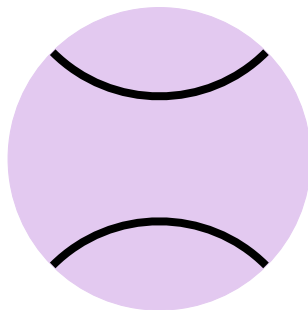
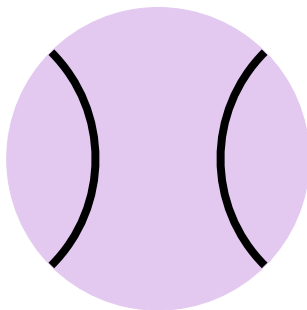
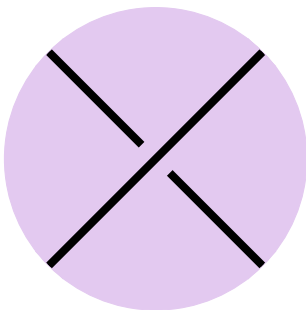


Temperley–Lieb times four

Or: Invariant theory, magnetism, subfactors and skein

Daniel Tubbenhauer



March 2022

Throughout

Please convince yourself that I haven't messed up while picking my quotations from my stolen material

The Temperley–Lieb (TL) calculus is everywhere

The TL calculus was discovered several times, e.g.:

- ▶ Via valence bond theory Rumer–Teller–Weyl (RTW) ~1932
 - ▶ Via the Potts model Temperley–Lieb ~1971
 - ▶ Via subfactors Jones ~1983
 - ▶ Via skein theory Kauffman ~1987
-

Eine für die Valenztheorie geeignete Basis
der binären Vektorinvarianten.

Von

G. Rumer (Moskau), **E. Teller** und **H. Weyl** (Göttingen).

Vorgelegt von H. WEYL in der Sitzung am 28 Oktober 1932.

Proc. Roy. Soc. Lond. A. 322, 251–280 (1971)
Printed in Great Britain

Relations between the ‘percolation’ and ‘colouring’
problem and other graph-theoretical problems associated with
regular planar lattices: some exact results for
the ‘percolation’ problem

BY H. N. V. TEMPERLEY

Department of Applied Mathematics, University College, Swansea, Wales, U.K.

AND E. H. LIEB†

*Department of Mathematics, Massachusetts Institute of Technology,
Cambridge, Mass., U.S.A.*

(Communicated by S. F. Edwards, F.R.S.—Received 11 November 1970)

The Temperley–Lieb (TL) calculus is everywhere

The TL calculus was discovered several times, e.g.:

- ▶ Via valence bond theory Rumer–Teller–Weyl (RTW) ~1932

Warning

I consider the two 1932 papers below as one

RUMER, G., Zur Theorie der Spinvalenz	337
– TELLER, E., und WEYL, H., Eine für die Valenztheorie geeignete Basis der binären Vektorinvarianten	499

They are quite similar and appeared in the same issue of
Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen
Mathematisch-Physikalische Klasse
(not continue from 1933 onward)

Von

G. Rumer (Moskau), **E. Teller** und **H. Weyl** (Göttingen).

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the ‘percolation’ problem

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The Temperley

The TL calcul

► Via valenc

► Via the P

► Via subfac

► Via skein

Eine für die
der bi

G. Rumer (Moska

Vorgelegt von H. WEYL in der Sitzung am 28 Oktober 1932.

What we will see today

(2) Quantum chemistry

(3) Statistical mechanics

(4) Operator theory

(1) Quantum topology

The collage contains several diagrams:

- Top Left:** Molecular orbital diagram showing three carbon atoms with p-orbitals. A central yellow region is labeled "Sigma bond (1 pair of electron)". A blue region above and below is labeled "pi bond (1 pair of electron)".
- Top Right:** A box labeled "Box of Gas" containing a collection of black dots representing particles. A cartoon character stands next to it with a speech bubble containing "U, V, N".
- Bottom Left:** A circular diagram divided into four quadrants, with two white circles and two grey circles, each marked with a star.
- Bottom Right:** A lattice of diagrams showing various mathematical expressions involving q and q^{-1} , such as $q^{i(i+q^{-1})}$ and $q^{i(i+q)}$.

percolation' and 'colouring'
oretical problems associated with
es: some exact results for
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V. TEMPERLEY
University College, Swansea, Wales, U.K.
J. H. LIEB†
Massachusetts Institute of Technology,
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What we will see today

(2) Quantum chemistry

(3) Statistical mechanics

(4) Operator theory

(1) Quantum topology

The collage contains several diagrams:

- Top Left:** Molecular orbitals showing a **Sigma bond (1 pair of electron)** and a **pi bond (1 pair of electron)** between two atoms.
- Top Right:** A **Box of Gas** containing particles, with a scientist character and a speech bubble containing U, V, N .
- Bottom Left:** A circular diagram with shaded sectors and stars, possibly representing a quantum state or symmetry.
- Bottom Right:** A lattice of nodes connected by lines, with mathematical labels such as $g^{(i_1, \dots, i_n)}$ and $g^{(i_1, \dots, i_{n-1})}$.

percolation' and 'colouring'
oretical problems associated with
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V. TEMPERLEY
University College, Swansea, Wales, U.K.
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Massachusetts Institute of Technology,
Mass., U.S.A.

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(1) is the newest incarnation of the TL calculus but easiest to explain, so I start with (1)

The Temperley–Lieb (TL) calculus is everywhere

The TL calculus was discovered several times, e.g.:

▶ Via valence bond theory: Rumer, Teller, Wall (RTW) — 1932

▶ Via the Potts

▶ Via subfactor

▶ Via skein theory

Not discussed today, but honorable mentions

The TL calculus also appears in...

- ...the theory of quantum groups
- ...integrable models
- ...representation theory of reductive groups
- ...categorical quantum mechanics
- ...logic and computation
- ...probably more that I am not aware of

Eine für die Valenz
der binären Vektorinvarianten.

Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the 'percolation' problem

V. TEMPERLEY
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Cambridge, Mass., U.S.A.
† F.R.S.—Received 11 November 1970)

Example (of a folk theorem in quantum group theory)

The TL calculus is equivalent to $\mathit{Tilt}_{\mathbb{K}}(U_q(\mathfrak{sl}_2))$
(after additive + idempotent completion)

G. Rumer (Moskva)

Vorgelegt von H.

The Temperley–Lieb (TL) calculus is everywhere

Today's talk is based on:

my memory (horrible reference...)

On the Number of Rumer Diagrams

Valentin Vankov Iliev *

The Increasingly Popular Potts Model

or

A Graph Theorist Does Physics (!)

Jo Ellis-Monaghan

Subfactors

in Memory of Vaughan Jones

Zhengwei Liu

Tsinghua University

Math-Science Literature Lecture Series

November 23, 2020, Harvard CMSA and Tsinghua YMSC

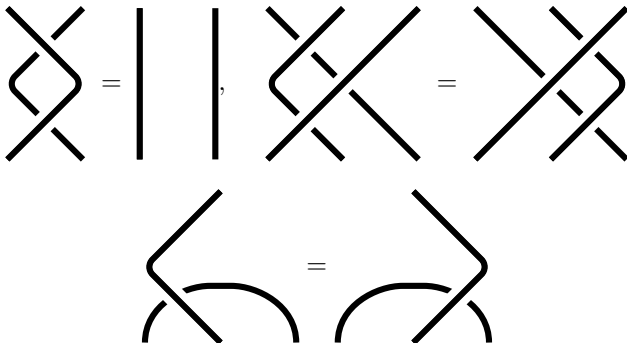
The above are easy to google (its worth it!)

Kauffman's construction ~1987

Step 1 Take the framed tangle calculus **Tan** with generators



and relations being the usual tangle relations, e.g.



Warning

I do not want to be precise what “calculus” means since it doesn't matter and is a bit messy in the literature, e.g.:

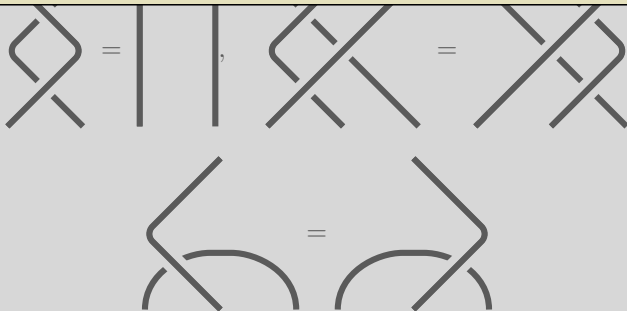
RTW never used any precise formulation

TL used algebras, but not using that terminology

Jones used algebras and operads, but not using the latter terminology

Kauffman used operads, but not using that terminology

Other researchers might prefer monoidal categories (e.g. following Turaev ~1990)



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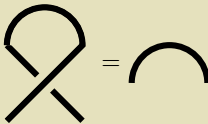
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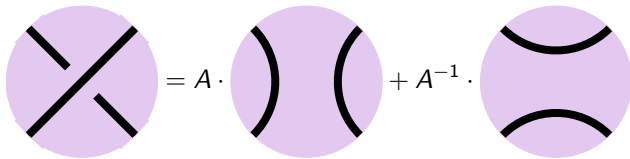
Other researchers might prefer monoidal categories (e.g. following Turaev ~1990)

Note that *Tan* is framed
so no relations of the form



Kauffman's construction ~ 1987

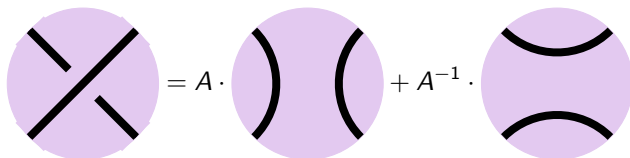
Step 2 Make **Tan** $\mathbb{Z}[A, A^{-1}]$ -linear and impose



Kauffman skein relation

Kauffman's construction ~ 1987

Step 2 Make **Tan** $\mathbb{Z}[A, A^{-1}]$ -linear and impose



Kauffman skein relation

Kauffman skein relation
 \longleftrightarrow
averaging over ways to get rid of the crossings

Here I am faithfully reproducing a constant disagreement in the literature over the meaning of the “quantum parameter”
In quantum group theory $q = A^2$

Why the A?

The A in Kauffman's formula only became clear in the light of Jones' paper

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 1, January 1985

A POLYNOMIAL INVARIANT FOR KNOTS VIA VON NEUMANN ALGEBRAS¹

BY VAUGHAN F. R. JONES²

that appeared a bit earlier than Kauffman's paper

STATE MODELS AND THE JONES POLYNOMIAL

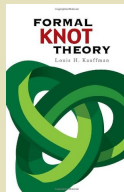
LOUIS H. KAUFFMAN

(Received in revised form 1 September 1986)

§1. INTRODUCTION

IN THIS PAPER I construct a state model for the (original) Jones polynomial [5]. In [6] a state model was constructed for the Conway polynomial.

Before 1985 Kauffman tried but didn't quite get there; [6] ~1983 is:



Kauffman's construction ~ 1987

Step 3 Realize that one also need impose to impose

$$\bigcirc = \delta = -A^2 - A^{-2}$$

Then you are done and we have the TL calculus $\mathbf{TL}_{\mathbb{Z}[A, A^{-1}]}(\delta)$

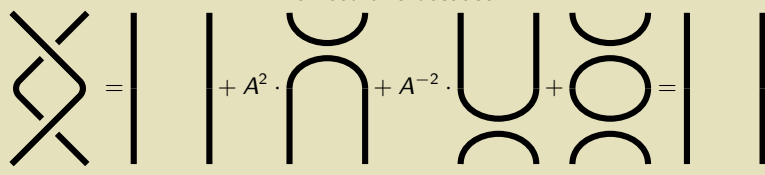
Kauffman's construction ~ 1987

Step 3 Realize that one also need impose to impose

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We need this because



implies $\delta = -A^2 - A^{-2}$

This is Kauffman's famous calculation

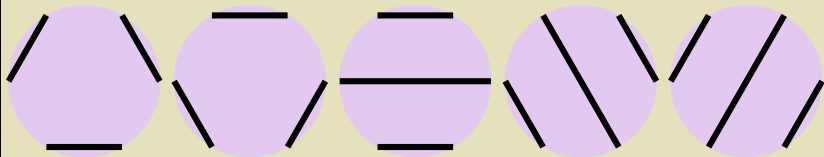
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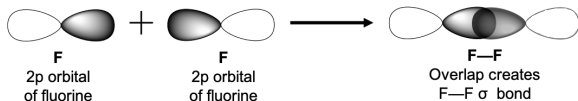
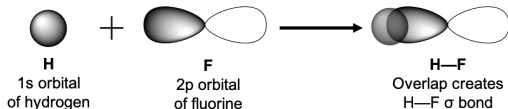
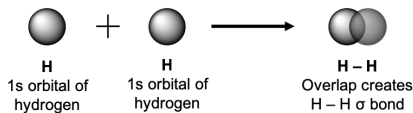
The

Example (6 points)



This is a basis of the six strand case
In general, the Catalan numbers give the dimension

The RTW construction ~ 1932



- ▶ **Problem** Find a model for chemical bonding
- ▶ Valence bond theory uses methods of quantum mechanics to explain bonding
- ▶ The RTW paper models valence bonds using $SL_2(\mathbb{C})$

The RTW construction ~ 1932

- ▶ Each atom is a vector $x = (x_1, x_2) \in \mathbb{C}^2$
- ▶ Each bond $[x, y]$ is a matrix determinant

$$[x, y] = \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = x_1 y_2 - x_2 y_1$$

viewed as a polynomial with four variables

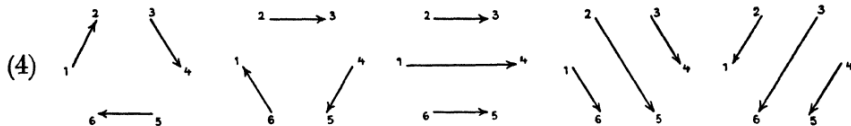
- ▶ These determinants are the building blocks of all $f: \mathbb{C}^{2n} \rightarrow \mathbb{C}$ that are invariant under transformation with determinant 1
- ▶ First fundamental theorem of invariant theory

Any $SL_2(\mathbb{C})$ -invariant function is a linear combination of products

$$[x^{(1)}, y^{(1)}] \cdot \dots \cdot [x^{(k)}, y^{(k)}]$$

The RTW construction ~ 1932

3. Als zweites Beispiel behandeln wir die zyklische einvalentige Kette mit sechs Atomen. Die Valenzfunktionen sind



- ▶ RTW now put the atoms on a circle
- ▶ Then RTW draw bonds as lines
- ▶ The result is TL diagrams coming from valence theory:

atoms=points and bonds=strands

The Kauffman bracket via valence bonds

aus N Strichen zwischen den n Punkten x, y, \dots, z . Wir stützen uns darauf, daß man mit Hilfe der Relation (2):

$$(3) \quad \begin{array}{c} x & z \\ \diagdown & / \\ & \diagup & \diagdown \\ y & l \end{array} = \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} + \begin{array}{c} \circ \text{---} \\ \circ \text{---} \end{array},$$

Kreuzungen auflösen kann¹⁾. Natürlich ist mit dieser Bemerkung

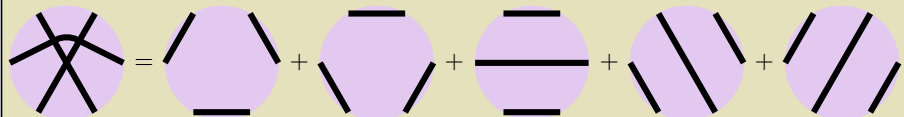
The Kauffman bracket follows easily from the RTW setting:

$$[x, l][y, z] = (x_1 l_2 - x_2 l_1)(y_1 z_2 - y_2 z_1) = [x, z][y, l] + [x, y][l, z]$$

► RTW now put the atoms on a **circle**

► Then RTW draw bonds as lines

Example



The Kauffman bracket via valence bonds

aus N Strichen zwischen den n Punkten x, y, \dots, z . Wir stützen uns darauf, daß man mit Hilfe der Relation (2):

$$(3) \quad \begin{array}{c} x & z \\ \diagdown & / \\ & \\ / & \diagdown \\ y & t \end{array} = \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \end{array} + \begin{array}{c} | \\ | \end{array},$$

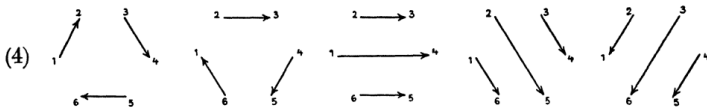
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Second fundamental theorem of invariant theory via valence bonds

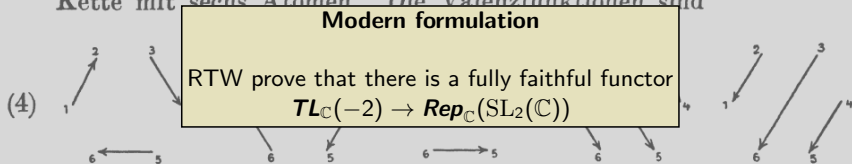
3. Als zweites Beispiel behandeln wir die zyklische einvalentige Kette mit sechs Atomen. Die Valenzfunktionen sind



RTW also prove that crossingless matching form a basis

The RTW construction ~1932

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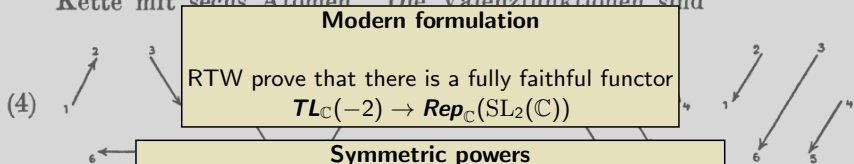


- ▶ RTW now put the atoms on a circle
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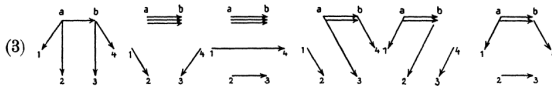
atoms=points and bonds=strands

The RTW construction ~1932

3. Als zweites Beispiel behandeln wir die zyklische einvalentige Kette mit sechs Atomen. Die Valenzfunktionen sind



Als Beispiel betrachten wir das Hydrazin NH_2-NH_2 . Wir bezeichnen mit a, b die beiden N-Atome, mit 1, 2, 3, 4 die vier H-Atome. Ordnen wir die Atome auf einem Kreis an, so erhalten wir nach der Anweisung folgende sechs Valenzzustände als Basis³⁾:



Actually it is more general:

RTW also address these questions for symmetric powers with $Sym^k \mathbb{C}^2$ corresponding to a k -valence bond

- ▶ RTW nov
- ▶ Then RTW
- ▶ The result

3. Als zweites Beispiel behandeln wir die zyklische einvalentige Kette mit sechs Atomen. Die Valenzfunktionen sind

Let's ask SAGEMath whether the RTW basis has the correct number of elements:

Type some Sage code below and press Evaluate.

```
1 A1 = WeylCharacterRing(['A', 1])
2 A1(1,0)*A1(1,0)*A1(1,0)
```

Evaluate

$2 \cdot A1(2,1) + A1(3,0)$

Type some Sage code below and press Evaluate.

```
1 A1 = WeylCharacterRing(['A', 1])
2 A1(3,0)*A1(1,0)*A1(1,0)
```

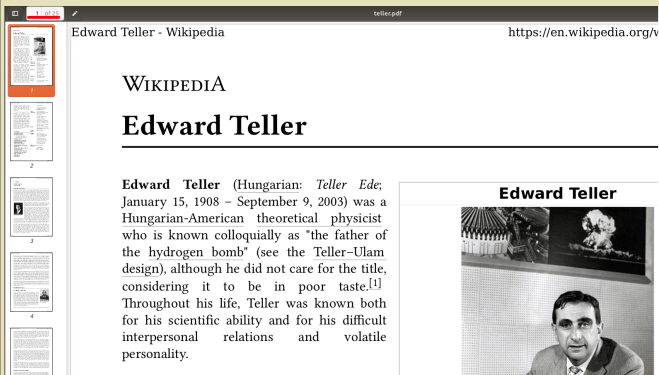
Evaluate

$A1(3,2) + 2 \cdot A1(4,1) + A1(5,0)$

Indeed, $1^2 + 2^2 = 5$ and $1^2 + 2^2 + 1^2 = 6$

3. Als zweites Beispiel behandeln wir die zyklische einvalentige Kette mit sechs Atomen. Die Valenzfunktionen sind

Edward Teller is the big name here

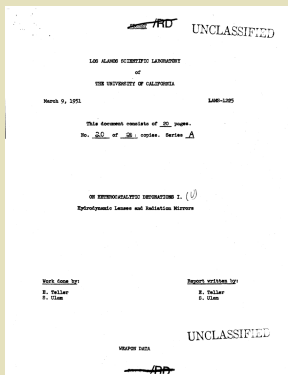


Edward Teller - Wikipedia <https://en.wikipedia.org/v>

WIKIPEDIA

Edward Teller

Edward Teller (Hungarian: *Teller Ede*; January 15, 1908 – September 9, 2003) was a Hungarian-American theoretical physicist who is known colloquially as "the father of the hydrogen bomb" (see the Teller-Ulam design), although he did not care for the title, considering it to be in poor taste.^[1] Throughout his life, Teller was known both for his scientific ability and for his difficult interpersonal relations and volatile personality.



UNCLASSIFIED

LOS ALAMOS SCIENTIFIC LABORATORY
OF
THE UNIVERSITY OF CALIFORNIA

March 9, 1951 LAM-1295

This document consists of 20 pages.
No. 20 of 20 copies. Series A

ON HYDROGENATED DEVIATIONS I. (1)
Hydrodynamic Lenses and Radiation Mirrors

Work done by:
E. Teller
S. Glas

Report written by:
E. Teller
S. Glas

UNCLASSIFIED

WEAPON DATA

Teller's Wikipedia page has 25 printed pages (01.Mar.2022); it is very readable

The Ising Model

Consider a sheet of metal:



It has the property that at low temperatures it is magnetized, but as the temperature increases, the magnetism “melts away”.*

We would like to model this behavior. We make some simplifying assumptions to do so.

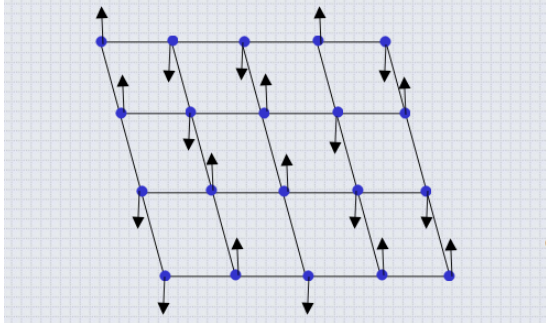
- The individual atoms have a “spin”, i.e., they act like little bar magnets, and can either point up (a spin of +1), or down (a spin of -1).
- Neighboring atoms with the same spins have an interaction energy, which we will assume is constant.
- The atoms are arranged in a regular lattice.

▶ The Ising model’s interpretation is explained above **Magnetism**

▶ The Potts model is a generalization of the Ising model

▶ TL studied these **Solid-state physics**

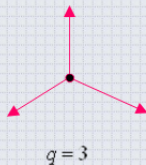
A choice of 'spin' at each lattice point.



- ▶ The Ising model is a lattice model
- ▶ The states are spins (up and down)
- ▶ Recall that what we want to know is Z_S Partition function

$q=Q!$ The Potts Model

Now let there be q possible states....

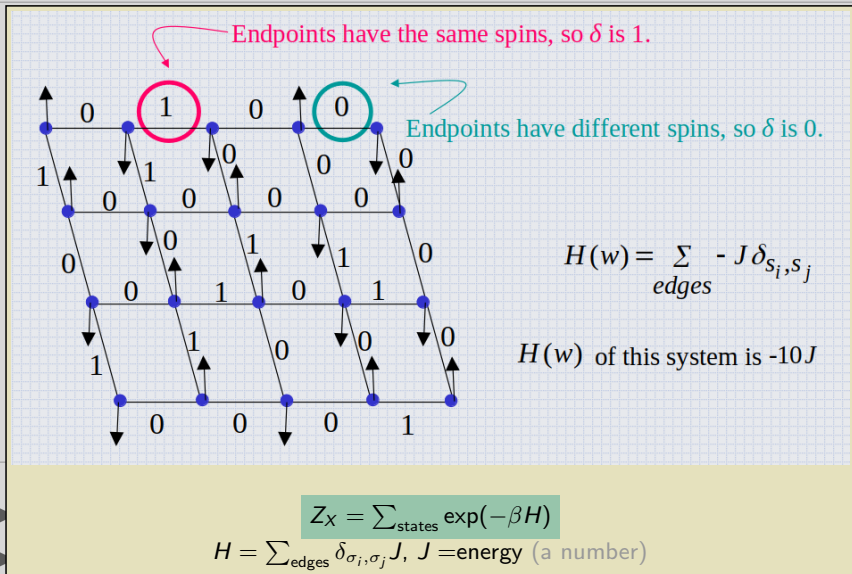


Orthogonal vectors,
with δ replaced by dot
product

Colorings of the points
with q colors

- ▶ The Potts model is a lattice model
- ▶ The states are “spins” from 1 to Q (Ising $Q = 2$)
- ▶ We want to know $Z_S = Z_S(\beta = 1/kT)$ Partition function

The TL construction ~1971



► We want to know $Z_S = Z_S(\beta = 1/kT)$ Partition function

The Potts model is very applicable:

Applications of the Potts Model

- **Liquid-gas transitions**
- **Foam behaviors**
- **Magnetism**
- **Biological Membranes**
- **Social Behavior**
- **Separation in binary alloys**
- **Spin glasses**
- **Neural Networks**
- **Flocking birds**
- **Beating heart cells**

These are all complex systems with nearest neighbor interactions.

These microscale interactions determine the macroscale behaviors of the system, in particular phase transitions.

▶ The

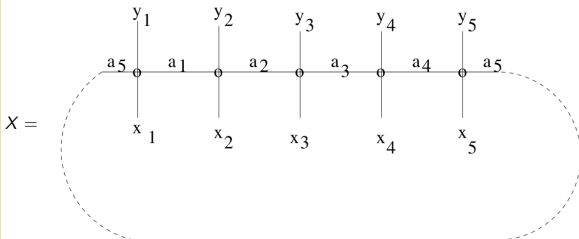
▶ The

For all of these there is some form of the TL calculus

▶ We want to know $Z_S = Z_S(\beta = 1/kT)$ Partition function

Recall from last time that solving the model “is equivalent to having good expressions for transfer matrices”

Transfer matrices



$$T^{\text{row}} = \sum_{a_i} R(a_n, a_1 | x_1, y_1) \dots R(a_{n-1}, a_n | x_n, y_n)$$

- ▶ For \mathbb{Z}^2 with periodic boundary use transfer matrix above
- ▶ **Problem** Computing the largest eigenvalue becomes infeasible

▶ The states are “spins” from 1 to Q (Ising $Q = 2$)

▶ We want to know $Z_S = Z_S(\beta = 1/kT)$ **Partition function**

The TL construction ~1971

$$\left. \begin{aligned} (1 - T_{12})[1\ 2] &= 2[1\ 2], \\ (1 - T_{23})[1\ 2][3\ 4] &= [4\ 1][2\ 3], \\ (1 - T_{23})[4\ 1][2\ 3] &= 2[4\ 1][2\ 3]. \end{aligned} \right\}$$

- ▶ $V = \mathbb{C}^Q$; the operators below are on tensor powers of V
- ▶ $p =$ multiplication by $1/\sqrt{Q}$, $d_{i,i+1}(v_i \otimes v_j) = \delta_{ij} v_i \otimes v_j$
 $Q = (A + A^{-1})^2$, e.g. $Q = 2$ implies $A = (-1)^{1/4}$
- ▶ $E_{2i-1} = 1 \otimes \dots \otimes 1 \otimes p \otimes 1 \otimes \dots \otimes 1$ (p in the i th entry)
- ▶ $E_{2i} = 1 \otimes \dots \otimes 1 \otimes d_{i,i+1} \otimes 1 \otimes \dots \otimes 1$ ($d_{i,i+1}$ in the i th entry)
- ▶ Up to scaling, $E_k = 1 - T_{k(k+1)}$ **Kauffman bracket**
- ▶ The transfer matrix with free horizontal boundary conditions is a multiple of $(\prod_{i=1}^{n-1} aE_{2i} + 1)(\prod_{i=1}^n bE_{2i-1} + 1)$ where a and b determined by the boundary condition

The operators satisfy the TL relations

$$E_k \leftrightarrow \text{cup with } k \text{ strands}$$

$$E_k E_k = \sqrt{Q} E_k \leftrightarrow \text{cup over cup} = \sqrt{Q} \text{cup}$$

$$E_k E_{k\pm 1} E_k = E_k \leftrightarrow \text{cup over cup over cup} = \text{cup}$$

$$E_k E_l = E_l E_k \text{ for } |k - l| > 1 \leftrightarrow \text{far commutativity}$$

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▶ $p = \text{multiplic}$

$$Q = (A + A^{-1})$$

▶ $E_{2i-1} = 1 \otimes$

▶ $E_{2i} = 1 \otimes \dots$

▶ Up to scalin

▶ The transfer

is a multiple of $(\prod_{i=1}^{n-1} aE_{2i} + 1)(\prod_{i=1}^n bE_{2i-1} + 1)$ where a and b determined by the boundary condition

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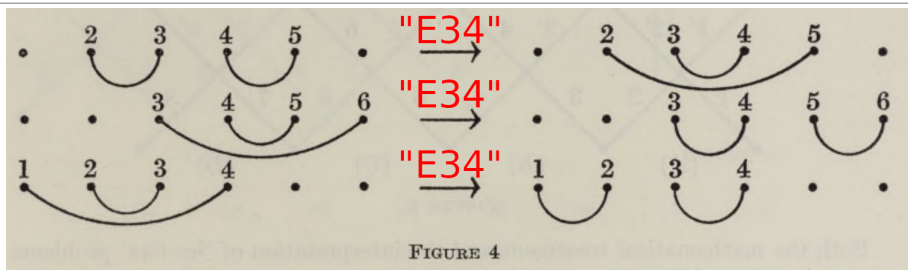
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TL then show that Z_S is determined by the TL relations

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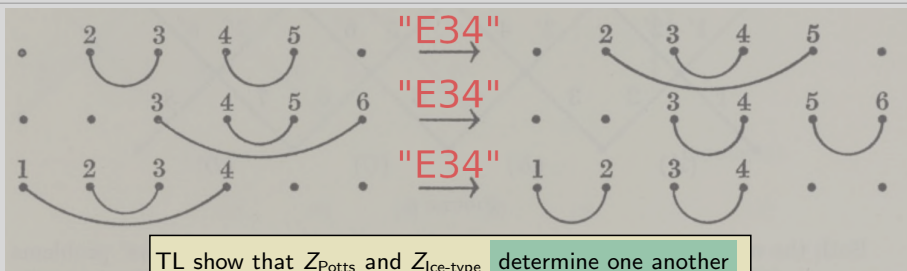
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The TL construction ~1971



- ▶ TL also write down and study cell modules
- ▶ They use the usual diagrammatics to describe these
- ▶ They did not use diagrammatics to describe $TL_C(\sqrt{Q})$ itself
- ▶ They do not compute the dimension of $TL_C(\sqrt{Q})$

The TL construction ~1971



General solution of the Potts model

$$Z_G \approx T(a, b)$$

where $T(x, y)$ is the Tutte polynomial

for $a = (|Q| + \exp(\beta J) - 1) / (\exp(\beta J) - 1)$ and $b = \exp(\beta J)$

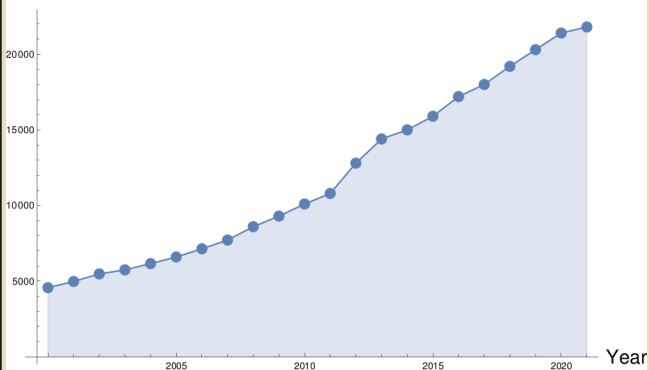
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- ▶ They do not compute the dimension of $\text{TL}_{\mathbb{C}}(\sqrt{Q})$

The TL construction ~1971

Impressive!

Articles mentioning Potts models found by google scholar 01.Mar.2022

Count



More than 250000 hits in total

This is a widely spread incarnation of the TL calculus

- ▶ TL also
- ▶ They u
- ▶ They d
- ▶ They d

Index for Subfactors

V.F.R. Jones

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA

- ▶ Factor = von Neumann algebra with trivial center
- ▶ A subfactor is an inclusion of factors $N \subset M$
- ▶ Murray–von Neumann ~1930+ classified factors by types: I , II_1 , II_∞ and III
- ▶ II_1 are the “most exciting” ones They have a unique trace!
We will stick with these (I drop the “of type II_1 ” – it should appear everywhere)

Invent. math. 72, 1–25 (1983)

The word “algebra” is highlighted because there will be representations!

*Inventiones
mathematicae*

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Example

M is a factor, G a nice group, then $M^G \subset M$ is a subfactor

Vague slogan Subfactors \leftrightarrow fixed points of a “quantum group” G action

The is also a version of Galois correspondence
and one can recover G from $M^G \subset M$

In this sense subfactor theory generalizes finite group theory

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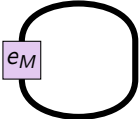
▶ A subfactor is an inclusion of factors $N \subset M$

▶ Jones' paper was one of the starting point of transferring subfactors from functional analysis to algebra/combinatorics and III

▶ II_1 are the “most exciting” ones They have a unique trace!

We will stick with these (I drop the “of type II_1 ” – it should appear everywhere)

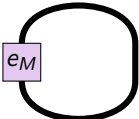
Jones' construction ~ 1983

$$[M : N] \leftrightarrow \text{tr}(e_M) \in \mathbb{R}_{\geq 0}$$


- ▶ Subfactor $N \subset M$, M is a N -module by left multiplication
- ▶ Assume that M is finitely generated projective N -module
The index $[M : N] \in \mathbb{R}_{\geq 0}$ is the trace of the idempotent e_M for M
- ▶ Jones' index theorem ~ 1983 The index is an invariant of $N \subset M$ and

$$[M : N] \in \left\{ 4 \cos^2\left(\frac{\pi}{k+2}\right) \mid k \in \mathbb{N} \right\} \cup [4, \infty]$$

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Note the “quantization” below 4:

This was a weird/exciting result!

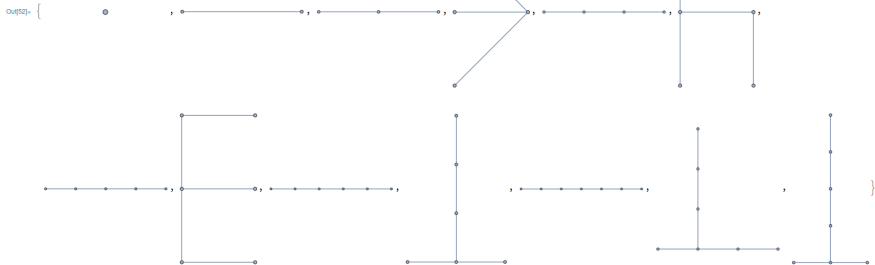
Jones: The most challenging part is constructing these subfactors

Sketch of the quantization argument

Associated a graph G to $N \subset M$ and " $[M : N] = pf(G)$ "

Then use Kronecker's theorem

```
In[51]:= findgraphs4[n_] := Select[GraphData /@ Flatten[Table[GraphData["Connected", m], {m, 1, n}], 1], Last[Sort[Eigenvalues[AdjacencyMatrix[#]], Less]] < 2 &];
findgraphs4[7]
```



§4. Possible Values of the Index

§4.1. Certain Algebras Generated by Projections

Jones' projectors satisfy the scaled TL relations for $\delta = [M : N] = 4 \cos^2\left(\frac{\pi}{k+2}\right)$

$$e_k \iff \frac{1}{[M:N]} \text{ (cup with } k \text{ strands)}$$

$$\blacktriangleright e_k e_k = e_k \iff \text{ (two cups) } = \text{ (one cup)}$$

$$\blacktriangleright e_k e_{k\pm 1} e_k = \frac{1}{[M:N]} e_k \iff \text{ (cup, crossing, cup) } = \frac{1}{[M:N]} \text{ (cup) } \cdot \text{ (cup)}$$

$$\blacktriangleright e_k e_l = e_l e_k \text{ for } |k - l| > 1 \iff \text{ far commutativity}$$

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$\{e_i | i \geq n\}$ generate a factor R_n
 $R_2 \subset R_1$ is a subfactor of index $4 \cos^2\left(\frac{\pi}{k+2}\right)$

$$\blacktriangleright e_k e_{k\pm 1} e_k = \frac{1}{[M:N]} e_k \iff \text{ (cup and cap) } = \frac{1}{[M:N]} \text{ (cup)}$$

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The Markov property!

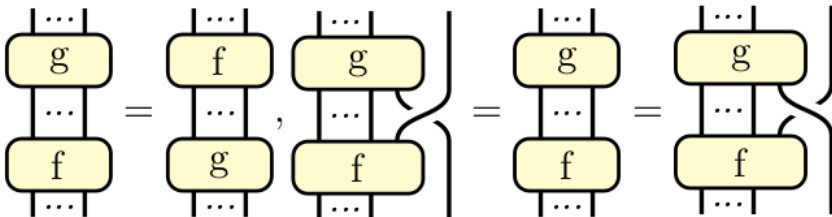
They satisfy the relations

- (I) $e_i^2 = e_i, e_i^* = e_i,$
 (II) $e_i e_{i\pm 1} e_i = t/(1+t)^2 e_i,$ **TL relations**
 (III) $e_i e_j = e_j e_i$ if $|i-j| \geq 2.$

Here t is a complex number. It has been shown by H. Wenzl [24] that an arbitrarily large family of such projections can only exist if t is either real and positive or $e^{\pm 2\pi i/k}$ for some $k = 3, 4, 5, \dots$. When t is one of these numbers, there exists such an algebra for all n possessing a trace $\text{tr}: A_n \rightarrow \mathbb{C}$ completely determined by the normalization $\text{tr}(1) = 1$ and

- (IV) $\text{tr}(ab) = \text{tr}(ba),$
 (V) $\text{tr}(we_{n+1}) = t/(1+t)^2 \text{tr}(w)$ if w is in $A_n,$
 (VI) $\text{tr}(a^*a) > 0$ if $a \neq 0$ **Markov trace**
 (note $A_0 = \mathbb{C}$).

In braid pictures (crossing is given by Kauffman skein formula)



In hindsight the crucial result Jones' proved about $TL_C(\delta)$ is the existence of a Markov trace

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In hindsight the crucial result Jones' proved about $TL_C(\delta)$ is the existence of a Markov trace

8.4 Possible Values of the Index

Why? Well, because of the (Birman-)Jones polynomial:

Jones Polynomial

Jones met Birman in 1984:

Markov trace on the braid group \Rightarrow knot invariant

It was surprising that the Markov trace naturally comes from the trace of the II_1 factor,

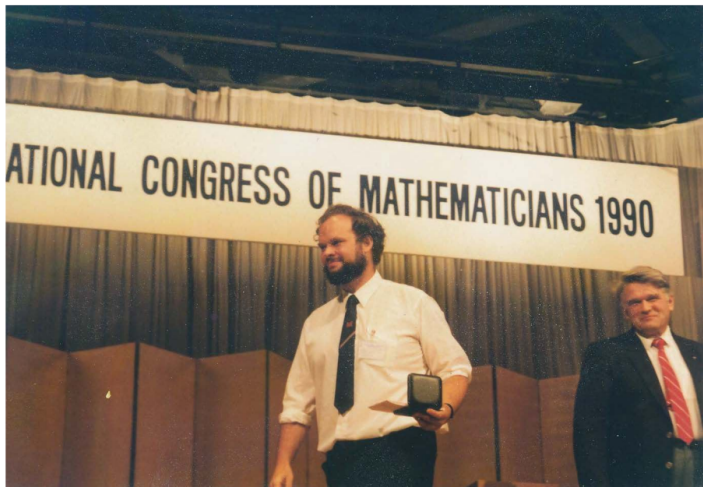
$$\tau(x\sigma_n) = \tau(x), \forall x \in TL_n, \longrightarrow \text{Reidemeister move I} = \left| \begin{array}{c} \text{crossing} \\ \text{vertical line} \end{array} \right|$$

therefore leading to a knot invariant, well-known as the Jones polynomial, by which Jones answered a series of old questions in knot theory in 1985.

$$\text{Diagram 1} = t + t^3 - t^4 \qquad \text{Diagram 2} = t^{-1} + t^{-3} - t^{-4}$$

Reflection: $t \rightarrow t^{-1}$.

Jones was awarded the Fields Medal at Kyoto in 1990 for these breakthroughs.



► $e_k e_l = e_l e_k$ for $|k - l| > 1 \iff$ far commutativity

§4. Possible Values of the Index

§4.1. Certain Algebras Generated by Projections

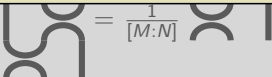
Jones also implicitly coined the name "TL algebra" ~ 1985 :

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 1, January 1985

A POLYNOMIAL INVARIANT FOR KNOTS VIA VON NEUMANN ALGEBRAS¹

BY VAUGHAN F. R. JONES²

For real t , D. Evans pointed out that an explicit representation of A_n on \mathbb{C}^{2n+2} was discovered by H. Temperley and E. Lieb [23], who used it to show the equivalence of the Potts and ice-type models of statistical mechanics. A

▶ $e_k e_{k\pm 1} e_k = \frac{1}{[M:N]} e_k \iff$  $= \frac{1}{[M:N]} \text{ (crossing) }$

▶ $e_k e_l = e_l e_k$ for $|k - l| > 1 \iff$ far commutativity

The Temperley-Lieb (TL) calculus is everywhere

The TL calculus was discovered several times, e.g.:

- Via valence bond theory **Rumer-Teller-Maxwell (RTM) –1932**
- Via the Potts model **Temperley-Lieb –1971**
- Via subfactor **Jones –1983**
- Via skein theory **Kauffman –1987**

Eine für die Valenztheorie geeignete Basis der dualen Vektorvariante.

Von
O. Rumer (Moskau), E. Teller und H. West (Stuttgart)
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What we will see today

- (2) Quantum chemistry
- (3) Statistical mechanics
- (4) Operator theory
- (5) Quantum topology

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Kauffman's construction –1987

Step 2: Make $\text{Yao} \sum(A, A^{-1})$ -linear and impose



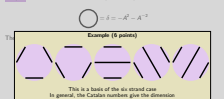
Kauffman skein relation
 averaging over ways to get rid of the crossing

Here I am faithfully reproducing a constant diagram in the literature over the meaning of the "quantum parameter" in quantum group theory $q = d^{-1}$.

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Kauffman's construction –1987

Step 3: Realize that one also needs impose to impose



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The RTM

The Kauffman bracket via valence bonds

on n P-Bindern verbindet die n Punkte $n, n-1, \dots, 1$. Wir abstrahieren davon, daß man sich 180° die Rotation (R) ersinnen darf. Einmal links, einmal rechts.

The Kauffman bracket follows easily from the RTM setting: $\langle R, R \rangle = \langle L, L \rangle = \langle R, R \rangle + \langle L, L \rangle$

RTM now put the strands on a **circle**



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The TL construction –1971

$q=Q!$ The Potts Model



- The Potts model is a lattice model
- The states are "spins" from 1 to Q (hinc $Q=2$)
- We want to know $Z_q = Z_q(\beta=1/AT)$ **Partition function**

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The TL construction –1971

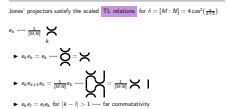


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Jones' construction –1983

14 Possible Values of the Index
 J.A. Conon Algebra Generated by Projections



Jones' construction –1983

In **highlight**, the crucial result: Jones proved about $TL_n(\delta)$ is the existence of a Markov trace

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Markov trace on the braid group \rightarrow knot invariant

It was surprising that the Markov trace naturally comes from the trace of the H_q factor.

$\tau(\sigma_{ij}) = \tau(\sigma_{ji}), \forall i, j \in \{1, \dots, n\}$ (Homomorphism more!)

therefore leading to a knot invariant, well known as the Jones polynomial, which Jones announced a series of cell questions in last theory in 1985.

Reflection: $\tau = \tau^{-1}$

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There is still much to do...

