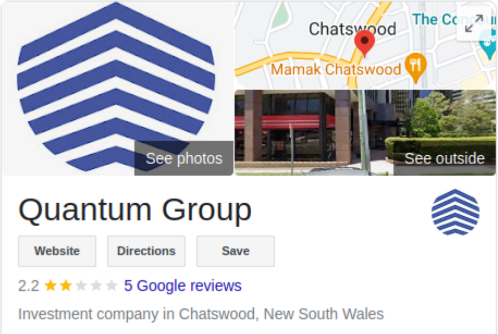
A brief, incomplete, and mostly wrong history of quantum groups

Or: From ice to R-matrices

Daniel Tubbenhauer



Hmm, what a “quantum group” is appears debatable. Nevertheless, I’ll give it a go!

Neither quantum nor group...?

Throughout

Please convince yourself that I haven't messed up while picking my quotations from my stolen material

Neither quantum nor group...?

Quantum groups arose in the 1980s from attempts to...

- ...construct solutions of the Yang–Baxter equation (YBE) Faddeev's school

More as we go!

- ...find examples of noncommutative+noncocommutative Hopf algebras via deforming $U(\mathfrak{g})$ Drinfel'd and Jimbo (in parallel)

Proceedings of the International Congress of Mathematicians
Berkeley, California, USA, 1986

Quantum Groups

V. G. DRINFEL'D

mutative) Hopf algebra. So the notions of Hopf algebra and quantum group are in fact equivalent, but the second one has some geometric flavor.

It is important that a quantum group is not a group, nor even a group object in the category of quantum spaces. This is because for noncommutative algebras the tensor product is not a coproduct in the sense of category theory.

A q-Difference Analogue of $U(\mathfrak{g})$ and the Yang–Baxter Equation

MICHIO JIMBO
Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606, Japan

Note added in Proof. After completing the manuscript, the author learned that the same algebra (3A–E) has also been introduced in the recent work of V. G. Drinfel'd (*Doklady Akad. Nauk. SSSR*, 1985). He would like to thank Prof. L. D. Faddeev for drawing his

Neither quantum nor group...?

Quantum gro

- ...constru ... deev's school

- ...find ex ... algebras via deformin

Proceedings of the In
Berkeley, California,

Qua

V.

... U(g) and the

mutative) Hopf algebra. So the
in fact equivalent, but the seco

It is important that a quantum group is not a group, nor even a group object in the category of quantum spaces. This is because for noncommutative algebras the tensor product i

...ersity, Kyoto, 606, Japan

Note added in Proof. After completing the manuscript, the author learned that the same algebra (3A–E) has also been introduced in the recent work of V. G. Drinfel'd (Doklady ...deev for drawing his

History and Perspectives of Quantum Groups

L. D. Faddeev

The combination of terms "Quantum Group" was introduced by V. Drinfeld 20 years ago and appeared in his invited talk at the ICM 1986 in Berkeley[1] [11]. While universally adopted now, it was considered as a misnomer by many purists. Indeed the object in question is neither a group nor does it belongs to quantum theory. However Drinfeld used the term "quantization" as a synonym of "deformation", referring to the recent realization that the algebra of observables of a quantum mechanical system is a noncommutative deformation of the corresponding classical algebra of functions on symplectic phase space, see e.g. [4].

It is remarkable that a corresponding procedure was inspired by mathematical physics, more exactly by the theory of quantum integrable models [18] and exactly soluble models of statistical physics [3]. Altogether this history is an instructive example of the interinfluence of mathematics and mathematical physics.

Today

I will motivate the story from the perspective of statistical mechanics

Neither quantum nor group...?

Quantum

▶ ...co ... s school

▶ ...fi ... ras via defo

Proceed
Berkeley

and the

mutative) Hopf algel
in fact equivalent, bu
It is important tha
in the category of qua
the tensor product is

. 606, Japan
learned that the same
G. Drinfel'd (*Doklady*
ddeev for drawing his

Since their introduction quantum groups have appear "everywhere":

The theory of quantum groups developed in mid 1980s (Faddeev's school, Drinfeld, Jimbo) from attempts to construct and understand solutions of the quantum Yang-Baxter equation arising in quantum field theory and statistical mechanics. Since then, it's grown into a vast subject with deep links to many areas:

- representation theory
- the Langlands program
- low-dimensional topology
- category theory
- enumerative geometry
- quantum computation
- algebraic combinatorics
- conformal field theory
- integrable systems
- integrable probability

This slide is shamelessly stolen from:

A brief introduction to quantum groups

Pavel Etingof

MIT

May 5, 2020

but I won't go into this (my apologies)

Today's talk is based on:

my memory (horrible reference...)

In and around the origin of quantum groups.

Vaughan F.R. Jones *

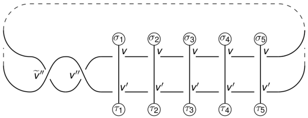
26th October 2018

Lecture 6

Ask google: "Lectures on quantum groups Bump"

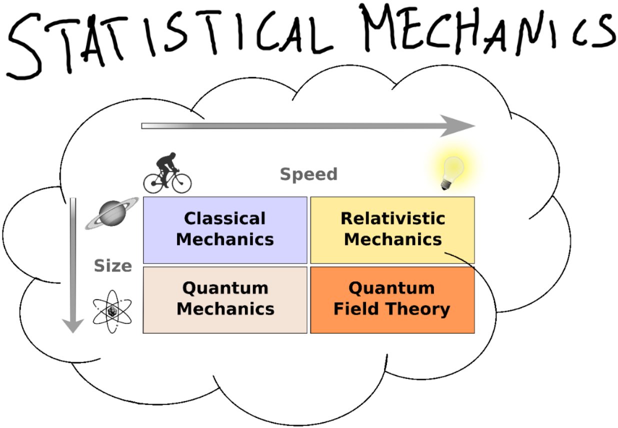
Daniel Bump

May 28, 2019



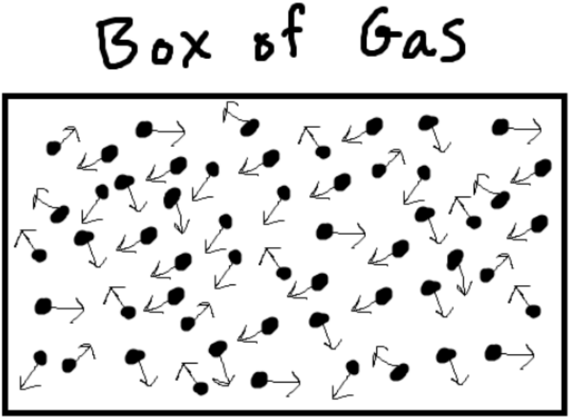
The above are easy to google (its worth it!)

Statistical mechanics in a nutshell



- Statistical mechanics is a branch of physics that pervades all other branches
- Its exact incarnation is different in each quadrant, but the basics are identical
- Instead of microstates σ describe a set Ω of microstates, the macrostates

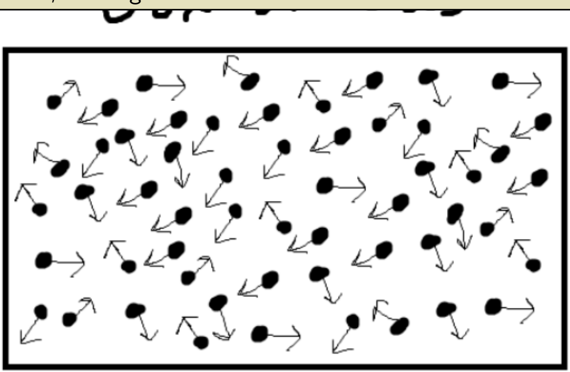
Statistical mechanics in a nutshell



- Example Describe the behavior of gas molecules globally
- The point is to model a system at hand
- The models we will use are lattice models

Statistical mechanics in a nutshell

I will now motivate the whole story of statistical mechanics in general, although we will not use some of the involved notions

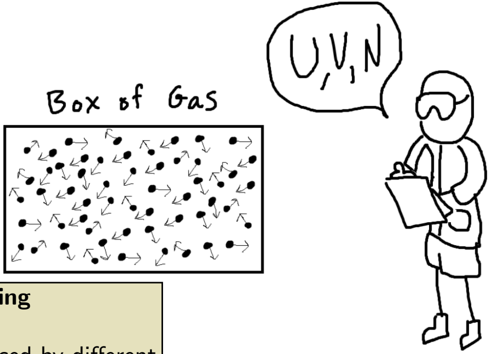


- Example Describe the behavior of gas molecules globally
- The point is to model a system at hand
- The models we will use are lattice models

Statistical mechanics in a nutshell

too complicated: $(x_1, y_1, z_1, p_{x1}, p_{y1}, p_{z1}, \dots x_N, y_N, z_N, p_{xN}, p_{yN}, p_{zN}) \in \mathbb{R}^{6N}$

good:



Warning

V and N are replaced by different quantities in different situations

- Position+momentum of N particles are 6N values Often not doable
- Macroscopic we have fewer parameters needed to be solved
- Example For a box filled with gas we have U (energy), V (volume) and N

Statistical mechanics in a nutshell

T=Temperature , k=Boltzmann constant (for normalizing units)
$\exp(-E_\sigma/kT)$ = Boltzmann weight

- System S, macrostate set Ω of microstates σ
- An energy E_σ is assigned to $\sigma \in \Omega$ according to the model Fixed numbers
- Partition function $Z_S(T) = Z_S = \sum_{\sigma\in\Omega} \exp(-E_\sigma/kT) = \sum_{\sigma\in\Omega} \exp(-\beta E_\sigma)$

Enter, the partition function

Z_S has many amazing properties. For one, it can be used to write an endless number of clever identities

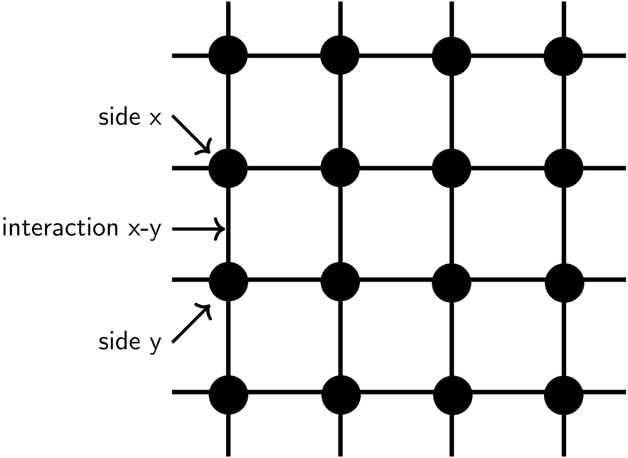
Example Expected energy $\langle E \rangle = U = -\partial/\partial\beta \log Z_S$

Example Entropy $S_S = k(1 - \beta\partial/\partial\beta) \log Z_S$

Solving the model = finding a good expression of Z_S

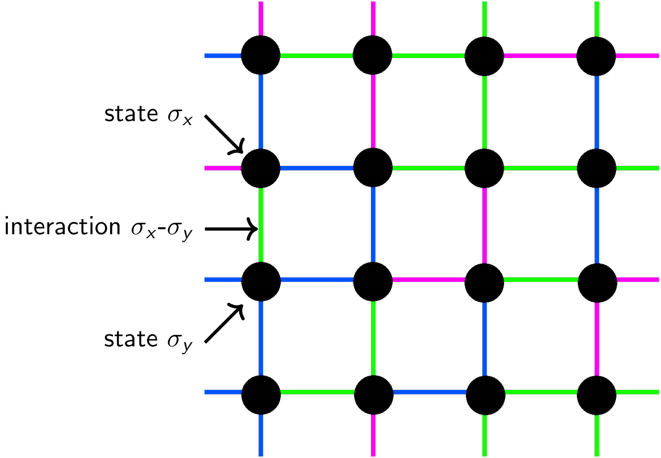
- System S, macrostate set Ω of microstates σ
- An energy E_σ is assigned to $\sigma \in \Omega$ according to the model Fixed numbers
- Partition function $Z_S(T) = Z_S = \sum_{\sigma\in\Omega} \exp(-E_\sigma/kT) = \sum_{\sigma\in\Omega} \exp(-\beta E_\sigma)$

Statistical mechanics in a nutshell

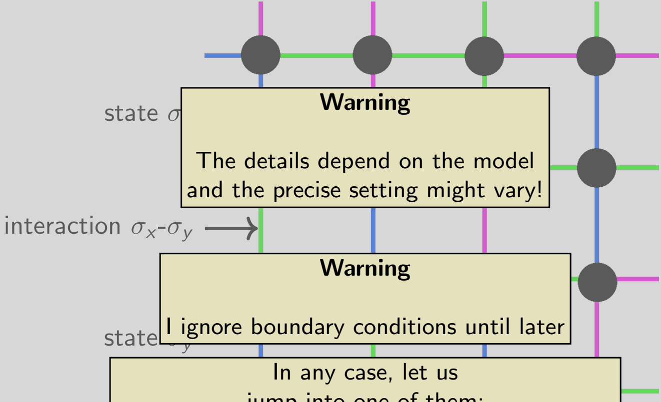


- Often models involve collections of locally interacting sites on some lattice
- The partition function Z_X makes sense for a finite subset X of the lattice
- Then consider an increasing family of subsets whose union is the whole system

Statistical mechanics in a nutshell



- In these models often state="colors on edges" for X with N sides
- Energy of σ is $\prod_{\text{edges}} E(\sigma_x, \sigma_y, \text{color}) \Rightarrow$ Local partition function Z_X
- **Goal** Find a good expression of $\lim_{X \to S} Z_X$



Warning

The details depend on the model and the precise setting might vary!

Warning

I ignore boundary conditions until later

In any case, let us
jump into one of them:
the vertex model (proposed by Pauling ~1935)

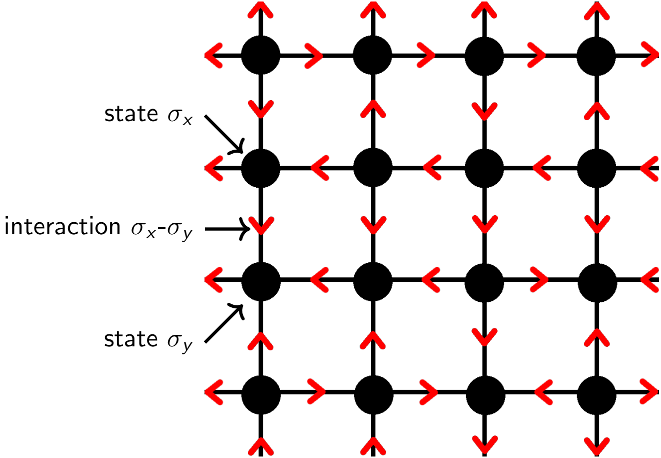
[Contribution from the Gates Chemical Laboratory, California Institute of Technology, No. 506]

The Structure and Entropy of Ice and of Other Crystals with Some Randomness of Atomic Arrangement

By Linus Pauling

- In these mod … W sides
- Energy of σ i … $\prod_{\text{edges}} E(\sigma_x, \sigma_y, \text{color})$ … ction Z_X
- Goal Find a good expression of $\lim_{X \to S} Z_X$

The ice-type vertex model

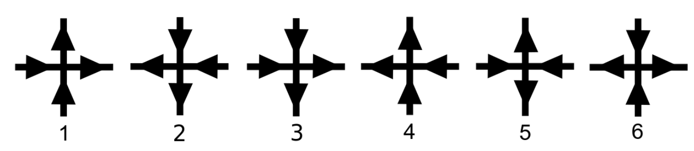


- Ice-type = each edge gets an orientation
- Makes sense for any lattice but lets restrict to the square lattice $\mathbb{Z}^2$
- Want to compute Z_X

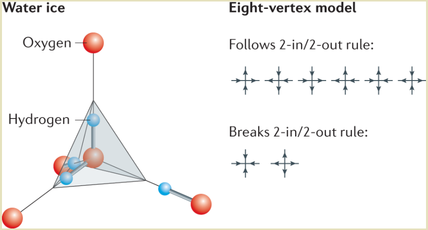
The

Warning

Strictly speaking there are a few variants
like a six-vertex model, allowing only six local orientation configurations, e.g.

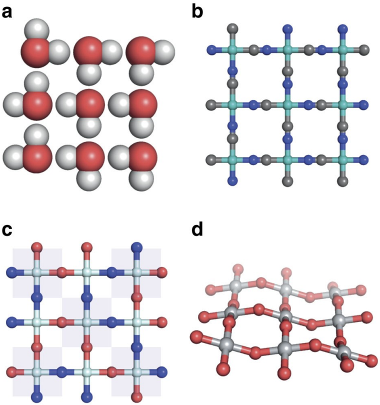


We will come back to this 6-vertex version later



- ▶
- Makes sense for any lattice but lets restrict to the square lattice $\mathbb{Z}^2$
- Want to compute Z_X

The ice-type vertex model



- This models ice lattices and other real crystals with hydrogen bonds
- Lieb found an exact solution for $\mathbb{Z}^2$ $\sim$1967; $\mathbb{Z}^3$ is still open

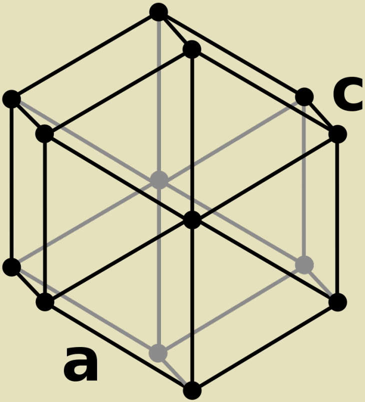

Residual Entropy of Square Ice

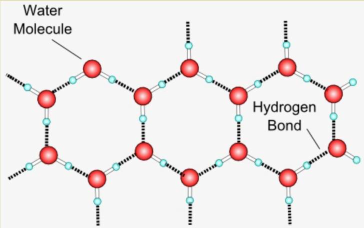
Elliott H. Lieb
Phys. Rev. **162**, 162 – Published 5 October 1967

Warning

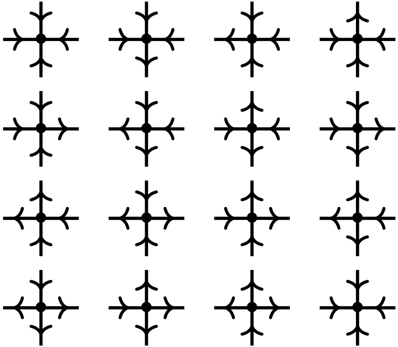
Actually hexagonal lattices would be better
but they are harder so lets ignore them

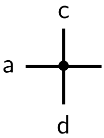
Most common ice lattice:





The ice-type vertex model

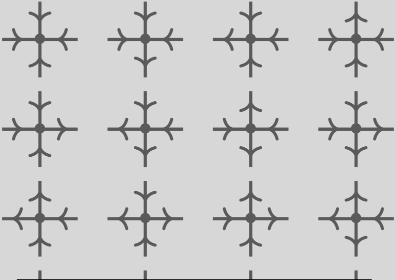


$R(a,b|c,d) = \exp(-\beta\varepsilon_{a,b}^{c,d}) \leftrightsquigarrow$ 

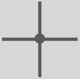
for $a,b,c,d \in \{\uparrow,\downarrow\}$

- There a sixteen local configurations
- $Z_X = \sum_{\text{states}} \prod_{\text{vertices}} R(a,b|c,d)$

The ice-type vertex model



The tool we need to compute Z_X are transfer matrices

$R(a,b|c,d) = \exp(-\beta\varepsilon_{a,b}^{c,d}) \leftrightsquigarrow$ a  b for $a,b,c,d \in \{\uparrow,\downarrow\}$

- There a sixteen local configurations
- $Z_X = \sum_{\text{states}} \prod_{\text{vertices}} R(a,b|c,d)$

Transfer matrices

$X =$ 

$$Z_X = \sum_{a,b,c,d} R(a,b)R(b,c)R(c,d)$$

$$R = \begin{pmatrix} 0 & R(a,b) & 0 & 0 \\ R(a,b) & 0 & R(b,c) & 0 \\ 0 & R(b,c) & 0 & R(c,d) \\ 0 & 0 & R(c,d) & 0 \end{pmatrix}, \quad T = \sum_{a,b,c,d} R$$

$a - d$ entry of R^3 is $R(a,b)R(b,c)R(c,d)$

- Toy-model: ice-type for $\mathbb{Z}$ with fixed boundary
- Slogan Summation over indices in Z_X becomes the summation over indices in matrix multiplication
- The boundary entry of R^N encodes the partition function

Transfer matrices

$X =$ 

$$Z_X = \sum_{a,b,c,d} R(a,b)R(b,c)R(c,d)$$

Upshot

Use linear algebra to understand Z_X
and study the asymptotic behavior of R^N
Keyword: Largest eigenvalue

$R = \Big(0 \ldots ; R(a \ldots ; 0 \ldots ; 0 \quad 0 \quad R(c,d) \quad 0 \Big) \quad \ldots T = \sum_{a,b,c,d} R$

$$a - d \text{ entry of } R^3 \text{ is } R(a,b)R(b,c)R(c,d)$$

- Toy-model: ice-type for $\mathbb{Z}$ with fixed boundary
- Slogan Summation over indices in Z_X becomes the summation over indices in matrix multiplication
- The boundary entry of R^N encodes the partition function

Transfer matrices

Perron–Frobenius theorem (Perron ∼1907, Frobenius ∼1912)

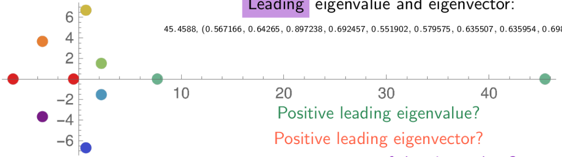
(Note quite relevant for the vertex model, but still cool)

$$\begin{pmatrix} 3 & 0 & 5 & 1 & 8 & 7 & 0 & 1 & 4 & 7 \\ 4 & 8 & 0 & 6 & 3 & 4 & 2 & 6 & 8 & 3 \\ 8 & 6 & 6 & 7 & 6 & 0 & 9 & 4 & 8 & 5 \\ 3 & 7 & 7 & 1 & 5 & 6 & 4 & 1 & 7 & 4 \\ 4 & 0 & 3 & 4 & 4 & 8 & 8 & 1 & 4 & 2 \\ 0 & 3 & 7 & 3 & 2 & 4 & 2 & 2 & 3 & 8 \\ 6 & 3 & 6 & 1 & 5 & 6 & 1 & 6 & 4 & 4 \\ 2 & 4 & 0 & 2 & 8 & 8 & 1 & 4 & 8 & 6 \\ 6 & 7 & 6 & 3 & 4 & 2 & 9 & 6 & 5 & 0 \\ 0 & 6 & 9 & 9 & 8 & 3 & 9 & 9 & 1 & 9 \end{pmatrix}$$

What on earth is going on? Strange patterns with the eigenvalues and vectors:

Leading eigenvalue and eigenvector:

45.4588, (0.567166, 0.64265, 0.897238, 0.692457, 0.551902, 0.579575, 0.635507, 0.635954, 0.698596, 1)



Positive leading eigenvalue?

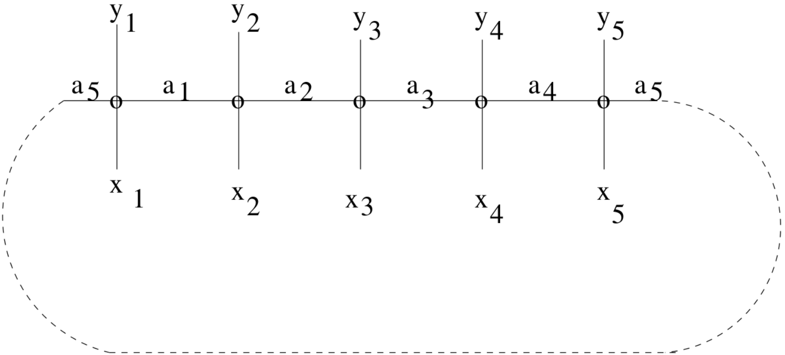
Positive leading eigenvector?

Irreducible matrices with entries from $\mathbb{R}_{\geq 0}$
have an eigenvalue $pf \in \mathbb{R}_{\geq 0}$ and
an associated eigenvector $\vec{pf} \in \mathbb{R}^n_{\geq 0}$
The growth rate of R^N is roughly given by pf^N

R R

- Toy-m
- Sloga ... er indices in matrix
- The boundary entry of R^N encodes the partition function

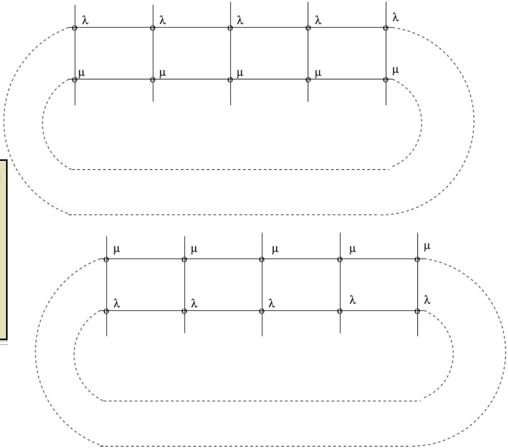
Transfer matrices

$X =$ 

$$T^{\mathrm{row}} = \sum_{a_i} R(a_n, a_1 | x_1, y_1) ... R(a_{n-1}, a_n | x_n, y_n)$$

- For $\mathbb{Z}^2$ with periodic boundary use transfer matrix above
- Problem Computing the largest eigenvalue becomes infeasible

Commuting transfer matrices

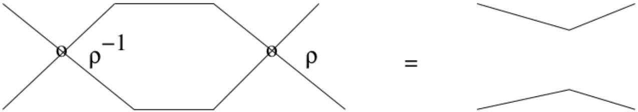


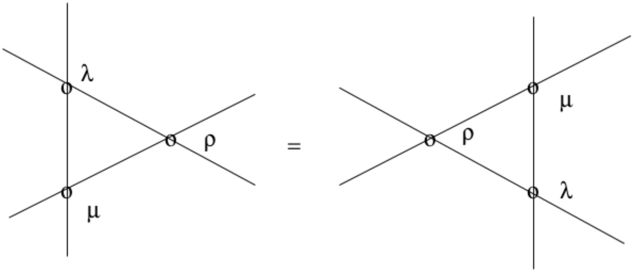
Shorthand notation

λ, μ are the spectral parameters of the transfer matrices (largest eigenvalues)

- Idea (Baxter ∼1975) Use only commuting transfer matrices
- Upshot Have a common eigenvector and, using it, are relatively easy to study

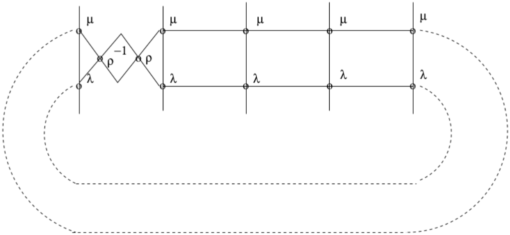
Commuting transfer matrices

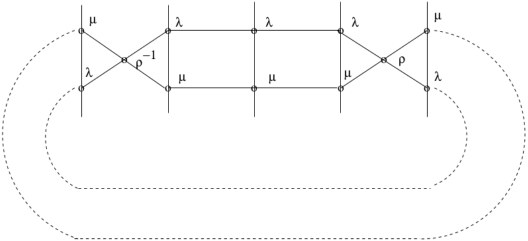




- First Invertibility
- Second Yang–Baxter equation (YBE)

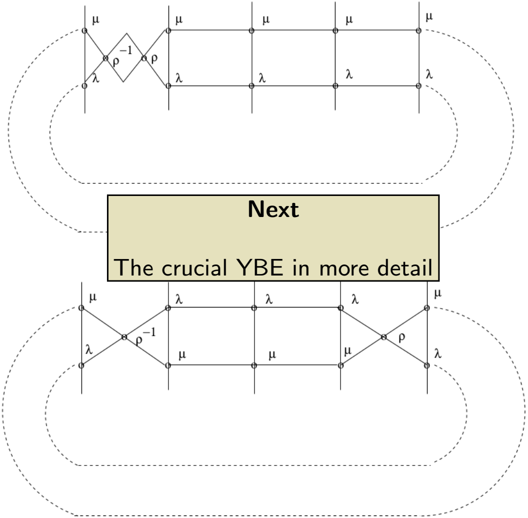
Commuting transfer matrices





These two relations ensure commutativity , see above

Commuting transfer matrices

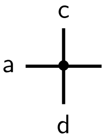


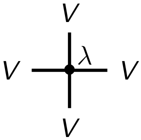
Next

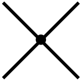
The crucial YBE in more detail

These two relations ensure commutativity, see above

Yang–Baxter equation

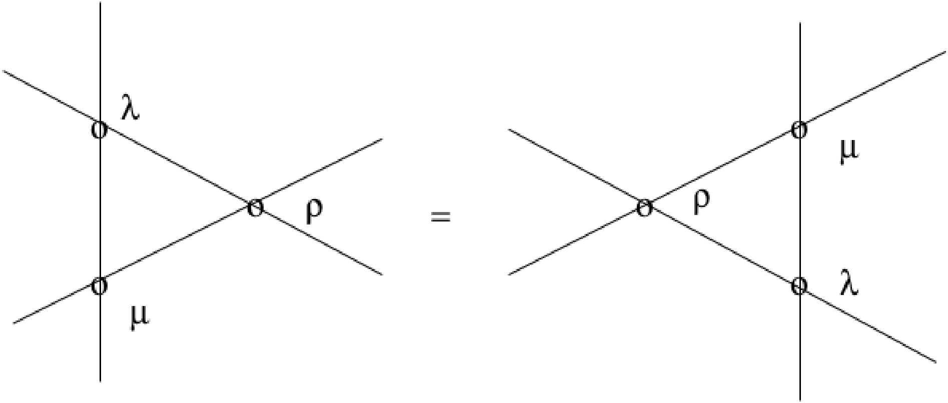
$R(a,b|c,d) = \exp(-\beta\varepsilon_{a,b}^{c,d}) \leftrightsquigarrow$  for $a,b,c,d \in \{\uparrow,\downarrow\}$

$R \leftrightsquigarrow$ 

nowadays more often drawn as 

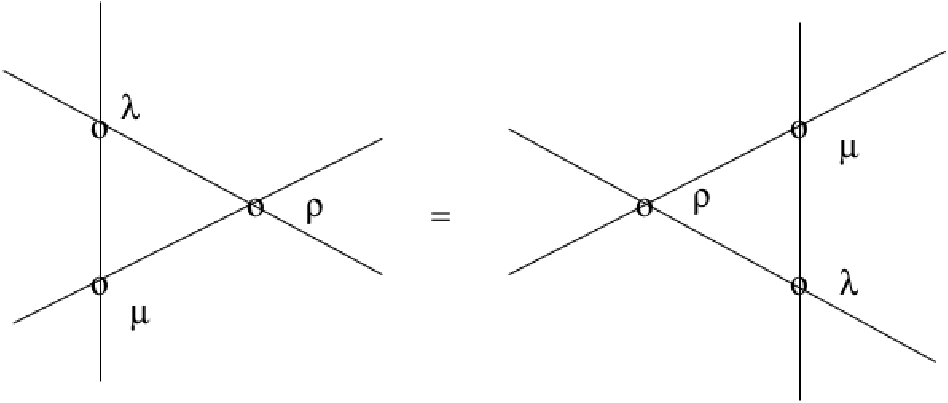
- The R-matrix can be interpreted as a map $V \otimes V \to V \otimes V$
- $V =$ vector space with basis $\{\uparrow,\downarrow\}$

Yang–Baxter equation



The YBE can be interpreted as an equation of maps $V \otimes V \otimes V \to V \otimes V \otimes V$

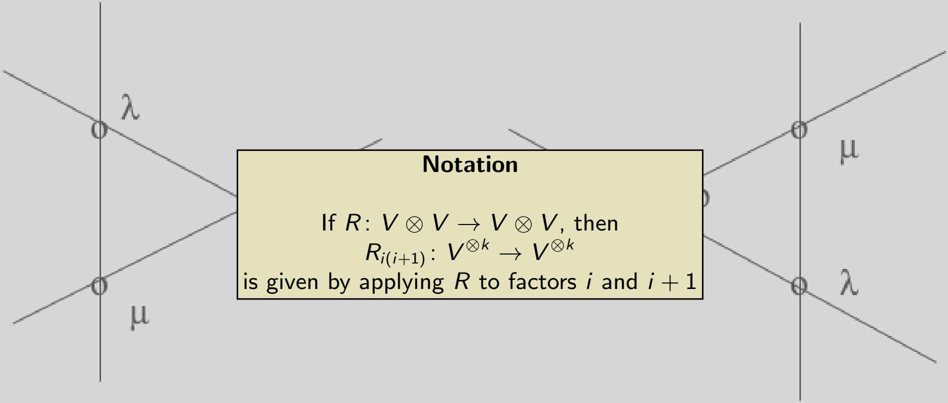
Yang–Baxter equation



The YBE reads

$$R_{12}(\lambda)R_{23}(\rho)R_{12}(\mu) = R_{23}(\mu)R_{12}(\rho)R_{23}(\lambda)$$

where $R(x)$ = transfer matrix for x



Notation

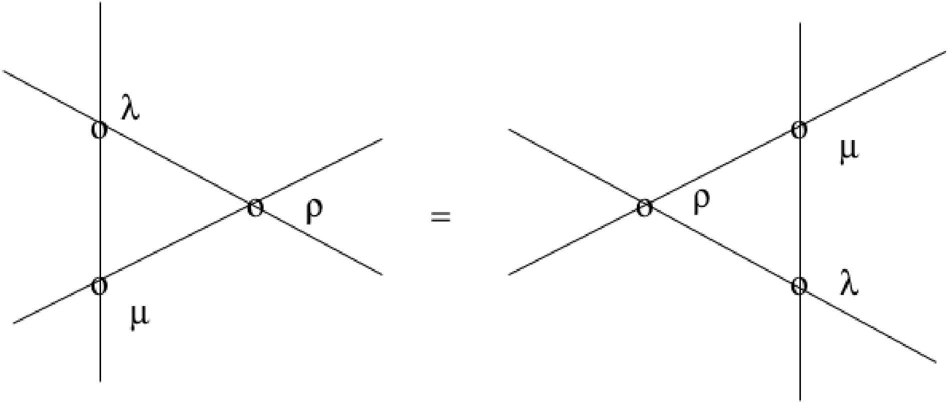
If $R\colon V\otimes V\to V\otimes V$, then
$R_{i(i+1)}\colon V^{\otimes k}\to V^{\otimes k}$
is given by applying R to factors i and $i+1$

The YBE reads

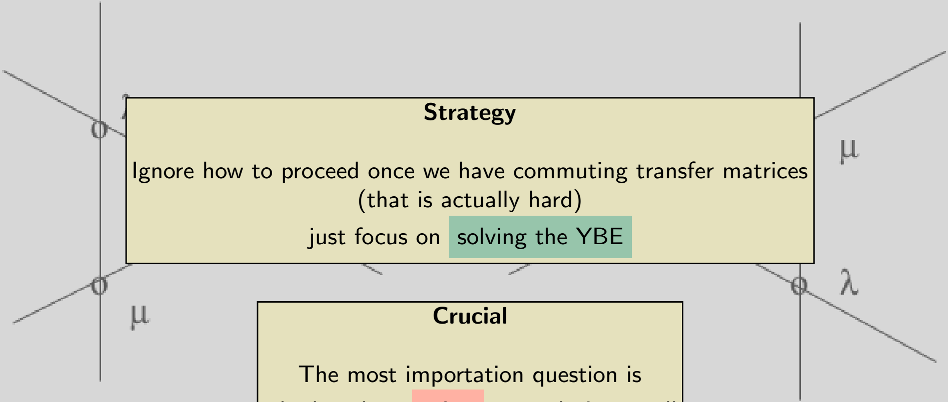
$$R_{12}(\lambda)R_{23}(\rho)R_{12}(\mu)=R_{23}(\mu)R_{12}(\rho)R_{23}(\lambda)$$

where $R(x)$ = transfer matrix for x

Yang–Baxter equation



- The equations represented by the diagram form a huge system of highly non-linear equations
- The YBE involves only three vertices



Strategy

Ignore how to proceed once we have commuting transfer matrices (that is actually hard)
just focus on solving the YBE

Crucial

The most importation question is whether there exists any solution at all

- The equations represented by the diagram form a huge system of highly non-linear equations
- The YBE involves only three vertices

Solving YBE (for the six vertex model)

$$R(z) = \frac{1}{zq - z^{-1}q^{-1}} \begin{pmatrix} zq^{-1} - z^{-1}q & 0 & 0 & 0 \\ 0 & z^{-1}(q^{-1} - q) & z - z^{-1} & 0 \\ 0 & z - z^{-1} & z(q^{-1} - q) & 0 \\ 0 & 0 & 0 & zq^{-1} - z^{-1}q \end{pmatrix}$$

$$z = \exp(-\lambda), q = \exp(\theta)$$

$$R = \lim_{z \to 0} R(z)$$

$$R = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & q^2 - 1 & q & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & q^2 \end{pmatrix}$$

- The above is a solution for the YBE
- The matrix R is what one often sees in quantum groups $\lambda = \infty$ limit
- I will now explain where this solution comes from

Solving YBE (for the six vertex model)

$$R(z) = \frac{1}{zq - z^{-1}q^{-1}} \begin{pmatrix} zq^{-1} - z^{-1}q & 0 & 0 & 0 \\ 0 & z^{-1}(q^{-1} - q) & z - z^{-1} & 0 \\ 0 & z - z^{-1} & z(q^{-1} - q) & 0 \\ 0 & 0 & 0 & zq^{-1} - z^{-1}q \end{pmatrix}$$

Note that R is not a
solution to the YBE in general (since $\lambda = \infty$)
but only satisfies the braid relations:

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$$

With general parameters we want $R_{12}(z)R_{23}(yz)R_{12}(y) = R_{23}(y)R_{12}(yz)R_{23}(z)$

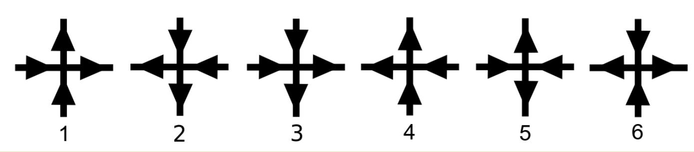
$$\begin{pmatrix} 0 & 0 & 0 & q^2 \end{pmatrix}$$

- The above is a solution for the YBE
- The matrix R is what one often sees in quantum groups $\lambda = \infty$ limit
- I will now explain where this solution comes from

Solving YBE (for the six vertex model)

$R(z) =$ … $\begin{pmatrix} zq^{-1} - z^{-1}q & 0 & 0 & 0 \\ \ldots & & & 0 \\ \ldots & & & 0 \\ \ldots & & & \ldots - z^{-1}q \end{pmatrix}$

$R(z)$ is a solution for the six vertex model



	$\uparrow\uparrow$	$\uparrow\downarrow$	$\downarrow\uparrow$	$\downarrow\downarrow$
$\uparrow\uparrow$	$zq^{-1} - z^{-1}q$	0	0	0
$\uparrow\downarrow$	0	$z^{-1}(q^{-1} - q)$	$z - z^{-1}$	0
$\downarrow\uparrow$	0	$z - z^{-1}$	$z(q^{-1} - q)$	0
$\downarrow\downarrow$	0	0	0	$zq^{-1} - z^{-1}q$

- The …
- The … quantum groups … limit
- I will now explain where this solution comes from

How to discover quantum $\mathfrak{sl}_2$

- The standard module $V = L(1)$ of $\mathfrak{sl}_2 = \langle E, F, H\rangle$ has basis vectors $\uparrow, \downarrow$
- With respect to this basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- $\mathfrak{sl}_2$ has one simple module $L(n)$ per $n \in \mathbb{N}$
- For $n = 4$ this module is

Before you ask: my ground field is $\mathbb{C}$

$$E \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, F \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, H \mapsto \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

- The category of finite dimensional $\mathfrak{sl}_2$-modules is semisimple

How to discover quantum $\mathfrak{sl}_2$

- The standard module $V = L(1)$ of $U_q(\mathfrak{sl}_2) = \langle E, F, K \rangle$ has basis vectors $\uparrow, \downarrow$
- With respect to this basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, K = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \leftrightsquigarrow q^H$$

- $U_q(\mathfrak{sl}_2)$ has one simple module $L(n)$ per $n \in \mathbb{N}$
- For $n = 4$ this module is

Before you ask: my ground field is $\mathbb{C}(q)$ for q a formal variable

$$E \mapsto \begin{pmatrix} 0 & [1] & 0 & 0 \\ 0 & 0 & [2] & 0 \\ 0 & 0 & 0 & [3] \\ 0 & 0 & 0 & 0 \end{pmatrix}, F \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ [3] & 0 & 0 & 0 \\ 0 & [2] & 0 & 0 \\ 0 & 0 & [1] & 0 \end{pmatrix}, K \mapsto \begin{pmatrix} q^3 & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & q^{-3} \end{pmatrix}$$

- The category of finite dimensional $U_q(\mathfrak{sl}_2)$-modules is semisimple

$$[n] = q^{n-1} + q^{n-3} + \dots + q^{-n+3} + q^{-n+1}$$

How to discover quantum $\mathfrak{sl}_2$

- $V \otimes V = \mathbb{C}(q)\{\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow\}$ (taken in this order); with respect to this basis

$$E \mapsto \begin{pmatrix} 0 & 1 & q & 0 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, F \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{-1} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & q & 0 \end{pmatrix}, K \mapsto \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-2} \end{pmatrix}$$

- The six vertex model gives

$$R = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$$

- Since it should commute with E and F it (up to scalars) has to be of the form

$$R = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & q^2 - q^3 b & q^2 b & 0 \\ 0 & q^2 b & q^2 - qb & 0 \\ 0 & 0 & 0 & q^2 \end{pmatrix}$$

- $b = q^{-1}$ is the solution from before! (Not quite – I messed up conventions along the way!)

▶ V … is basis

Stop!

You used the coproduct here:
$\Delta(K) = K \otimes K$, $\Delta(E) = E \otimes K + 1 \otimes E$ and $\Delta(F) = F \otimes 1 + K^{-1} \otimes F$

Where does that come from?

▶ The six vertex model gives

$$R = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \end{pmatrix}$$

▶ S … e form

I do not know!

Drinfel'd and Jimbo might say: "It makes $U_q(\mathfrak{sl}_2)$ a Hopf algebra and there are not many options to achieve that"

But that is an "in hindsight" interpretation, *e.g.* Faddeev writes:

[2] I can not help mentioning, that during my Les Houches lectures in 1982 Tierrie-Mieg, who was among the students, told me that the object I speak about is called Hopf algebra, but I unforgivenly neglected his comment.

Faddeev's school invented $U_q(\mathfrak{sl}_2)$ without knowing Hopf algebras!

▶ $b = q$ … is the solution from before! (Not quite – I messed up conventions along the way!)

How to discover quantum $\mathfrak{sl}_2$

asis

The comultiplication is the point!

Representations of SL_2

With a caveat, the following representation theories are the same:

- Finite-dimensional representations of $SL(2, \mathbb{R})$;
- Finite-dimensional representations of $SU(2)$;
- Finite-dimensional analytic representations of $SL(2, \mathbb{C})$;
- Finite-dimensional complex representations of their Lie algebras $\mathfrak{sl}_2(\mathbb{R})$, $\mathfrak{su}_2$, $\mathfrak{sl}_2(\mathbb{C})$;
- The enveloping algebras of these Lie algebras;
- The quantized enveloping algebra $U_q(\mathfrak{sl}_2)$, where q is not a root of unity.

But they are not equivalent as monoidal categories!

orm

ns

along the way!)

How to discover quantum $\mathfrak{sl}_2$

- The $U_q(\mathfrak{sl}_2)$ R-matrix is:

$$R = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & q^2-1 & q & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & q^2 \end{pmatrix}$$

- The R-matrix from before was

$$R(z) = \frac{1}{zq - z^{-1}q^{-1}} \begin{pmatrix} zq^{-1} - z^{-1}q & 0 & 0 & 0 \\ 0 & z^{-1}(q^{-1}-q) & z - z^{-1} & 0 \\ 0 & z - z^{-1} & z(q^{-1}-q) & 0 \\ 0 & 0 & 0 & zq^{-1} - z^{-1}q \end{pmatrix}$$

- The spectral parameter is still missing!

How to discover quantum $\mathfrak{sl}_2$

- The $U_q(\mathfrak{sl}_2)$ R-matrix is:

Idea

In order to have a spectral parameter
we need a quantum group with one two-dimensional module
L_z for $z \in \mathbb{C} \setminus \{0\}$

- The R-matrix from before was

$$R(z) = \frac{1}{zq - z^{-1}q^{-1}} \begin{pmatrix} zq^{-1} - z^{-1}q & 0 & 0 & 0 \\ 0 & z^{-1}(q^{-1} - q) & z - z^{-1} & 0 \\ 0 & z - z^{-1} & z(q^{-1} - q) & 0 \\ 0 & 0 & 0 & zq^{-1} - z^{-1}q \end{pmatrix}$$

- The spectral parameter is still missing!

- The $U_q(\mathfrak{sl}_2)$ R-matrix is:

Idea

In order to have a spectral parameter
we need a quantum group with one two-dimensional module
L_z for $z \in \mathbb{C} \setminus \{0\}$

- The R-matrix from before was

$R(z) = \frac{\ }{zq}$ ... 0 0 0 $-z^{-1}q$

The affine quantum Lie algebra $U_q(\hat{\mathfrak{sl}}_2)$ almost does the job!

$U_q(\hat{\mathfrak{sl}}_2)$ can be constructed as a central extension of $\mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}]$
and $X \otimes t^k$ acts on $v \in V$ as $z^k X . v$

$U'_q(\hat{\mathfrak{sl}}_2)$ = subalgebra supported on classical weights
The R-matrix is then in $\mathrm{End}_{U'_q(\hat{\mathfrak{sl}}_2)}(L_z \otimes L_z)$

- The

- The $U_q(\mathfrak{sl}_2)$ R-matrix is:

After some nontrivial algebra:

$$Z_S \sim \mathrm{ch} L_{\Lambda_0}$$

$$0 \overset{\infty}{\text{———}} 1$$

- The R-matrix fro

$L_{\Lambda_0} \leftrightsquigarrow$ vector representation of $U_q(\hat{\mathfrak{sl}}_2)$

$$R(z) = \frac{1}{zq - z^{-1}q^{-1}} \left(\cdots \; z^{-1} \; \cdots \; -q) \; \cdots \; 0 \quad 0 \; 0 \; 0 \; zq^{-1} - z^{-1}q \right)$$

$$Z_S \sim \mathrm{ch} L_{\Lambda_0} \sim \sum_{a \in 2\mathbb{Z}} \frac{q^{a^2/2}}{\prod_{n=1}^{\infty}(1-q^{2n})}$$

q is a function in T

This was the first instance of "Characters of simple modules of quantum affine Lie algebras turn up everywhere"

- The spectral param

How to discover quantum $\mathfrak{sl}_2$

- The $U_q(\mathfrak{sl}_2)$ R-matrix is:

$$R = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & q^2 - 1 & q & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & q^2 \end{pmatrix}$$

- The R-matrix

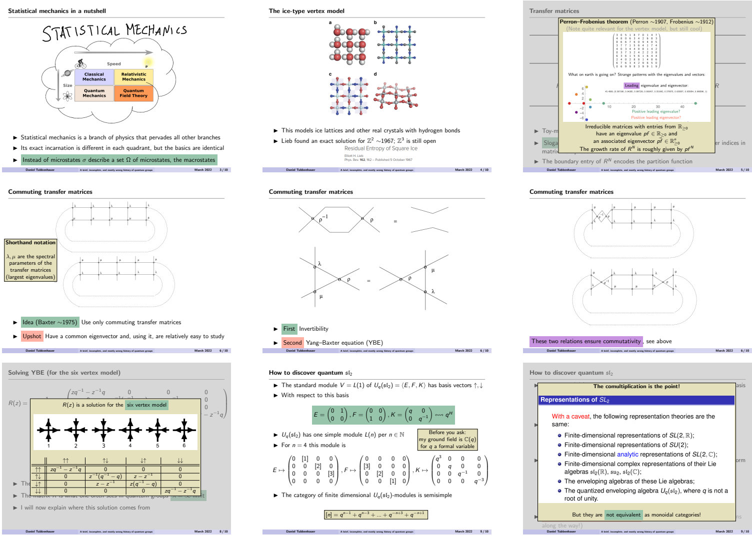
$$R(z) = \frac{1}{zq - z^{-1}q^{-1}} \begin{pmatrix} & & & 0 \\ & & q) & 0 \\ & & & 0 \\ 0 & 0 & 0 & zq^{-1} - z^{-1}q \end{pmatrix}$$

What came next?

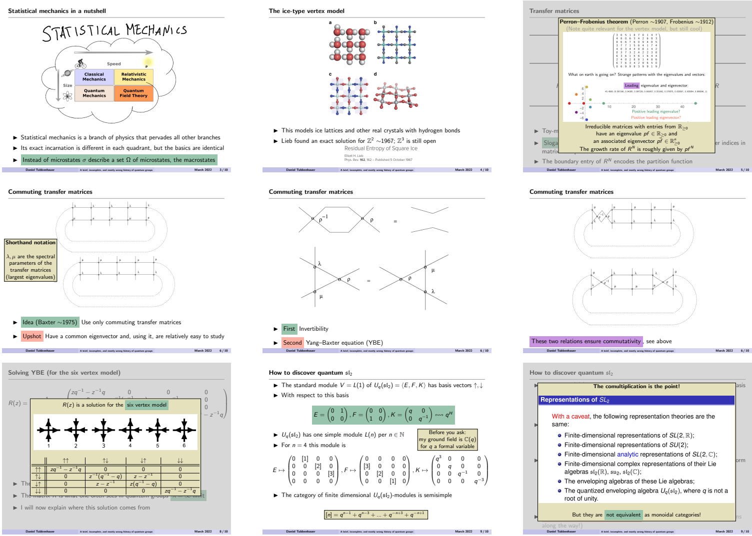
If you understand $\mathfrak{sl}_2$, then you should try $\mathfrak{g}$!

Initiated by Drinfel'd + Jimbo $\sim$1985+
and still very much in construction

- The spectral parameter is still missing!



There is still much to do...



Thanks for your attention!