorable mentions

The TL calculus also appears in...

...the theory of quantum groups
...integrable models
...representation theory of reductive groups
...categorical quantum mechanics
...logic and computation
...probably more that I am not aware of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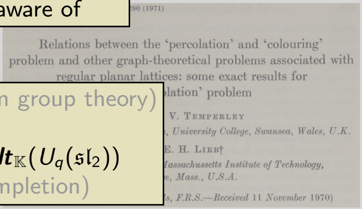
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Example (of a folk theorem in quantum group theory)

The TL calculus is equivalent to $\boldsymbol{Tilt}_{\mathbb{K}}(U_q(\mathfrak{sl}_2))$
(after additive + idempotent completion)

The Temperley–Lieb (TL) calculus is everywhere

Today's talk is based on:

my memory (horrible reference...)

On the Number of Rumer Diagrams

Valentin Vankov Iliev *

The Increasingly Popular Potts Model
or
A Graph Theorist Does Physics (!)

Jo Ellis-Monaghan

Subfactors
in Memory of Vaughan Jones

Zhengwei Liu

Tsinghua University

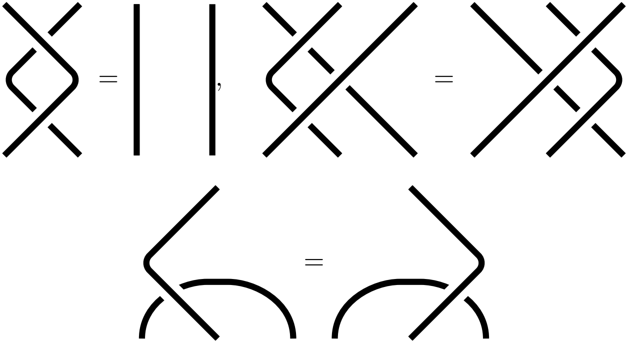
Math-Science Literature Lecture Series
November 23, 2020, Harvard CMSA and Tsinghua YMSC

The above are easy to google (its worth it!)

Kauffman's construction ∼1987

Step 1 Take the framed tangle calculus ***Tan*** with generators

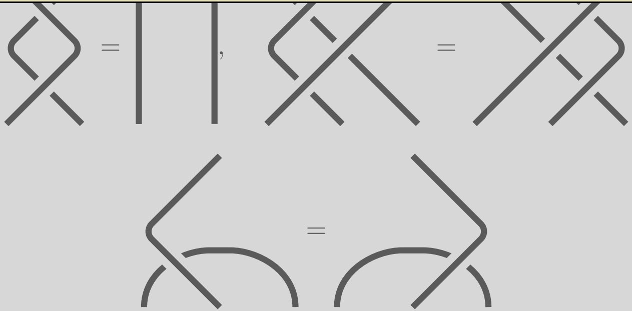
and relations being the usual tangle relations, *e.g.*



Kauffman's construction ∼1987

Warning

I do not want to be precise what "calculus" means since
it doesn't matter and is a bit messy in the literature, *e.g.*:
RTW never used any precise formulation
TL used algebras, but not using that terminology
Jones used algebras and operads, but not using the latter terminology
Kauffman used operads, but not using that terminology
Other researchers might prefer monoidal categories (*e.g.* following Turaev ∼1990)



St

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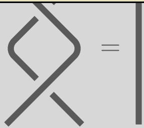
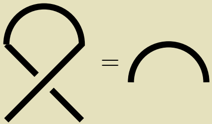
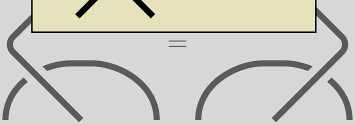
an



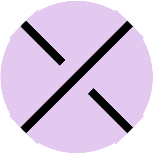
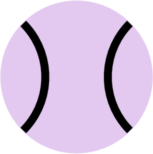
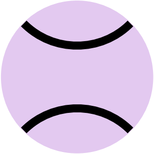
Note that ***Tan*** is framed so no relations of the form





Kauffman's construction $\sim$1987

Step 2 Make ***Tan*** $\mathbb{Z}[A, A^{-1}]$-linear and impose

 $= A \cdot$  $+ A^{-1} \cdot$ 

Kauffman skein relation

Kauffman's construction ∼1987

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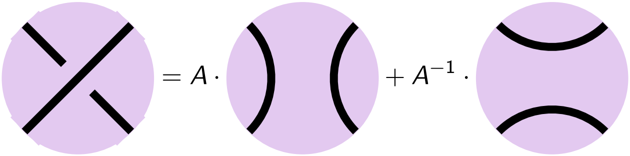


Kauffman skein relation

Kauffman skein relation
$\leftrightsquigarrow$
averaging over ways to get rid of the crossings

Here I am faithfully reproducing a constant
disagreement in the literature over the meaning of the "quantum parameter"
In quantum group theory $q = A^2$

Kauf

Step

Kauf

Why the A?

The A in Kauffman's formula only became clear in the light of Jones' paper

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 1, January 1985

A POLYNOMIAL INVARIANT FOR KNOTS VIA VON NEUMANN ALGEBRAS[1]

BY VAUGHAN F. R. JONES[2]

that appeared a bit earlier than Kauffman's paper

STATE MODELS AND THE JONES POLYNOMIAL

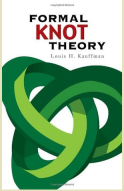
LOUIS H. KAUFFMAN

(*Received in revised form* 1 *September* 1986)

§1. INTRODUCTION

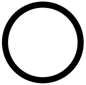
IN THIS PAPER I construct a state model for the (original) Jones polynomial [5]. (In [6] a state model was constructed for the Conway polynomial.)

Before 1985 Kauffman tried but didn't quite got there; [6] $\sim$1983 is:



Kauffman's construction ∼1987

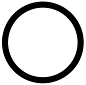
Step 3 Realize that one also need impose to impose



$$= \delta = -A^2 - A^{-2}$$

Then you are done and we have the TL calculus $\boldsymbol{TL}_{\mathbb{Z}[A,A^{-1}]}(\delta)$

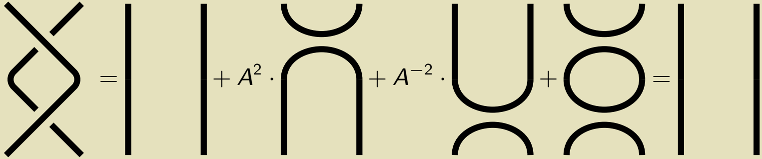
Step 3 Realize that one also need impose to impose



$$= \delta = -A^2 - A^{-2}$$

Then you are done and we have the TL calculus $\boldsymbol{TL}_{\mathbb{Z}[A,A^{-1}]}(\delta)$

We need this because



implies $\delta = -A^2 - A^{-2}$

This is Kauffman's famous calculation

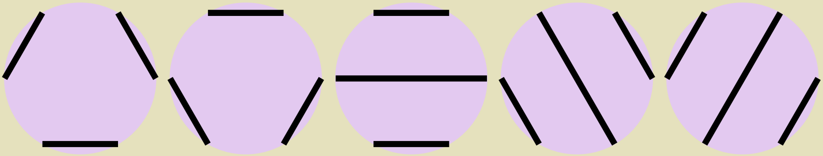
Kauffman's construction ∼1987

Step 3 Realize that one also need impose to impose

 $= \delta = -A^2 - A^{-2}$

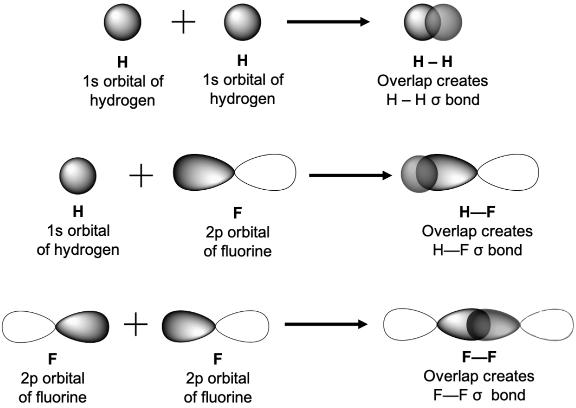
The

Example (6 points)



This is a basis of the six strand case
In general, the Catalan numbers give the dimension

The RTW construction ∼1932



- Problem Find a model for chemical bonding
- Valence bond theory uses methods of quantum mechanics to explain bonding
- The RTW paper models valence bonds using $\mathrm{SL}_2(\mathbb{C})$

The RTW construction ∼1932

- Each atom is a vector $x = (x_1, x_2) \in \mathbb{C}^2$
- Each bond $[x, y]$ is a matrix determinant

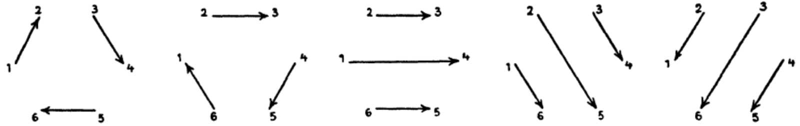
$$[x, y] = \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = x_1 y_2 - x_2 y_1$$

  viewed as a polynomial with four variables
- These determinants are the building blocks of all $f\colon \mathbb{C}^{2n} \to \mathbb{C}$ that are invariant under transformation with determinant 1
- First fundamental theorem of invariant theory
  Any $\mathrm{SL}_2(\mathbb{C})$-invariant function is a linear combination of products

$$[x^{(1)}, y^{(1)}] \cdot \ldots \cdot [x^{(k)}, y^{(k)}]$$

The RTW construction ∼1932

3. Als zweites Beispiel behandeln wir die zyklische einvalentige Kette mit sechs Atomen. Die Valenzfunktionen sind

(4) 

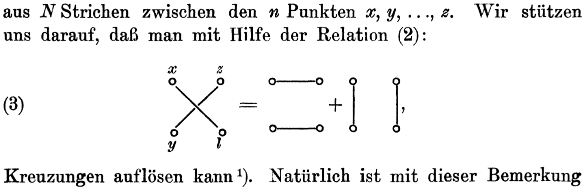
- RTW now put the atoms on a circle
- Then RTW draw bonds as lines
- The result is TL diagrams coming from valence theory:

atoms=points and bonds=strands

Ket … valentige

(4)

The Kauffman bracket via valence bonds


aus N Strichen zwischen den n Punkten $x, y, \ldots, z$. Wir stützen uns darauf, daß man mit Hilfe der Relation (2):

(3)

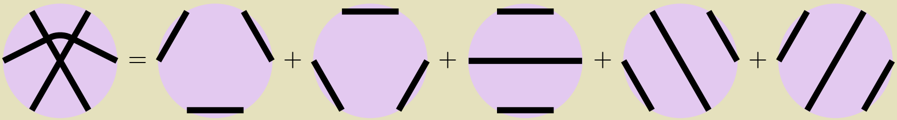
Kreuzungen auflösen kann [1]). Natürlich ist mit dieser Bemerkung

The Kauffman bracket follows easily from the RTW setting:

$$[x,l][y,z]=(x_1l_2-x_2l_1)(y_1z_2-y_2z_1)=[x,z][y,l]+[x,y][l,z]$$

- RTW now put the atoms on a circle
- Then RTW draw bonds as lines

Example



The RTW

The Kauffman bracket via valence bonds

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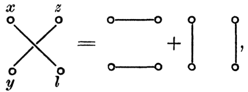
(3) 

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The Kauffman bracket follows easily from the RTW setting:

$$[x,l][y,z] = (x_1l_2 - x_2l_1)(y_1z_2 - y_2z_1) = [x,z][y,l] + [x,y][l,z]$$

- RT
- Th
- Th

Second fundamental theorem of invariant theory via valence bonds

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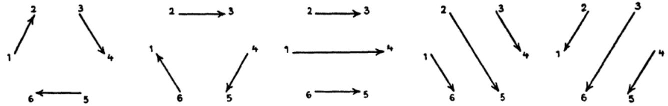
(4) 

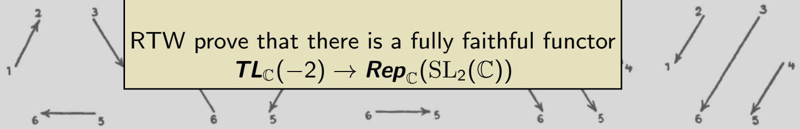
RTW also prove that crossingless matching form a basis

The RTW construction ∼1932

3. Als zweites Beispiel behandeln wir die zyklische einvalentige Kette mit sechs Atomen. Die Valenzfunktionen sind

Modern formulation

RTW prove that there is a fully faithful functor
$\textbf{\textit{TL}}_{\mathbb{C}}(-2) \to \textbf{\textit{Rep}}_{\mathbb{C}}(\mathrm{SL}_2(\mathbb{C}))$

(4) 

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(4)

Modern formulation

RTW prove that there is a fully faithful functor
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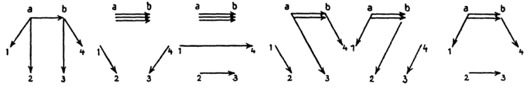
Symmetric powers

Als Beispiel betrachten wir das Hydrazin NH_2—NH_2. Wir bezeichnen mit a, b die beiden N-Atome, mit 1, 2, 3, 4 die vier H-Atome. Ordnen wir die Atome auf einem Kreis an, so erhalten wir nach der Anweisung folgende sechs Valenzzustände als Basis[3]):

(3) 

Actually it is more general:
RTW also address these questions for symmetric powers with $\mathrm{Sym}^k\mathbb{C}^2$ corresponding to a k-valence bond

- RTW now
- Then RT
- The resul

The RTW construction $\sim$1932

3. Als zweites Beispiel behandeln wir die zyklische einvalentige Kette mit sechs Atomen. Die Valenzfunktionen sind

Let's ask SAGEMath whether the RTW basis has the correct number of elements:

Type some Sage code below and press Evaluate.

```
A1 = WeylCharacterRing(['A',1])
A1(1,0)*A1(1,0)*A1(1,0)
```

Evaluate

```
2*A1(2,1) + A1(3,0)
```

Type some Sage code below and press Evaluate.

```
A1 = WeylCharacterRing(['A',1])
A1(3,0)*A1(1,0)*A1(1,0)
```

Evaluate

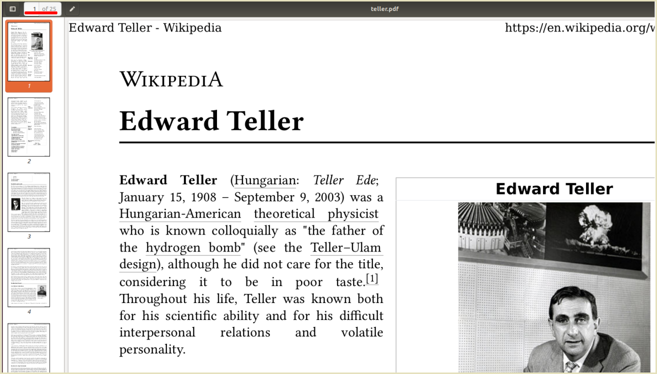
```
A1(3,2) + 2*A1(4,1) + A1(5,0)
```

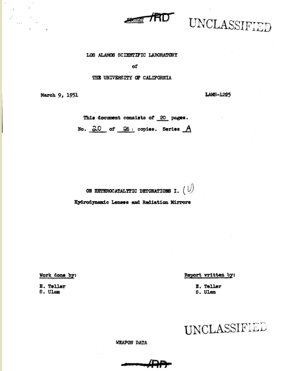
Indeed, $1^2 + 2^2 = 5$ and $1^2 + 2^2 + 1^2 = 6$

The RTW construction ∼1932

3. Als zweites Beispiel behandeln wir die zyklische einvalentige Kette mit sechs Atomen. Die Valenzfunktionen sind

Edward Teller is the big name here





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LOS ALAMOS SCIENTIFIC LABORATORY
of
THE UNIVERSITY OF CALIFORNIA

March 9, 1951 LAMS-1225

This document consists of 20 pages.
No. 20 of 30 copies. Series A

ON HETEROCATALYTIC DETONATIONS I. (U)
Hydrodynamic Lenses and Radiation Mirrors

Work done by:
E. Teller
S. Ulam

Report written by:
E. Teller
S. Ulam

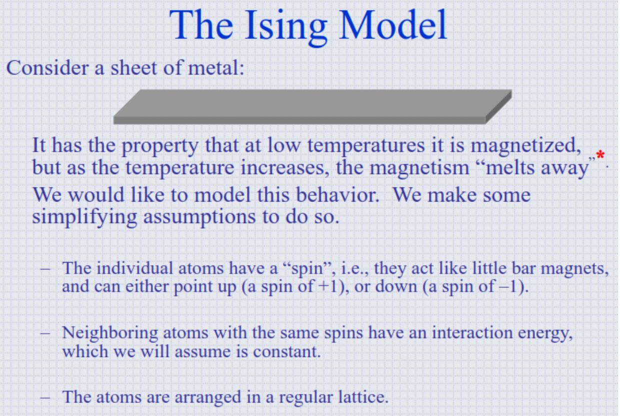
UNCLASSIFIED

WEAPON DATA

/RD

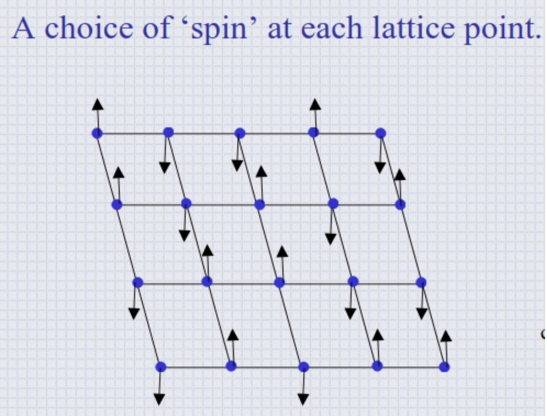
Teller's Wikipedia page has 25 printed pages (01.Mar.2022); it is very readable

The TL construction $\sim$1971

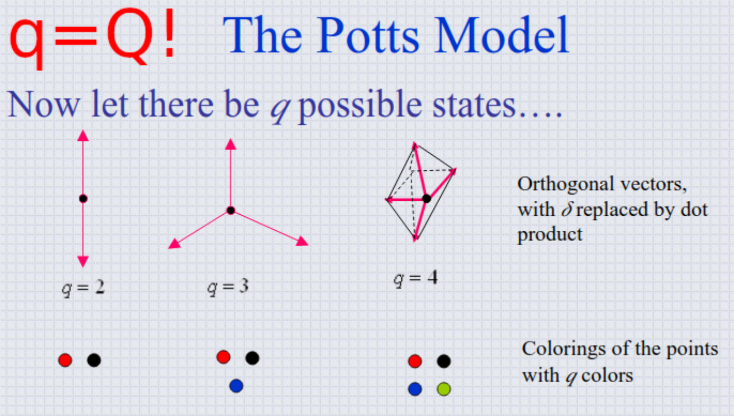


- The Ising model's interpretation is explained above Magnetism
- The Potts model is a generalization of the Ising model
- TL studied these Solid-state physics

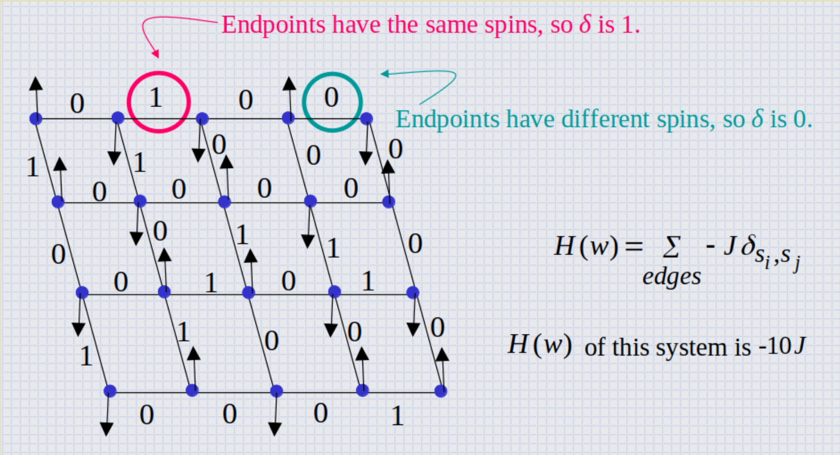
The TL construction ∼1971



- The Ising model is a lattice model
- The states are spins (up and down)
- Recall that what we want to know is Z_S Partition function



- The Potts model is a lattice model
- The states are "spins" from 1 to Q (Ising $Q = 2$)
- We want to know $Z_S = Z_S(\beta = 1/kT)$ Partition function



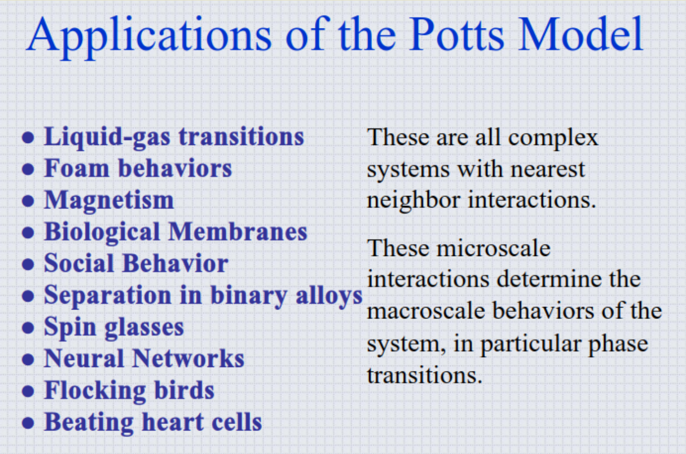
$$Z_X = \sum_{\text{states}} \exp(-\beta H)$$

$H = \sum_{\text{edges}} \delta_{\sigma_i,\sigma_j} J$, $J =$energy (a number)

- We want to know $Z_S = Z_S(\beta = 1/kT)$ Partition function

The TL construction ∼1971

The Potts model is very applicable:



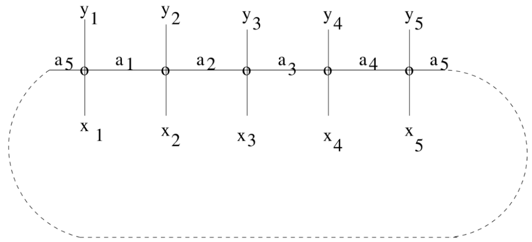
For all of these there is some form of the TL calculus

- The
- The
- We want to know $Z_S = Z_S(\beta = 1/kT)$ Partition function

Recall from last time that solving
the model "is equivalent to having good expressions for transfer matrices"

Transfer matrices

$X =$



$$T^{\text{row}} = \sum_{a_i} R(a_n, a_1 | x_1, y_1) \dots R(a_{n-1}, a_n | x_n, y_n)$$

- For $\mathbb{Z}^2$ with periodic boundary use transfer matrix above
- Problem Computing the largest eigenvalue becomes infeasible

- T
- The states are "spins" from 1 to Q (Ising $Q = 2$)
- We want to know $Z_S = Z_S(\beta = 1/kT)$ Partition function

The TL construction ∼1971

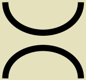
$$\left.\begin{aligned} (1-T_{12})[1\,2] &= 2[1\,2], \\ (1-T_{23})[1\,2][3\,4] &= [4\,1][2\,3], \\ (1-T_{23})[4\,1][2\,3] &= 2[4\,1][2\,3]. \end{aligned}\right\}$$

- $V=\mathbb{C}^Q$; the operators below are on tensor powers of V
- $p=$multiplication by $1/\sqrt{Q}$, $d_{i,i+1}(v_i\otimes v_j)=\delta_{ij}v_i\otimes v_i$

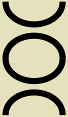
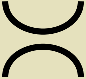
  $Q=(A+A^{-1})^2$, *e.g.* $Q=2$ implies $A=(-1)^{1/4}$
- $E_{2i-1}=1\otimes\ldots\otimes 1\otimes p\otimes 1\otimes\ldots\otimes 1$ (p in the ith entry)
- $E_{2i}=1\otimes\ldots\otimes 1\otimes d_{i,i+1}\otimes 1\otimes\ldots\otimes 1$ ($d_{i,i+1}$ in the ith entry)
- Up to scaling, $E_k=1-T_{k(k+1)}$ Kauffman bracket
- The transfer matrix with free horizontal boundary conditions is a multiple of $\big(\prod_{i=1}^{n-1}aE_{2i}+1\big)\big(\prod_{i=1}^{n}bE_{2i-1}+1\big)$ where a and b determined by the boundary condition

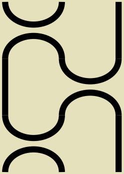
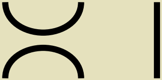
The TL constru

The operators satisfy the TL relations

$E_k \leftrightsquigarrow$ 

k

$E_k E_k = \sqrt{Q} E_k \leftrightsquigarrow$  $= \sqrt{Q}$ 

$E_k E_{k\pm 1} E_k = E_k \leftrightsquigarrow$  $=$ 

$E_k E_l = E_l E_k$ for $|k - l| > 1 \leftrightsquigarrow$ far commutativity

- $V = \mathbb{C}^Q$; th
- $p =$multiplic

  $Q = (A + A$
- $E_{2i-1} = 1 \otimes$
- $E_{2i} = 1 \otimes \dots$ y)
- Up to scalin
- The transfer s is a multiple of $\left(\prod_{i=1}^{n-1} aE_{2i} + 1\right)\left(\prod_{i=1}^{n} bE_{2i-1} + 1\right)$ where a and b determined by the boundary condition

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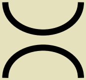
$E_k \leftrightsquigarrow$ 

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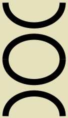
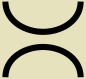
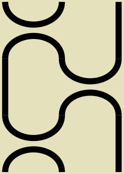
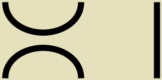
$E_kE_k = \sqrt{Q}E_k \leftrightsquigarrow$  $= \sqrt{Q}$ 

$E_kE_{k\pm1}E_k = E_k \leftrightsquigarrow$  $=$ 

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TL then show that

Z_S is determined by the TL relations

The TL construction ∼1971

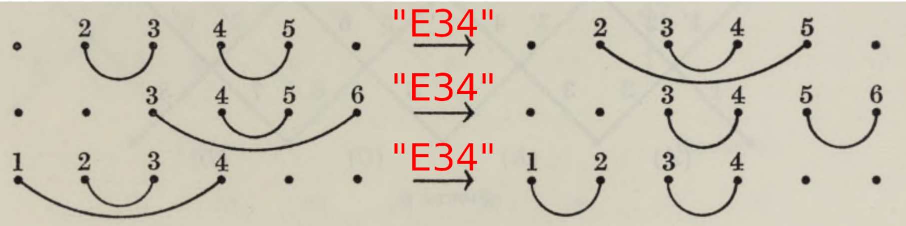
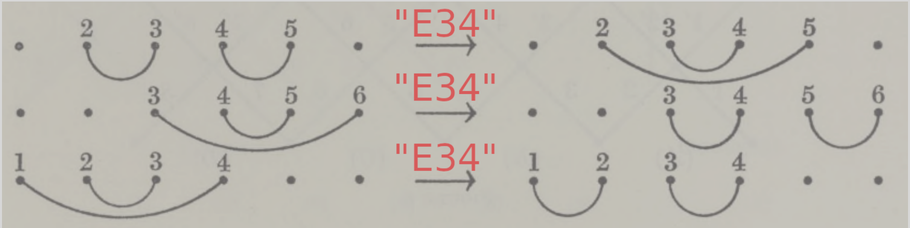


FIGURE 4

- TL also write down and study cell modules
- They use the usual diagrammatics to describe these
- They did not use diagrammatics to describe $\boldsymbol{TL}_{\mathbb{C}}(\sqrt{Q})$ itself
- They do not compute the dimension of $\boldsymbol{TL}_{\mathbb{C}}(\sqrt{Q})$



TL show that Z_{Potts} and $Z_{\text{Ice-type}}$ determine one another

General solution of the Potts model

$$Z_G \approx T(a,b)$$

where $T(x,y)$ is the Tutte polynomial for $a = (|Q| + \exp(\beta J) - 1)/(\exp(\beta J) - 1)$ and $b = \exp(\beta J)$

- TL also
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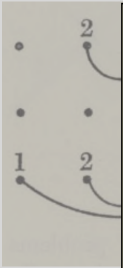
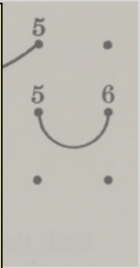
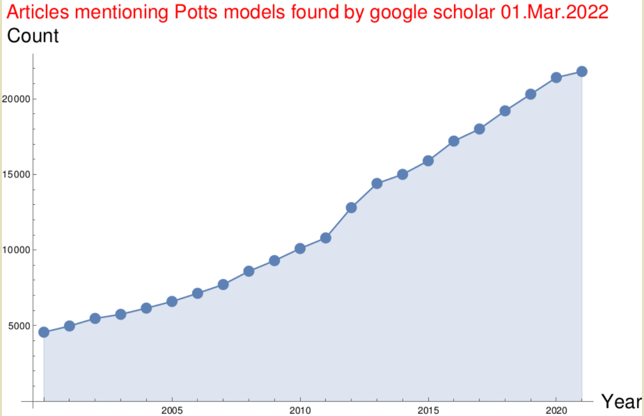
The TL construction ∼1971





Impressive!



More than 250000 hits in total

This is a widely spread incarnation of the TL calculus

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- They u
- They c
- They c

Invent. math. 72, 1–25 (1983)

Inventiones mathematicae

Index for Subfactors

V.F.R. Jones

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA

- Factor = von Neumann algebra with trivial center
- A subfactor is an inclusion of factors $N \subset M$
- Murray–von Neumann ∼1930+ classified factors by types: I, II_1, II_∞ and III
- II_1 are the "most exciting" ones They have a unique trace!
  We will stick with these (I drop the "of type II_1" – it should appear everywhere)

Invent. math. 72, 1–25 (198

The word "algebra" is highlighted because there will be representations!

Inventiones mathematicae

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Inventiones mathematicae

Inde

V.F.R
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Example

M is a factor, G a nice group, then $M^G \subset M$ is a subfactor

Vague slogan Subfactors $\leftrightsquigarrow$ fixed points of a "quantum group" G action

The is also a version of Galois correspondence
and one can recover G from $M^G \subset M$
In this sense subfactor theory generalizes finite group theory

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- M … and III

Jones' paper was one of the starting point of
transferring subfactors from functional analysis to algebra/combinatorics

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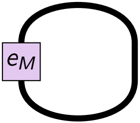
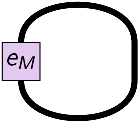
Jones' construction ∼1983

$[M : N] \rightsquigarrow$  $\in \mathbb{R}_{\geq 0}$

- Subfactor $N \subset M$, M is a N-module by left multiplication
- Assume that M is finitely generated projective N-module
  The index $[M : N] \in \mathbb{R}_{\geq 0}$ is the trace of the idempotent e_M for M
- Jones' index theorem ∼1983 The index is an invariant of $N \subset M$ and

$$[M : N] \in \{4\cos^2(\tfrac{\pi}{k+2}) | k \in \mathbb{N}\} \cup [4, \infty]$$

Jones' construction ~1983

$[M : N] \leftrightsquigarrow$ 

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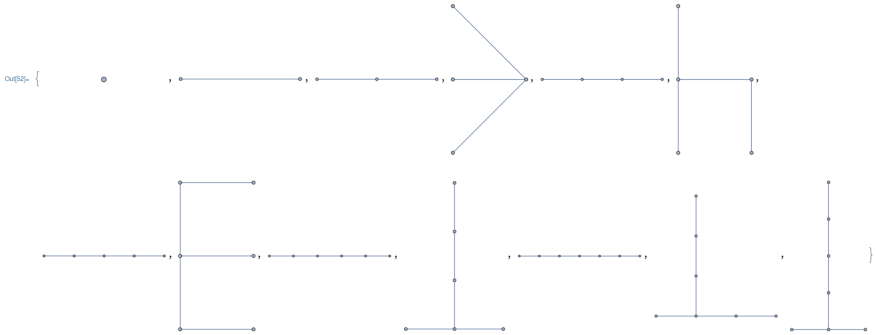
Note the "quantization" below 4:
This was a weird/exciting result!
Jones: The most challenging part is constructing these subfactors

Sketch of the quantization argument

Associated a graph G to $N \subset M$ and "$[M : N] = pf(G)$"
Then use Kronecker's theorem

In[51]:=
```
findgraphs4[n_] := Select[GraphData /@ Flatten[Table[GraphData["Connected", m], {m, 1, n}], 1], Last[Sort[Eigenvalues[AdjacencyMatrix[#]], Less]] < 2 &];
findgraphs4[7]
```



Jones' construction ∼1983

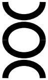
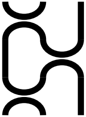
§4. Possible Values of the Index

§4.1. Certain Algebras Generated by Projections

Jones' projectors satisfy the scaled TL relations for $\delta = [M:N] = 4\cos^2(\frac{\pi}{k+2})$

$e_k \leftrightsquigarrow \frac{1}{[M:N]}$ 

k

- $e_k e_k = e_k \leftrightsquigarrow$  $=$ 
- $e_k e_{k\pm1} e_k = \frac{1}{[M:N]} e_k \leftrightsquigarrow$  $= \frac{1}{[M:N]}$ 
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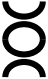
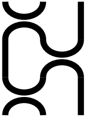
- $e_k e_k = e_k \leftrightsquigarrow$  $=$ 

$\{e_i | i \geq n\}$ generate a factor R_n
$R_2 \subset R_1$ is a subfactor of index $4\cos^2(\frac{\pi}{k+2})$

- $e_k e_{k\pm1} e_k = \frac{1}{[M:N]} e_k \leftrightsquigarrow$  $= \frac{1}{[M:N]}$ 
- $e_k e_l = e_l e_k$ for $|k-l| > 1 \leftrightsquigarrow$ far commutativity

Jones' construction ∼1983

The Markov property!

They satisfy the relations

(I) $e_i^2 = e_i, e_i^* = e_i$,

(II) $e_i e_{i\pm 1} e_i = t/(1+t)^2 e_i$, TL relations

(III) $e_i e_j = e_j e_i$ if $|i-j| \geq 2$.

Here t is a complex number. It has been shown by H. Wenzl [**24**] that an arbitrarily large family of such projections can only exist if t is either real and positive or $e^{\pm 2\pi i/k}$ for some $k = 3, 4, 5, \ldots$. When t is one of these numbers, there exists such an algebra for all n possessing a trace $\mathrm{tr}\colon A_n \to \mathbf{C}$ completely determined by the normalization $\mathrm{tr}(1) = 1$ and

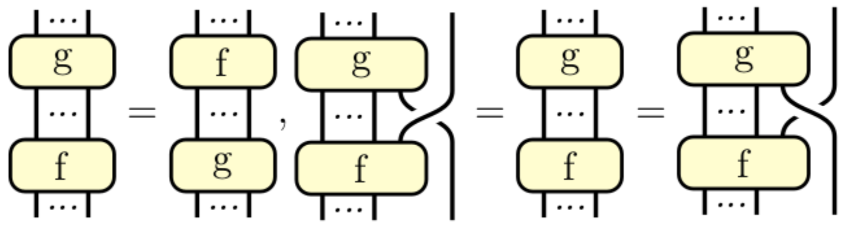
(IV) $\mathrm{tr}(ab) = \mathrm{tr}(ba)$,

(V) $\mathrm{tr}(we_{n+1}) = t/(1+t)^2\, \mathrm{tr}(w)$ if w is in A_n,

(VI) $\mathrm{tr}(a^*a) > 0$ if $a \neq 0$ Markov trace

(note $A_0 = \mathbf{C}$).

In braid pictures (crossing is given by Kauffman skein formula)



Jones' construction ~1

In hindsight the crucial result
Jones' proved about $\boldsymbol{TL}_{\mathbb{C}}(\delta)$
is the existence of a Markov trace

§4. Possible Values of the Index

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- $e_k e_k = e_k \leftrightsquigarrow$  $=$ 
- $e_k e_{k\pm1} e_k = \frac{1}{[M:N]} e_k \leftrightsquigarrow$  $= \frac{1}{[M:N]}$ 
- $e_k e_l = e_l e_k$ for $|k-l| > 1 \leftrightsquigarrow$ far commutativity

In hindsight the crucial result Jones' proved about $\boldsymbol{TL}_{\mathbb{C}}(\delta)$ is the existence of a Markov trace

§4. Possible Values of the Index

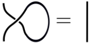
Why? Well, because of the (Birman–)Jones polynomial:

Jones Polynomial

Jones met Birman in 1984:

Markov trace on the braid group $\Rightarrow$ knot invariant

It was surprising that the Markov trace naturally comes from the trace of the II_1 factor,

$\tau(x\sigma_n) = \tau(x)$, $\forall x \in TL_n$, $\longrightarrow$  , (Reidemeister move I)

therefore leading to a knot invariant, well-known as the Jones polynomial, by which Jones answered a series of old questions in knot theory in 1985.

 $= t + t^3 - t^4$.  $= t^{-1} + t^{-3} - t^{-4}$.

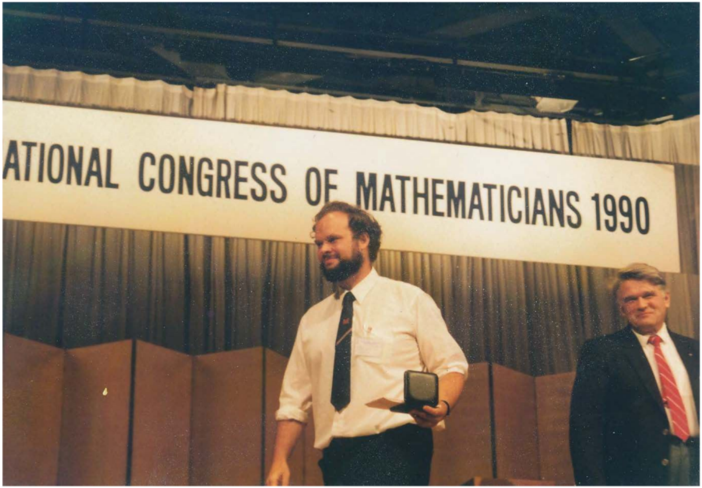
Reflection: $t \to t^{-1}$.

Jones' proj ... $\cos^2(\frac{\pi}{k+2})$

$e_k \rightsquigarrow \frac{1}{[M:}$

- $e_k e_k =$
- $e_k e_{k\pm}$
- $e_k e_l =$

Jones was awarded the Fields Medal at Kyoto in 1990 for these breakthroughs.



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§4. Possible Values of the Index

§4.1. Certain Algebras Generated by Projections

Jones' proj ... $\cos^2(\frac{\pi}{k+2})$

$e_k \leftrightsquigarrow \frac{1}{[M:N]}$ ...

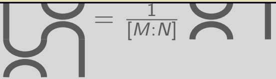
Jones also implicitly coined the name "TL algebra" ∼1985:

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 1, January 1985

A POLYNOMIAL INVARIANT FOR KNOTS VIA VON NEUMANN ALGEBRAS[1]

BY VAUGHAN F. R. JONES[2]

For real t, D. Evans pointed out that an explicit representation of A_n on $\mathbf{C}^{2n+2}$ was discovered by H. Temperley and E. Lieb [**23**], who used it to show the equivalence of the Potts and ice-type models of statistical mechanics. A

- $e_k e_k$ = ...
- $e_k e_{k\pm1} e_k = \frac{1}{[M:N]} e_k \leftrightsquigarrow$ 
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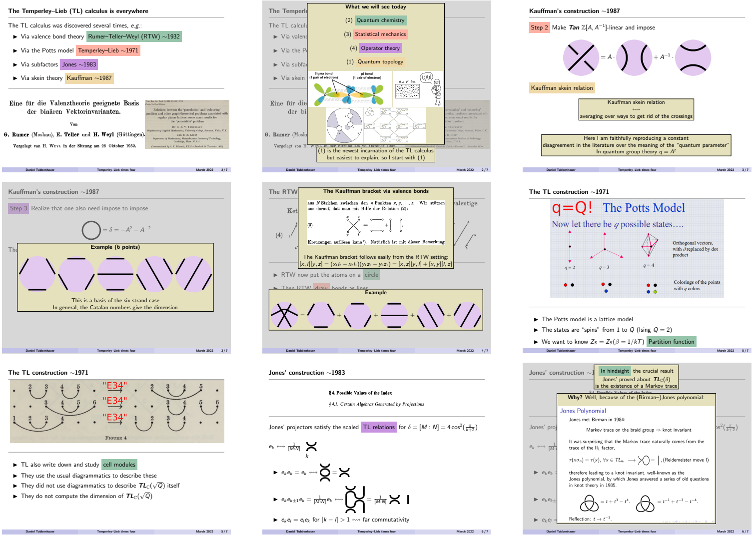
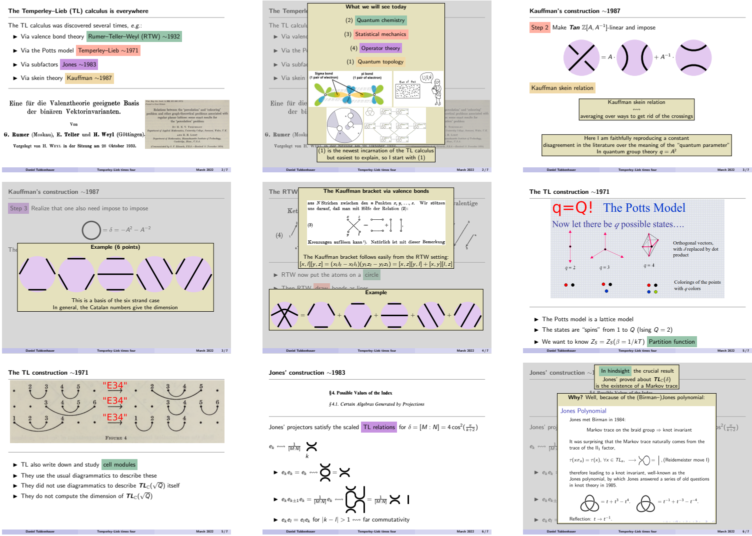


There is still much to do...



Thanks for your attention!