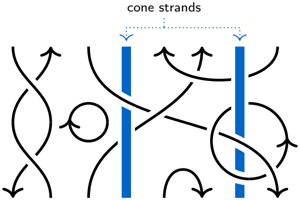
Link invariants and $\mathbb{Z}/2\mathbb{Z}$-orbifolds

Or: What makes types ABCD special?

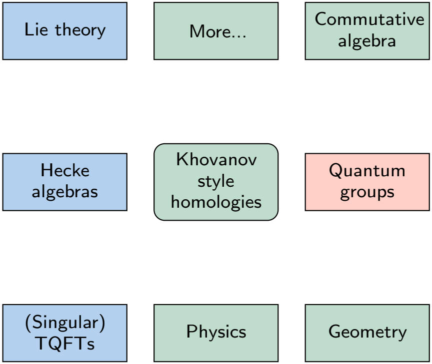
Daniel Tubbenhauer



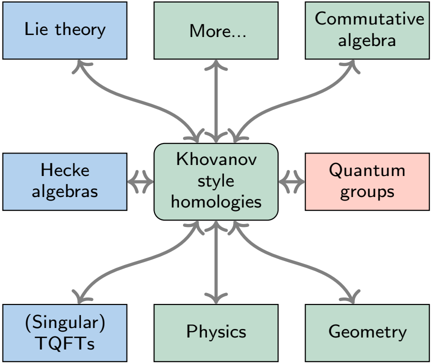
Joint work in progress (take it with a grain of salt) with Catharina Stroppel and Arik Wilbert
(Based on an idea of Mikhail Khovanov)

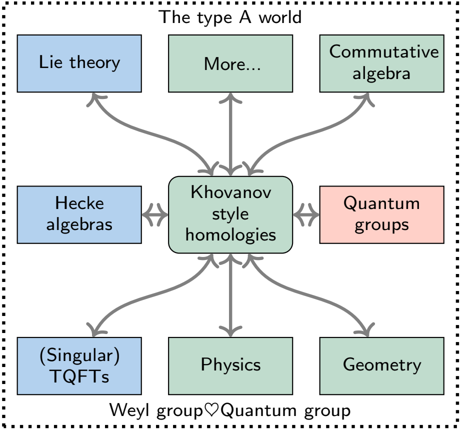
January 2018

Khovanov
style
homologies


Lie theory
More...
Commutative algebra
Hecke algebras
Khovanov style homologies
Quantum groups
(Singular) TQFTs
Physics
Geometry

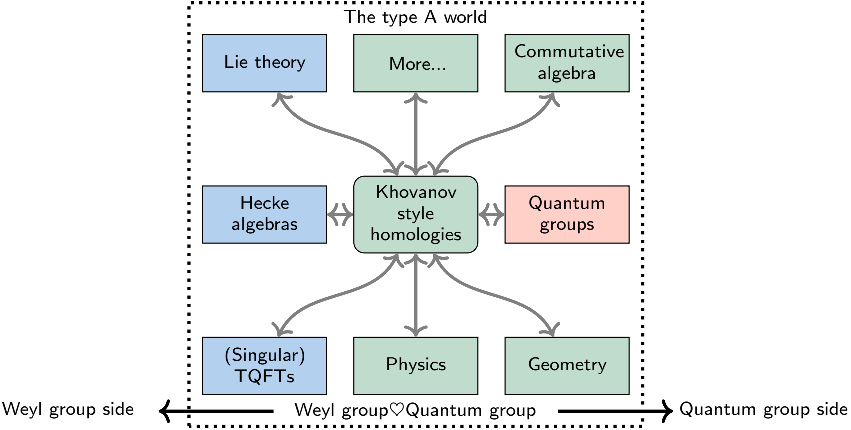
My beloved gadget
with many connections.





A quantum group of type E_7 is type A-braided!?

Outside of type A



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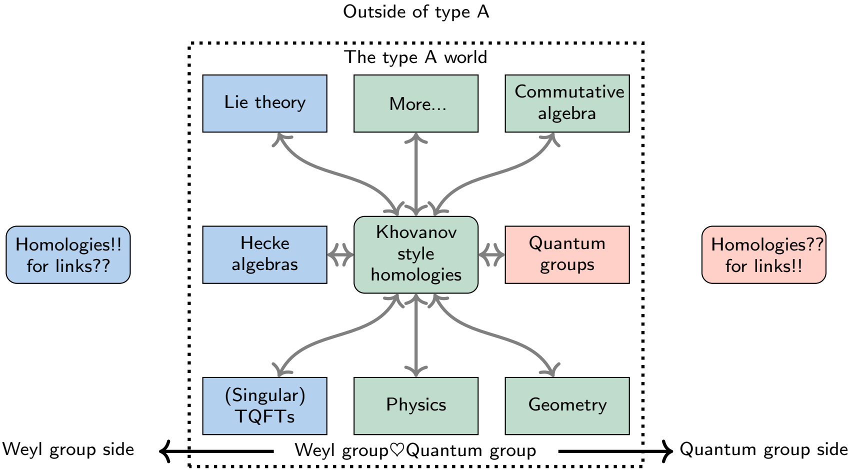
Outside of type A
The type A world
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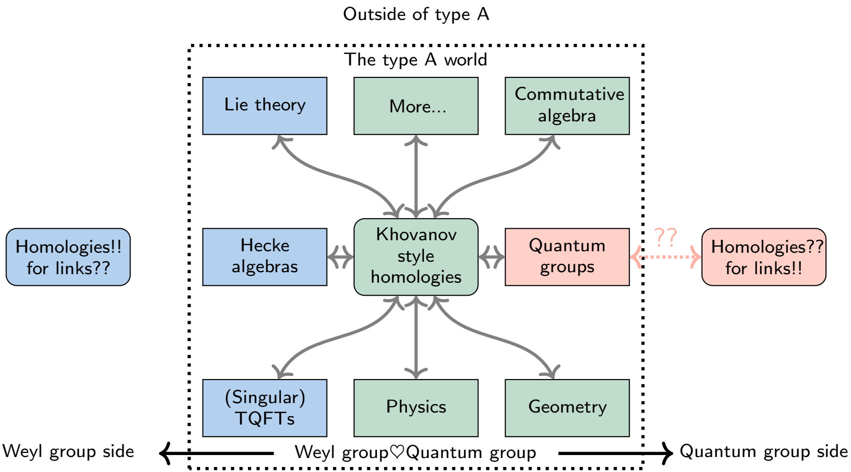
Weyl group side
Weyl group♡Quantum group
Quantum group side



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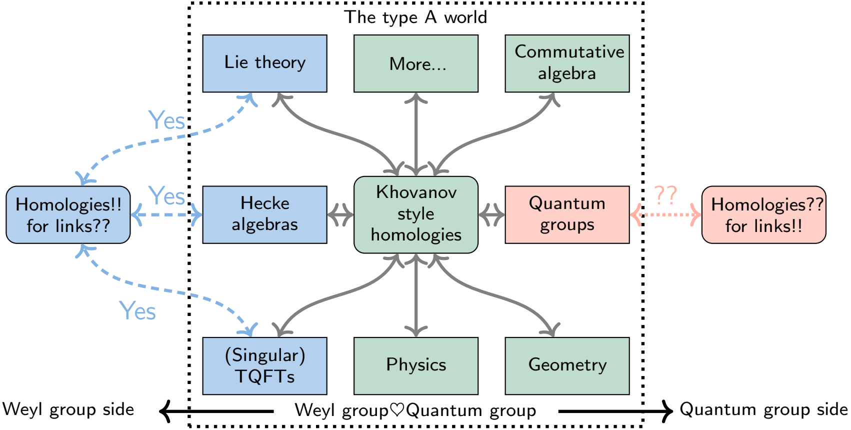

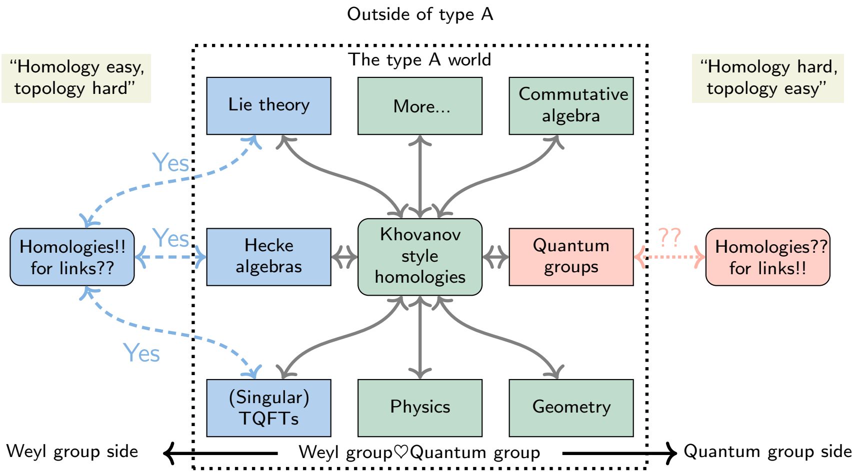
Outside of type A
The type A world
"Homology easy, topology hard"
"Homology hard, topology easy"
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1. **Tangle diagrams of $\mathbb{Z}/2\mathbb{Z}$-orbifold tangles**
   - Diagrams
   - Tangles in $\mathbb{Z}/2\mathbb{Z}$-orbifolds

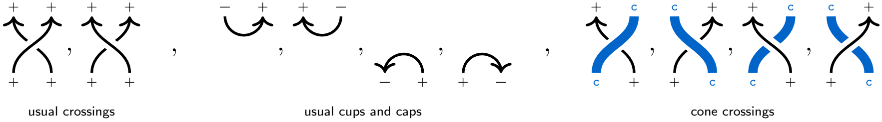
2. **Topology of Artin braid groups**
   - The Artin braid groups: algebra
   - Hyperplanes vs. configuration spaces

3. **Invariants**
   - Reshetikhin–Turaev-like theory for some coideals
   - Polynomials and homologies for $\mathbb{Z}/2\mathbb{Z}$-orbifold tangles

Tangle diagrams with cone strands

Let $\mathrm{c}\mathcal{T}an$ be the monoidal category defined as follows.

Generators. Object generators $\{+,-,\mathrm{c}\}$, morphism generators

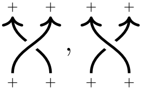


Relations. Reidemeister type relations, and the $\mathbb{Z}/_{2\mathbb{Z}}$-relations:

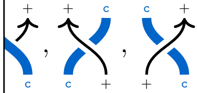
Tangle diagrams with cone strands

Let $\mathrm{c}\mathcal{T}an$ be the r

Generators. Obje



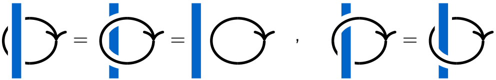
usual crossings



cone crossings

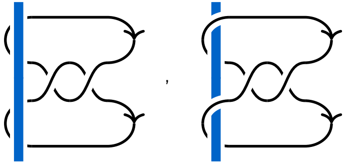
Relations. ▸ Reideme

Examples.



Unknot

Essential unknot



Hopf link

Essential Hopf link

Tangle diagrams with cone strands

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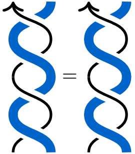
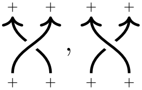
Example.



Generators. Object generator … n generators



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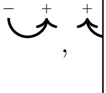
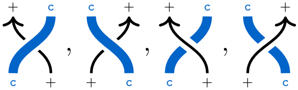
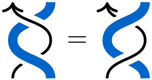
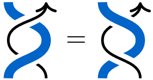


usual cups and caps



cone crossings

Relations. ‣ Reidemeister type relations, and the $\mathbb{Z}/2\mathbb{Z}$-relations:

 and 

Tangle diagrams with cone strands

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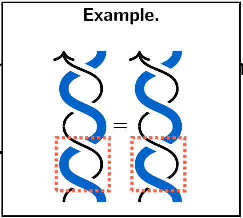
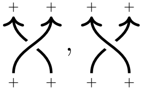


Generators. Object generator … n generators



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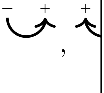
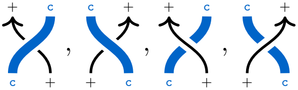


usual cups and caps



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Relations. ▸ Reidemeister type relations, and the $\mathbb{Z}/2\mathbb{Z}$-relations:

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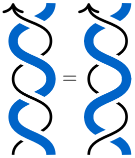
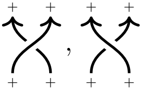


Generators. Object generator n generators



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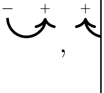

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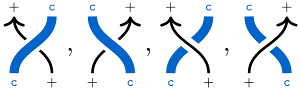

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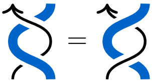

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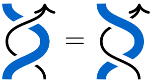

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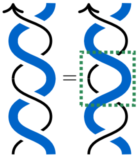
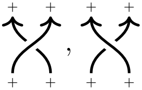


Generators. Object generator… n generators



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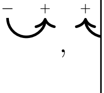

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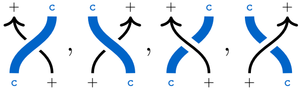

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Tangle diagrams with cone strands

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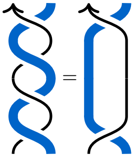

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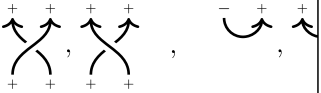 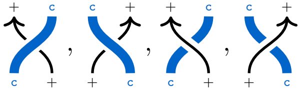

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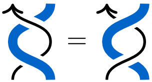
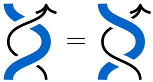
 and 

Tangle diagrams with cone strands

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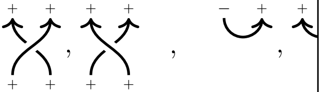
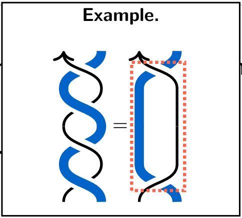
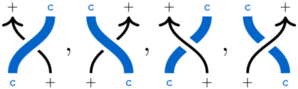
Generators. Object generator ... m generators







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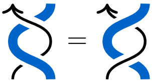

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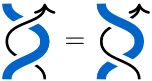
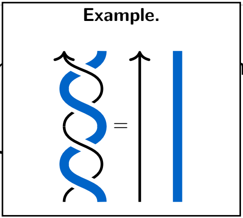
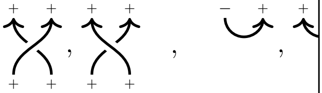


Tangle diagrams with cone strands

Let $\textcolor{blue}{c}\mathcal{T}an$ be the monoidal category defined as follows.



Generators. Object generator[…]m generators



usual crossings

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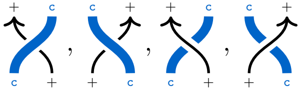

cone crossings

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Tangle diagrams with cone strands

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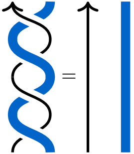
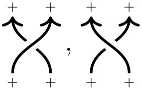
Example.



Generators. Object generator ... n generators



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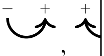
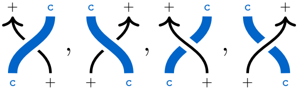




e crossings

Exercise. The relations are actually equivalent.

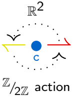
Relations. ▸ Reidemeister type relations, and the $\mathbb{Z}/_{2\mathbb{Z}}$-relations:

$\mathbb{Z}/2\mathbb{Z}$-orbifolds

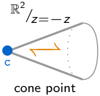
“Definition”. An ▸ orbifold is locally modeled on the standard Euclidean space modulo an action of some finite group.

Main example. $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{R}^2$ by rotation by π around a fixed point c:

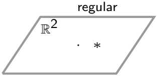
$$\mathrm{c}_1\mathtt{O}rb = \mathbb{R}^2 / {}^{\mathbb{Z}}/_{2\mathbb{Z}}$$

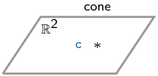


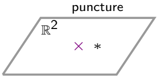
$$\rightsquigarrow \mathtt{X}_{\mathrm{c}_1\mathtt{O}rb} \approx$$



Philosophy. c is half-way in between a regular point and a puncture:







$\mathbb{Z}/2\mathbb{Z}$-orbifolds

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Main example. $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{R}^2$ by rotation by π around a fixed point c:

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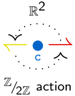

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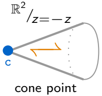

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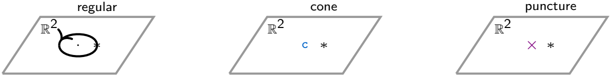

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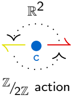
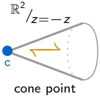
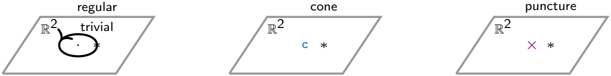


$$\rightsquigarrow \mathtt{X}_{\mathrm{c}_1\mathtt{O}rb} \approx$$



Philosophy. c is half-way in between a regular point and a puncture:



$\mathbb{Z}/2\mathbb{Z}$-orbifolds

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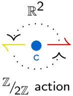

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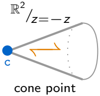

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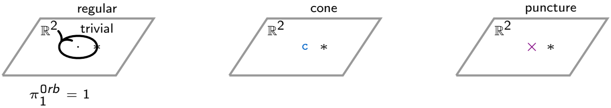

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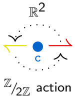

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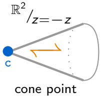

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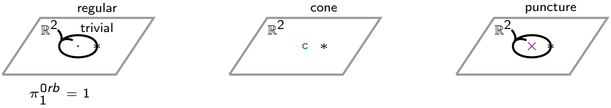

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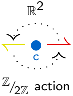

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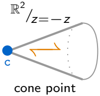

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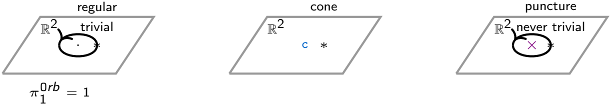

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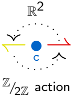

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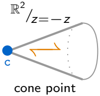

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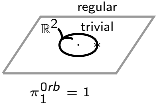

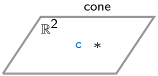

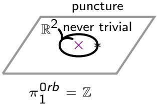

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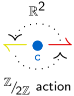

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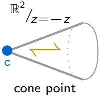

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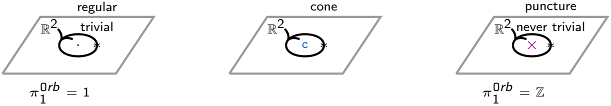

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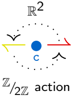

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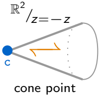

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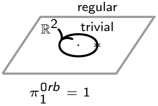

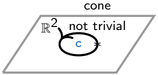

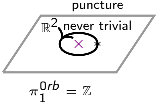

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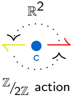

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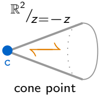

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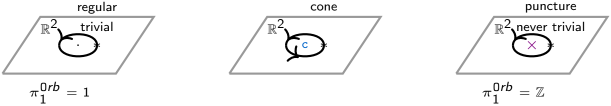

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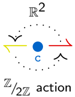

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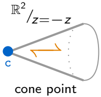

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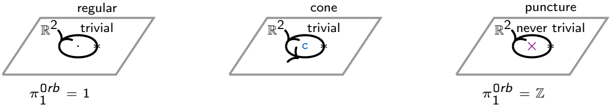

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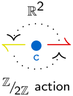
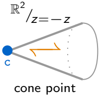
$c_1\mathsf{O}rb = \mathbb{R}^2 / \mathbb{Z}/2\mathbb{Z}$  $\rightsquigarrow \mathtt{X}_{c_1\mathsf{O}rb} \approx$ 

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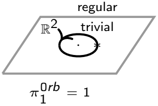

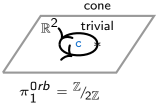

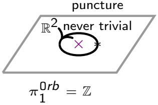

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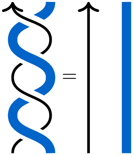
If we draw tangles in $c_1 \mathtt{O}rb \times [0,1]$, then:



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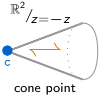
$c_1 \mathtt{O}rb =$



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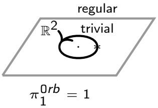

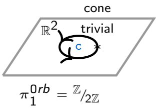

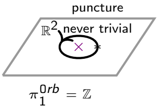

Pioneers of algebra

Let Γ be a ▸ Coxeter graph.

Artin ∼1925, Tits ∼1961++. The Artin braid groups and its Coxeter group quotients are given by generators-relations:

$$\mathcal{A}r_\Gamma = \langle b_i \mid \underbrace{\cdots b_i b_j b_i}_{m_{ij} \text{ factors}} = \underbrace{\cdots b_j b_i b_j}_{m_{ij} \text{ factors}} \rangle$$

$$\downarrow$$

$$\mathcal{W}_\Gamma = \langle s_i \mid s_i^2 = 1, \underbrace{\cdots s_i s_j s_i}_{m_{ij} \text{ factors}} = \underbrace{\cdots s_j s_i s_j}_{m_{ij} \text{ factors}} \rangle$$

Artin braid groups generalize classical braid groups, Coxeter groups Weyl groups.

We want to understand these better.

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Only algebra:
No “interpretation” yet.

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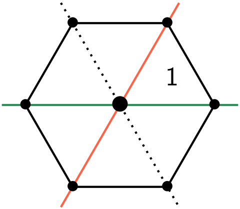
I follow hyperplanes

$\mathcal{W}_{A_2} = \langle s, t \rangle$ acts faithfully on $\mathbb{R}^2$ by reflecting in hyperplanes (for each reflection):



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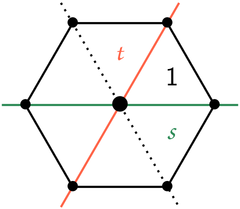

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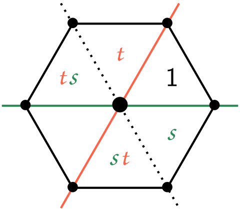

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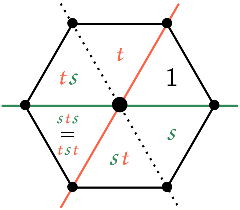

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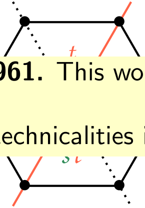

Coxeter ∼1934, Tits ∼1961. This works in ridiculous generality.

(Up to some minor technicalities in the infinite case.)

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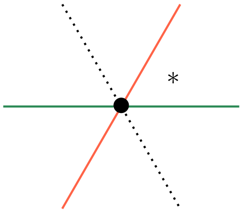

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$$\pi_1(\mathtt{N}^{\mathbb{C}}_{\mathrm{A}_2}) \cong \mathcal{A}r_{\mathrm{A}_2} = \langle b_s, b_t \mid b_s b_t b_s = b_t b_s b_t \rangle$$

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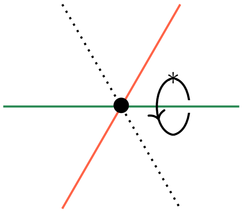

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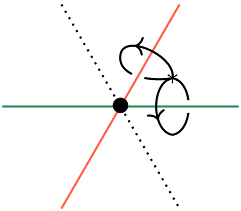
$$\pi_1(\mathtt{N}^{\mathbb{C}}_{A_2}) \cong \mathcal{A}r_{A_2} = \langle \mathbf{b}_s, \mathbf{b}_t \mid \mathbf{b}_s \mathbf{b}_t \mathbf{b}_s = \mathbf{b}_t \mathbf{b}_s \mathbf{b}_t \rangle$$

I follow hyperplanes

$\mathcal{W}_{\mathrm{A}_2} = \langle s, t \rangle$ acts faithfully on $\mathbb{R}^2$ by reflecting in hyperplanes (for each reflection):



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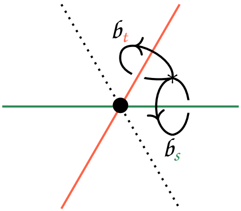

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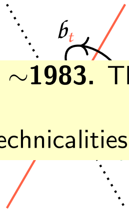

Brieskorn ∼1971, van der Lek ∼1983. This works in ridiculous generality.

(Up to some minor technicalities in the infinite case.)

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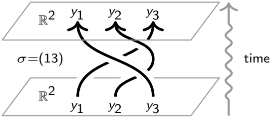
Configuration spaces

Artin ∼1925. There is a topological model of $\mathcal{A}r_{\mathrm{A}}$ via configuration spaces.

Example. Take $\mathtt{Conf}_{\mathrm{A}_2} = (\mathbb{R}^2)^3 \setminus \text{fat diagonal}/\mathrm{S}_3$. Then $\pi_1(\mathtt{Conf}_{\mathrm{A}_2}) \cong \mathcal{A}r_{\mathrm{A}_2}$.

Philosophy. Having a configuration spaces is the same as having braid diagrams:



a usual braid

Crucial. Note that – by explicitly calculating the ▸ equations defining the hyperplanes – one can directly check that:

"Hyperplane picture equals configuration space picture."

Configuration spaces

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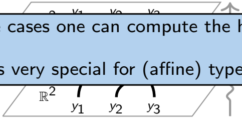
Example. Take Conf ... $(\mathbb{R}^2)^3 \setminus$ fat diagonal/∼. Then ... (Conf ...) $\cong \mathcal{A}r_{\mathrm{A}_2}$.

Lambropoulou ∼1993, tom Dieck ∼1998, Allcock ∼2002.

Type	A	$\widetilde{\mathrm{A}}$	B=C	$\widetilde{\mathrm{B}}$	$\widetilde{\mathrm{C}}$	D	$\widetilde{\mathrm{D}}$
Orbifold feature	none	×	×	×, c	×, ×	c	c, c

Additional inside: Works for tangles as well.

Philosophy. ... id diagrams:



In those cases one can compute the hyperplanes!

This is very special for (affine) types ABCD.

a usual braid

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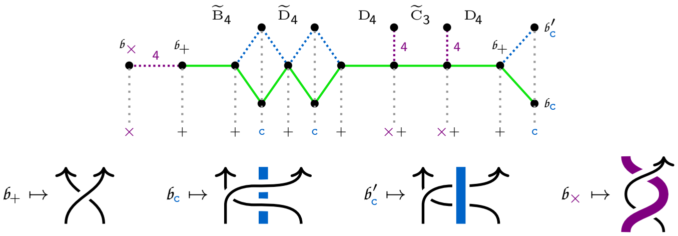
Example. … $\mathcal{A}r_{\mathrm{A}_2}$.

Philosoph… diagrams:

Hope.

The same works for Coxeter diagrams which are "locally type ABCD", e.g.:



a usual braid

But we can't compute the hyperplanes...

Crucial. Note that – by explicitly calculating the ▸ equations defining the hyperplanes – one can directly check that:

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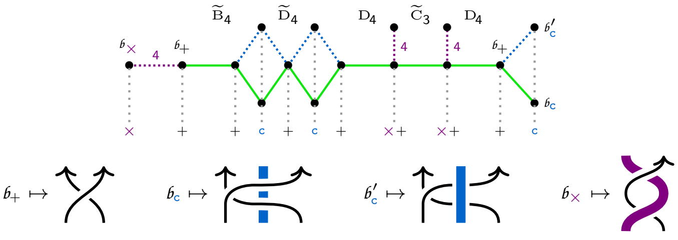

Example. ... $\mathcal{A}r_{\mathrm{A}_2}$.

Philosoph ... diagrams:

a usual braid

Crucial. ... - one can directly c...

In words: The $\mathbb{Z}/2\mathbb{Z}$-orbifolds provide the framework to study Artin braid groups of classical (affine) type and their "glued-generalizations".

"Hyperplane picture equals configuration space picture."

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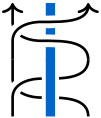
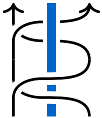
$\rightsquigarrow$





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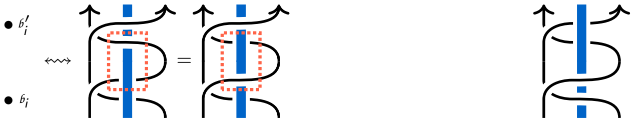

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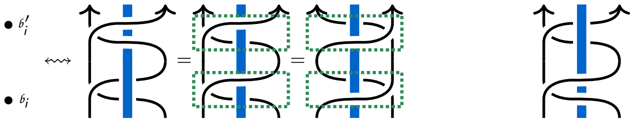

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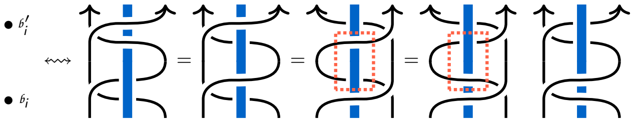

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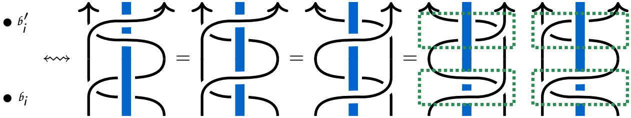

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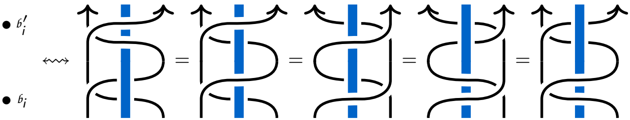

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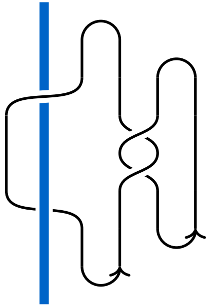
Reshetikhin–Turaev $\sim$1991. Construct link and tangle invariants as functors

$$\mathrm{u}\mathcal{RT}\colon \mathrm{u}\mathcal{T}an \to \text{well-behaved target category.}$$

Today: Target categories $= \mathcal{R}ep(\mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2))$ and friends.

Question. What could the $\mathbb{Z}/2\mathbb{Z}$-analog be?



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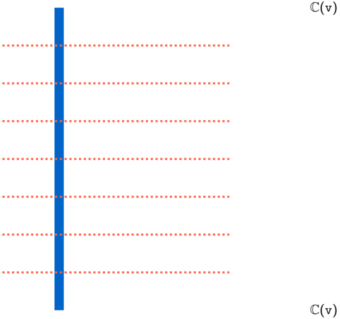

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$\mathbb{C}(\mathtt{v})$

$\mathbb{C}(\mathtt{v})$ = ground field,
$\mathtt{V}_{\mathtt{v}}$ = vector representation
of $\mathcal{U}_{\mathtt{v}} = \mathcal{U}_{\mathtt{v}}(\mathfrak{sl}_2)$.

$$\mathtt{V}_{\mathtt{v}} \otimes \mathtt{V}_{\mathtt{v}} \xleftarrow{\ \mathrm{ev}^*\ } \mathbb{C}(\mathtt{v})$$

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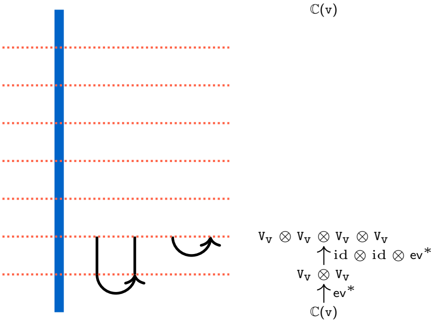

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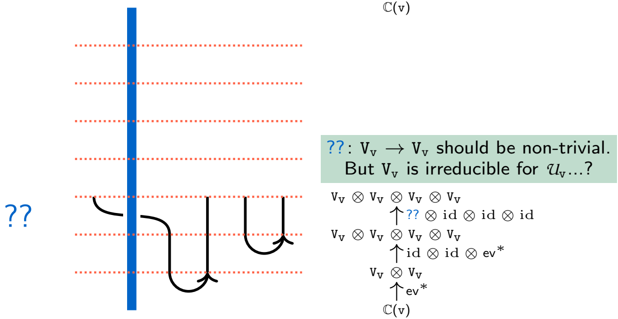

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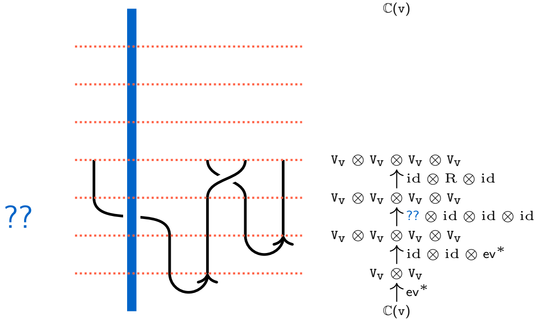

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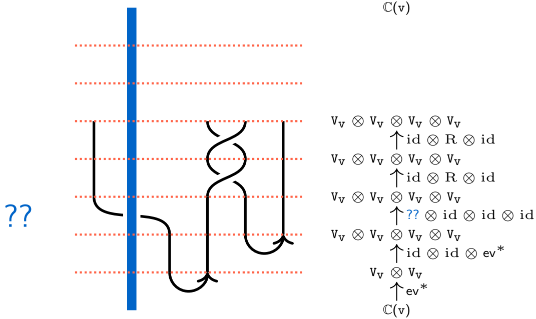

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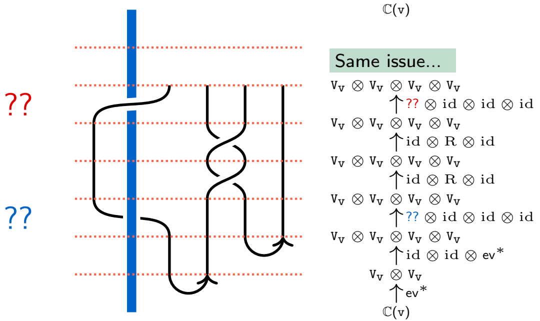

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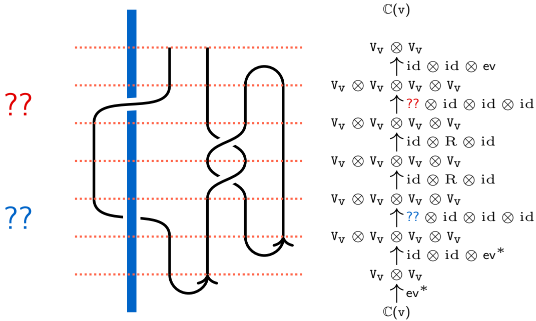

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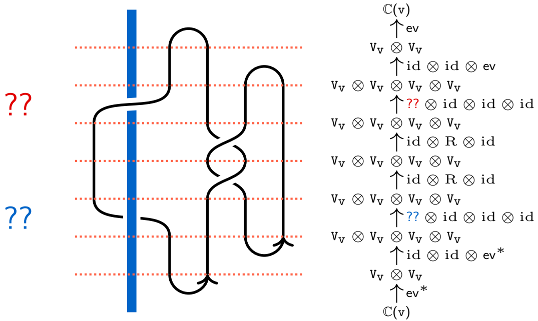

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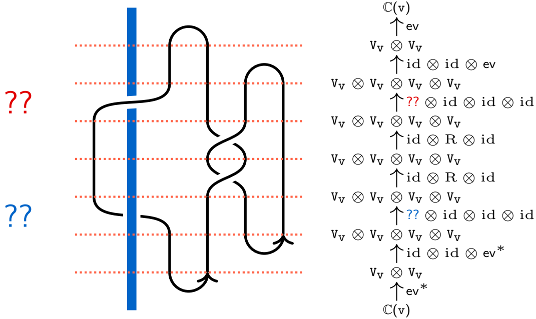
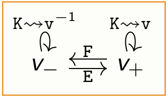
Orbifold-philosophy. We need something half-way in between $\mathbb{C}(\mathrm{v})$ and $\mathcal{U}_{\mathrm{v}}$.



Kulish–Reshetikhin ∼1981. $\mathcal{U}_{\mathrm{v}}$ is the associative, unital $\mathbb{C}(\mathrm{v})$-algebra generated by $\mathtt{E}, \mathtt{F}, \mathtt{K}^{\pm 1}$ subject to the usual relations.

Not really important...

$$\mathrm{V}_{\mathrm{v}}: \begin{matrix} \mathtt{E}v_+ = 0, & \mathtt{F}v_+ = v_-, & \mathtt{K}v_+ = \mathrm{v}v_+, \\ \mathtt{E}v_- = v_+, & \mathtt{F}v_- = 0, & \mathtt{K}v_- = \mathrm{v}^{-1}v_-. \end{matrix}$$



Define $\mathcal{U}_{\mathrm{v}}$-intertwiners:

$$\cup : \mathbb{C}(\mathrm{v}) \to \mathrm{V}_{\mathrm{v}} \otimes \mathrm{V}_{\mathrm{v}}, \quad 1 \mapsto v_- \otimes v_+ - \mathrm{v}^{-1} v_+ \otimes v_-,$$

$$\cap : \mathrm{V}_{\mathrm{v}} \otimes \mathrm{V}_{\mathrm{v}} \to \mathbb{C}(\mathrm{v}), \quad \begin{cases} v_+ \otimes v_+ \mapsto 0, & v_+ \otimes v_- \mapsto 1, \\ v_- \otimes v_+ \mapsto -\mathrm{v}, & v_- \otimes v_- \mapsto 0, \end{cases}$$

$$\times : \mathrm{V}_{\mathrm{v}} \otimes \mathrm{V}_{\mathrm{v}} \to \mathrm{V}_{\mathrm{v}} \otimes \mathrm{V}_{\mathrm{v}}, \quad \times = \mathrm{v} \,|\,| + \mathrm{v}^2 \asymp.$$

Half-way in between trivial $\subset$?? $\subset \mathcal{U}_{\mathtt{v}}$ – part I

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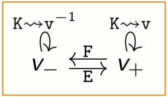

Fact. $\mathcal{U}_{\mathtt{v}}$ is a Hopf algebra $\Rightarrow$ We can tensor representations.

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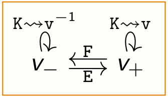
$$\times : \mathtt{V}_{\mathtt{v}} \otimes \mathtt{V}_{\mathtt{v}} \to \mathtt{V}_{\mathtt{v}} \otimes \mathtt{V}_{\mathtt{v}}, \quad \times = \mathtt{v} \,|\,| + \mathtt{v}^2 \,{}^{\cup}_{\cap}.$$

Half-way in between trivial $\subset$?? $\subset \mathcal{U}_{\mathrm{v}}$ – part I

Kulish–Reshetikhin $\sim$1981. $\mathcal{U}_{\mathrm{v}}$ is the associative, unital $\mathbb{C}(\mathrm{v})$-algebra generated by $\mathrm{E}, \mathrm{F}, \mathrm{K}^{\pm 1}$ subject to the usual relations.

$$\mathrm{V}_{\mathrm{v}}: \quad \begin{array}{lll} \mathrm{E}v_+ = 0, & \mathrm{F}v_+ = v_-, & \mathrm{K}v_+ = \mathrm{v}v_+, \\ \mathrm{E}v_- = v_+, & \mathrm{F}v_- = 0, & \mathrm{K}v_- = \mathrm{v}^{-1}v_-. \end{array}$$



Define $\mathcal{U}_{\mathrm{v}}$-intertwiners:

Example. $(\cap \circ \cup)(1) = \cap(v_- \otimes v_+) - \mathrm{v}^{-1}\cap(v_+ \otimes v_-) = -\mathrm{v} - \mathrm{v}^{-1}$.

$$\cup : \mathbb{C}(\mathrm{v}) \to \mathrm{V}_{\mathrm{v}} \otimes \mathrm{V}_{\mathrm{v}}, \quad 1 \mapsto v_- \otimes v_+ - \mathrm{v}^{-1} v_+ \otimes v_-,$$

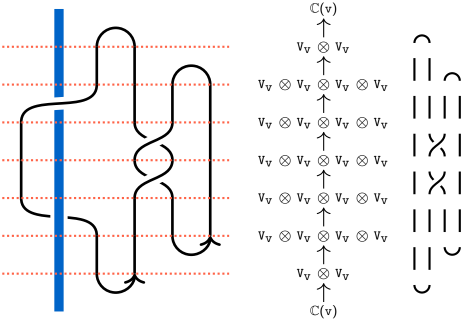
$$\cap : \mathrm{V}_{\mathrm{v}} \otimes \mathrm{V}_{\mathrm{v}} \to \mathbb{C}(\mathrm{v}), \quad \begin{cases} v_+ \otimes v_+ \mapsto 0, & v_+ \otimes v_- \mapsto 1, \\ v_- \otimes v_+ \mapsto -\mathrm{v}, & v_- \otimes v_- \mapsto 0, \end{cases}$$

$$\times : \mathrm{V}_{\mathrm{v}} \otimes \mathrm{V}_{\mathrm{v}} \to \mathrm{V}_{\mathrm{v}} \otimes \mathrm{V}_{\mathrm{v}}, \quad \times = \mathrm{v}\,|\,| + \mathrm{v}^2 \asymp.$$

Half-way in between trivial $\subset$?? $\subset \mathcal{U}_{\mathtt{v}}$ – part I

Kulish–Reshetikhin ∼1981. $\mathcal{U}_{\mathtt{v}}$ is the associative, unital $\mathbb{C}(\mathtt{v})$-algebra generated by $\mathtt{E}, \mathtt{F}, \mathtt{K}^{\pm 1}$ s

Example. We can not see the cone strands.



$\mathtt{V}_{\mathtt{v}}$:

$\mathtt{K} \rightsquigarrow \mathtt{v}$

v_+

Define $\mathcal{U}_{\mathtt{v}}$-int

$\mapsto 1,$

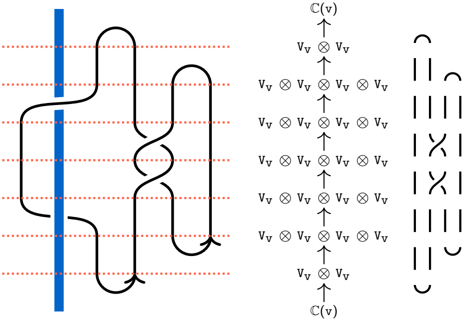
$\mapsto 0,$

$\mathtt{V}_{\mathtt{v}} \otimes \mathtt{V}_{\mathtt{v}} \to \mathtt{V}_{\mathtt{v}} \otimes \mathtt{V}_{\mathtt{v}}$, $= \mathtt{v} \,||\, + \mathtt{v}^2$.

Half-way in between trivial $\subset$?? $\subset \mathcal{U}_{\mathrm{v}}$ – part I

Kulish–Reshetikhin ∼1981. $\mathcal{U}_{\mathrm{v}}$ is the associative, unital $\mathbb{C}(\mathrm{v})$-algebra generated by $\mathrm{E}, \mathrm{F}, \mathrm{K}^{\pm 1}$ s

Example. We can not see the cone strands.



$\mathrm{V}_{\mathrm{v}}:$

$\mathrm{K} \rightsquigarrow \mathrm{v}$

v_+

Up to scalars no choice for | | | |

Define $\mathcal{U}_{\mathrm{v}}$-i

$\mapsto 1,$

$\mapsto 0,$

$: \mathrm{V}_{\mathrm{v}} \otimes \mathrm{V}_{\mathrm{v}} \to \mathrm{V}_{\mathrm{v}} \otimes \mathrm{V}_{\mathrm{v}}, \quad = \mathrm{v} \,|\,| + \mathrm{v}^2$

Half-way in between trivial $\subset$?? $\subset \mathcal{U}_{\mathrm{v}}$ – part II

Let $\mathrm{c}\mathcal{U}_{\mathrm{v}}$ be the ‣ coideal subalgebra of $\mathcal{U}_{\mathrm{v}}$ generated by $\mathtt{B} = \mathrm{v}^{-1}\mathtt{E}\mathtt{K}^{-1} + \mathtt{F}$.

$$\mathrm{V}_{\mathrm{v}}\colon\ \mathtt{B}v_{+} = v_{-},\quad \mathtt{B}v_{-} = v_{+}. \qquad v_{-} \underset{\mathtt{B}}{\overset{\mathtt{B}}{\rightleftarrows}} v_{+}$$

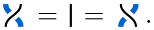
Define $\mathrm{c}\mathcal{U}_{\mathrm{v}}$-intertwiners:

 $: \mathrm{V}_{\mathrm{v}} \to \mathrm{V}_{\mathrm{v}},\quad v_{+} \mapsto v_{-},\ v_{-} \mapsto v_{+},$

 $: \mathbb{C}(\mathrm{v}) \to \mathrm{V}_{\mathrm{v}} \otimes \mathrm{V}_{\mathrm{v}},\quad 1 \mapsto v_{+} \otimes v_{+} - \mathrm{v}^{-1} v_{-} \otimes v_{-},$

 $: \mathrm{V}_{\mathrm{v}} \otimes \mathrm{V}_{\mathrm{v}} \to \mathbb{C}(\mathrm{v}),\quad \begin{cases} v_{+} \otimes v_{+} \mapsto -\mathrm{v}, & v_{+} \otimes v_{-} \mapsto 0, \\ v_{-} \otimes v_{+} \mapsto 0, & v_{-} \otimes v_{-} \mapsto 1, \end{cases}$

and



Aside. This drops out of a ‣ coideal version of Schur–Weyl duality.

Let $\mathtt{c}\mathcal{U}_{\mathtt{v}}$ be the ‣ coideal subalgebra of $\mathcal{U}_{\mathtt{v}}$ generated by $\mathtt{B} = \mathtt{v}^{-1}\mathtt{E}\mathtt{K}^{-1} + \mathtt{F}$.

$$\mathrm{V}_{\mathtt{v}}\colon\ \mathtt{B}v_{+} = v_{-},\quad \mathtt{B}v_{-} = v_{+}.\qquad v_{-} \overset{\mathtt{B}}{\underset{\mathtt{B}}{\rightleftarrows}} v_{+}$$

Define $\mathtt{c}\mathcal{U}_{\mathtt{v}}$-intertw

Observation. These are not $\mathcal{U}_{\mathtt{v}}$-equivariant, but ∪ and ∩ are $\mathtt{c}\mathcal{U}_{\mathtt{v}}$-equivariant.

$$: \mathbb{C}(\mathtt{v}) \to \mathrm{V}_{\mathtt{v}} \otimes \mathrm{V}_{\mathtt{v}},\quad 1 \mapsto v_{+} \otimes v_{+} - \mathtt{v}^{-1} v_{-} \otimes v_{-},$$

$$: \mathrm{V}_{\mathtt{v}} \otimes \mathrm{V}_{\mathtt{v}} \to \mathbb{C}(\mathtt{v}),\quad \begin{cases} v_{+} \otimes v_{+} \mapsto -\mathtt{v}, & v_{+} \otimes v_{-} \mapsto 0, \\ v_{-} \otimes v_{+} \mapsto 0, & v_{-} \otimes v_{-} \mapsto 1, \end{cases}$$

Aside. This drops out of a ‣ coideal version of Schur–Weyl duality.

Half-way in between trivial $\subset$?? $\subset \mathcal{U}_{\mathrm{v}}$ – part II

Let $\mathrm{c}\mathcal{U}_{\mathrm{v}}$ be the ▸ coideal subalgebra of $\mathcal{U}_{\mathrm{v}}$ generated by $\mathtt{B} = \mathrm{v}^{-1}\mathtt{E}\mathtt{K}^{-1} + \mathtt{F}$.

$$\mathrm{V}_{\mathrm{v}}\colon\ \mathtt{B}v_+ = v_-,\quad \mathtt{B}v_- = v_+.$$

$$v_- \underset{\mathtt{B}}{\overset{\mathtt{B}}{\rightleftarrows}} v_+$$

Define $\mathrm{c}\mathcal{U}$ **Example.** $(\cap \circ \cup)(1) = \cap(v_- \otimes v_+) - \mathrm{v}^{-1}\cap(v_+ \otimes v_-) = 0$

$| \circ | = |$ but $| \neq |$.

$\cup : \mathbb{C}(\mathrm{v}) \to \mathrm{V}_{\mathrm{v}} \otimes \mathrm{V}_{\mathrm{v}},\quad 1 \mapsto v_+ \otimes v_+ - \mathrm{v}^{-1} v_- \otimes v_-,$

$$\cap : \mathrm{V}_{\mathrm{v}} \otimes \mathrm{V}_{\mathrm{v}} \to \mathbb{C}(\mathrm{v}),\quad \begin{cases} v_+ \otimes v_+ \mapsto -\mathrm{v}, & v_+ \otimes v_- \mapsto 0, \\ v_- \otimes v_+ \mapsto 0, & v_- \otimes v_- \mapsto 1, \end{cases}$$

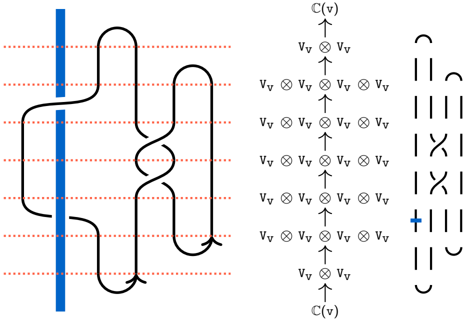
Aside. This drops out of a ▸ coideal version of Schur–Weyl duality.

Half-way in between trivial $\subset$?? $\subset \mathcal{U}_v$ – part II

Let $c\mathcal{U}_v$ be the coideal subalgebra of $\mathcal{U}_v$ generated by $B = v^{-1}EK^{-1} + F$.

Define $c\mathcal{U}_v$-ir

Example. We can see the cone strands.



),

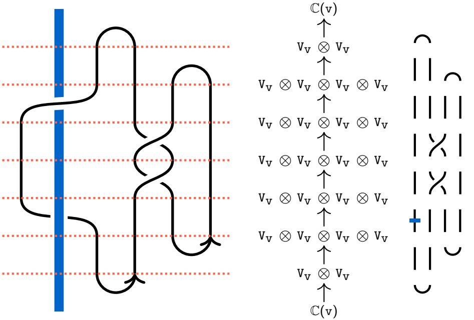
1,

Aside. This drops out of a coideal version of Schur–Weyl duality.

Half-way in between trivial $\subset$?? $\subset$ $\mathcal{U}_{\mathrm{v}}$ – part II

Let $\mathrm{c}\mathcal{U}_{\mathrm{v}}$ be the ▸ coideal subalgebra of $\mathcal{U}_{\mathrm{v}}$ generated by $\mathtt{B} = \mathrm{v}^{-1}\mathtt{E}\mathtt{K}^{-1} + \mathtt{F}$.

Example. We can see the cone strands.



Define $\mathrm{c}\mathcal{U}_{\mathrm{v}}$-i

We have now
$\neq$.

),

1,

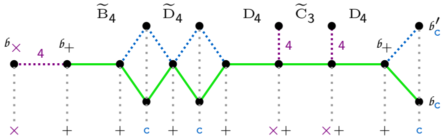
Aside. This drops out of a ▸ coideal version of Schur–Weyl duality.

Let $\mathtt{c}\mathcal{U}_{\mathrm{v}}$ be the ▸ coideal subalgebra of $\mathcal{U}_{\mathrm{v}}$ generated by $\mathtt{B} = \mathrm{v}^{-1}\mathtt{E}\mathtt{K}^{-1} + \mathtt{F}$.

Define $\mathtt{c}\mathcal{U}_{\mathrm{v}}$

Hope.

The same works for



But what is the replacement of $\mathtt{c}\mathcal{U}_{\mathrm{v}}$ outside of classical or affine classical type?

and

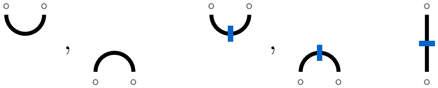
(Affine) ABCD are again very special.

Aside. This drops out of a ▸ coideal version of Schur–Weyl duality.

Back to diagrams

Let $\mathtt{m}\mathcal{A}rc$ be the monoidal category defined as follows.

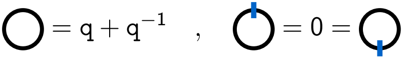
Generators. Object generator $\{\mathtt{o}\}$, morphism generators



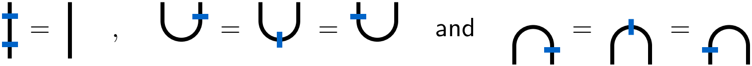
$\mathtt{o}$ cups and caps $\quad$ $\mathtt{m}$ cups and caps $\quad$ markers

Relations. "Coideal" relations:

A technicality: $\mathtt{q} = -\mathtt{v}$.



$\mathtt{o}$ circle removal $\quad$ $\mathtt{m}$ circle removals

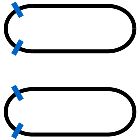


marker removal $\quad$ marker isotopies

Back to diagrams

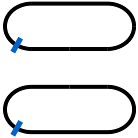
Let $\mathtt{m}\mathcal{A}rc$ be the monoidal category defined as follows.

Examples.



$= (\mathtt{q} + \mathtt{q}^{-1})^2$

But in contrast:



$= 0$

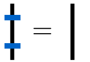
Generators. Object gener… rs

∘ cup

markers

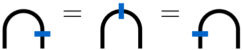
Relations. "Coideal" rela…

ls



marker removal

and



marker isotopies

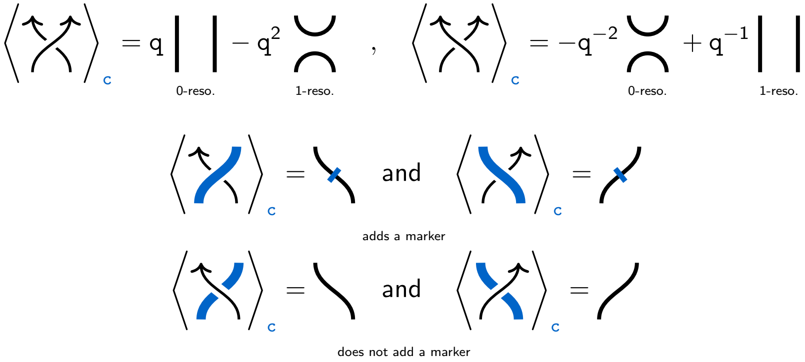
A polynomial invariant à la Jones & Kauffman

We define a monoidal functor $\langle\_\rangle_{\mathtt{c}} : \mathtt{c}\mathcal{T}an \to \mathtt{m}\mathcal{A}rc$ as follows. On objects,

$$\langle + \rangle_{\mathtt{c}} = \circ \quad , \quad \langle - \rangle_{\mathtt{c}} = \circ \quad , \quad \langle \mathtt{c} \rangle_{\mathtt{c}} = \varnothing$$

and on morphisms by

The skein relations.



A polynomial invariant à la Jones & Kauffman

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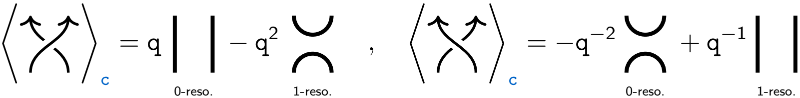
and on morphisms by

The skein relations.



The $\mathbb{Z}/2\mathbb{Z}$-skein relations.

adds a marker

does not add a marker

A polynomial invariant à la Jones & Kauffman

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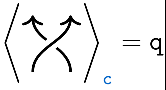
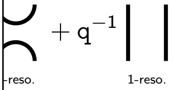
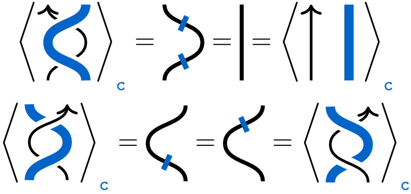
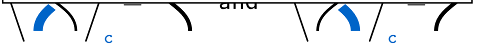




Theorem. This is a $\mathbb{Z}/2\mathbb{Z}$-tangle invariant.

Proof. Check relations, e.g.:





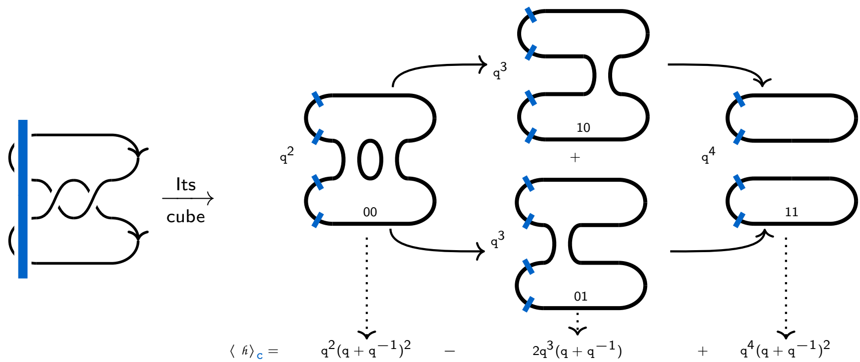
does not add a marker

A polynomial invariant à la Jones & Kauffman

We define a monoidal functor $\langle \_ \rangle_{c} : c\mathcal{T}an \to m\mathcal{A}rc$ as follows. On objects,

$$\langle + \rangle_{c} = \circ \quad , \quad \langle - \rangle_{c} = \circ \quad , \quad \langle c \rangle_{c} = \emptyset$$

Example. Here the Hopf link.



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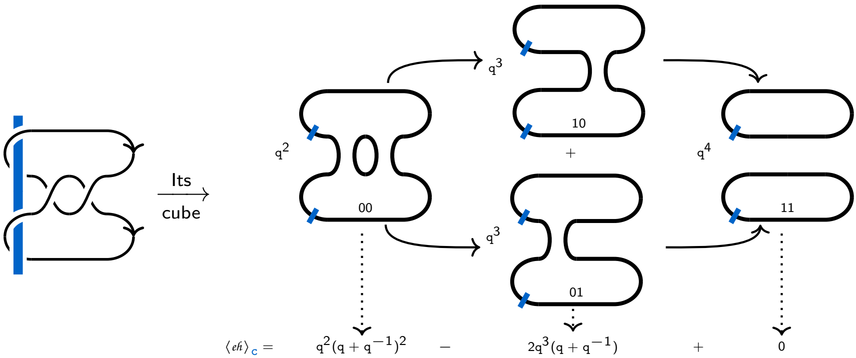
Example. Here the essential Hopf link.



does not add a marker

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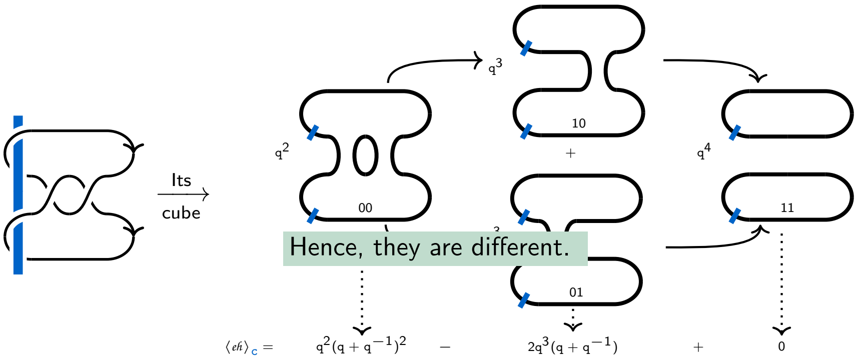


Hence, they are different.

does not add a marker

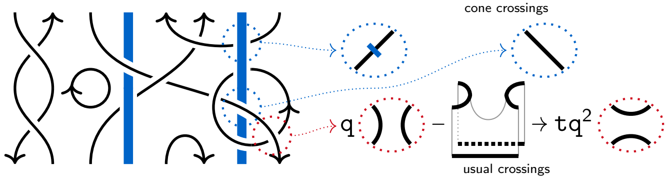
A polynomial invariant à la Jones & Kauffman

We define a monoidal functor $\langle \_ \rangle_{\mathtt{c}} : \mathtt{c}\mathcal{T}an \to \mathtt{m}\mathcal{A}rc$ as follows. On objects,

and on

A homological invariant à la Khovanov & Bar-Natan.

Works mutatis mutandis. Here is the picture:



$$\mathtt{m}\mathcal{Z}\left(\bigcirc_{m}\right) = \begin{cases} \mathbb{Z}[X]/(X^2), & \text{if } m \text{ is even,} \\ 0, & \text{if } m \text{ is odd,} \end{cases}$$

reso.

does not add a marker

A polynomial invariant à la Jones & Kauffman

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and on

A homological invariant à la Khovanov & Bar-Natan.

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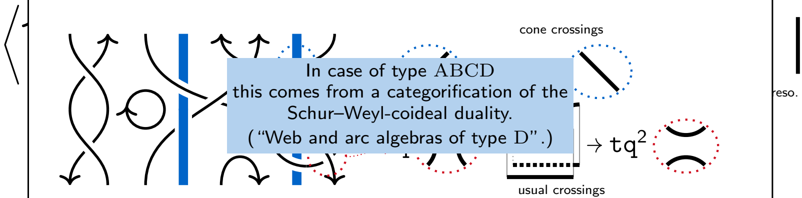

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A polynomial invariant à la Jones & Kauffman

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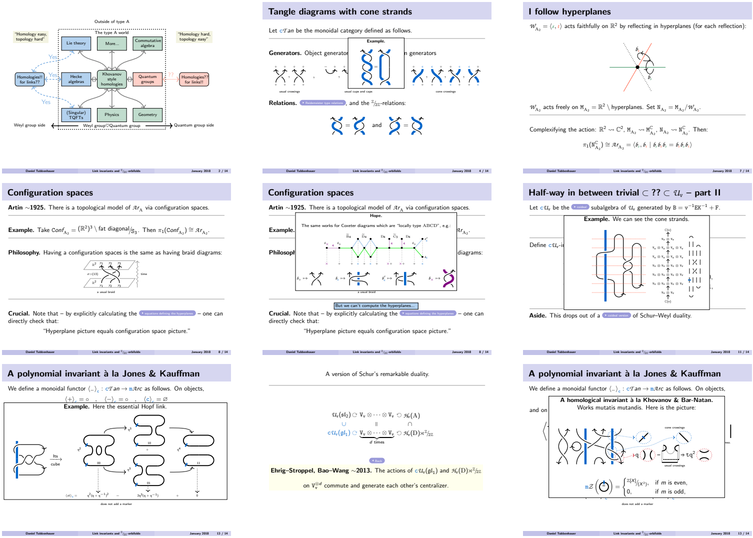
and on morphisms by

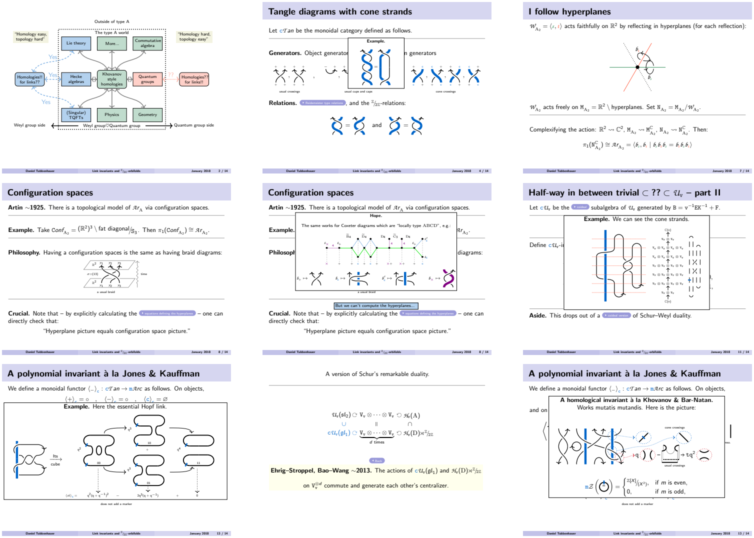
A homological invariant à la Khovanov & Rozansky.
Everything generalizes to higher ranks.
("Webs", "foams", etc.)

adds a marker

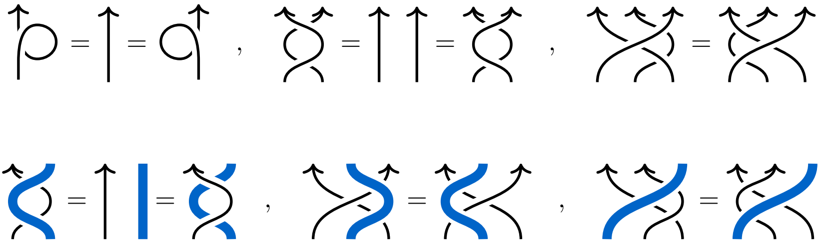
does not add a marker



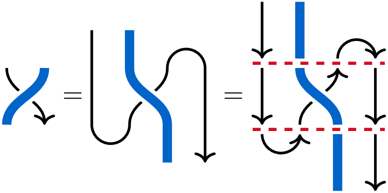
There is still much to do...



Thanks for your attention!

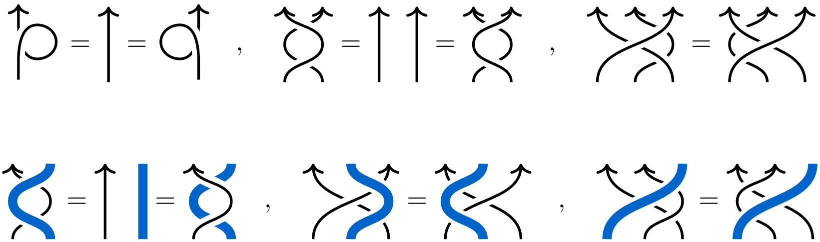


These guys and friends come for free.



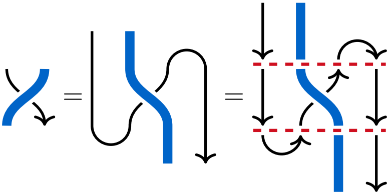
$\in \mathtt{Hom}_{c\mathcal{T}an}(c-, -c)$

Back



I see them as diagrams – no topological interpretation intended at the moment.

These guys and friends come for free.



$\in \mathtt{Hom}_{c\mathcal{T}an}(c-, -c)$

Back

Satake ∼1956 ("V-manifold"), Thurston ∼1978, Haefliger ∼1990 ("orbihedron"), etc. A triple $\mathtt{O}rb = (\mathtt{X}_{\mathtt{O}rb}, \cup_i \mathtt{U}_i, \mathtt{G}_i)$ of a Hausdorff space $\mathtt{X}_{\mathtt{O}rb}$, a covering $\cup_i \mathtt{U}_i$ of it (closed under finite intersections) and a collection of finite groups $\mathtt{G}_i$ is called an orbifold (of dimension m) if for each $\mathtt{U}_i$ there exists a open subset $\mathtt{V}_i \subset \mathbb{R}^m$ carrying an action of $\mathtt{G}_i$, and some compatibility conditions.

Fact. A two-dimensional ("smooth") orbifold is locally modeled on:

- ▷ Cone points $\leftrightsquigarrow$ rotation action of $\mathbb{Z}/l\mathbb{Z}$.
- ▷ Reflector corners $\leftrightsquigarrow$ reflection action of the dihedral group.
- ▷ Mirror points $\leftrightsquigarrow$ reflection action of $\mathbb{Z}/2\mathbb{Z}$.

◂ Back

Satake ∼1956 ("V-manifold"), Thurston ∼1978, Haefliger ∼1990

("or ..., a

cove ...

grou ... pen

Not super important. Only one thing to stress:
Topologically an orbifold is sometimes the same as its underlying space.
So all notions concerning orbifolds have to take this into account.

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Back

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Not super important. Only one thing to stress:
Topologically an orbifold is sometimes the same as its underlying space.
So all notions concerning orbifolds have to take this into account.

Quote by Thurston about the name orbifold:

“This terminology should not be blamed on me. It was obtained by a democratic process in my course of 1976-77. An orbifold is something with many folds; unfortunately, the word ‘manifold’ already has a different definition. I tried ‘foldamani’, which was quickly displaced by the suggestion of ‘manifolded’. After two months of patiently saying ‘no, not a manifold, a manifol**dead**,’ we held a vote, and ‘orbifold’ won.”

Back

Satake ∼1956 ("V-manifold"), Thurston ∼1978, Haefliger ∼1990 ("orbihedron"), etc. A triple $\mathtt{O}rb = (\mathtt{X}_{\mathtt{O}rb}, \cup_i \mathtt{U}_i, \mathtt{G}_i)$ of a Hausdorff space $\mathtt{X}_{\mathtt{O}rb}$, a covering $\cup_i \mathtt{U}_i$ of it (closed under finite intersections) and a collection of finite groups $\mathtt{G}_i$ is called an orbifold (of dimension m) if for each $\mathtt{U}_i$ there exists a open subset $\mathtt{V}_i \subset \mathbb{R}^m$ carrying an action of $\mathtt{G}_i$, and some compatibility conditions.

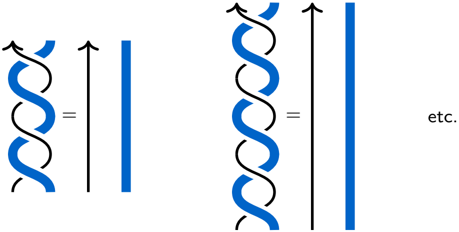
Fact. A two-dimensional ("smooth") orbifold is locally modeled on:

- Cone points $\leftrightsquigarrow$ rotation action of $\mathbb{Z}/l\mathbb{Z}$.
- Reflector corners $\leftrightsquigarrow$ reflection action of the dihedral group.
- Mirror points $\leftrightsquigarrow$ reflection action of $\mathbb{Z}/2\mathbb{Z}$.

Back

Satake $\sim$1956 (" … **r $\sim$1990**
("orbihedron"), e … orff space X_{Orb}, a
covering $\cup_i U_i$ of it … ection of finite
groups G_i is called … here exists a open
subset $V_i \subset \mathbb{R}^m$ ca … y conditions.

Examples.



A $\mathbb{Z}/2\mathbb{Z}$-orbifold tangle   A $\mathbb{Z}/3\mathbb{Z}$-orbifold tangle

"Puncture $= \lim_{l\to\infty}$ l-cone point".

Fact. A two-dime … d on:

- ▷ Cone points ⇜ …
- ▷ Reflector corners ⇝ reflection action of the dihedral group.
- ▷ Mirror points ⇝ reflection action of $\mathbb{Z}/2\mathbb{Z}$.

Back

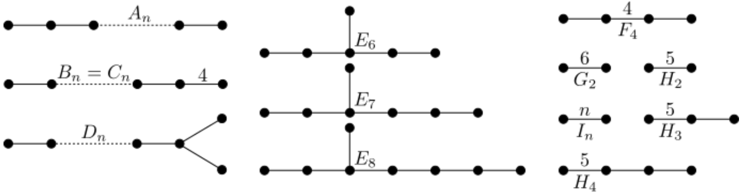


Figure: The Coxeter graphs of finite type.

Example. The type A family is given by the symmetric groups using the simple transpositions as generators.

(Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)

Back

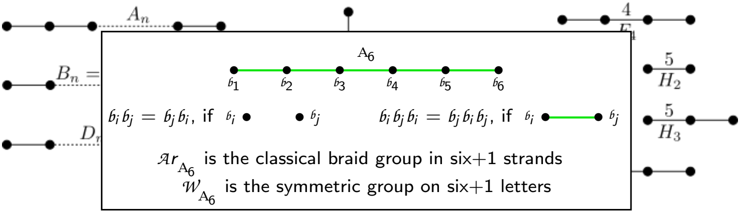
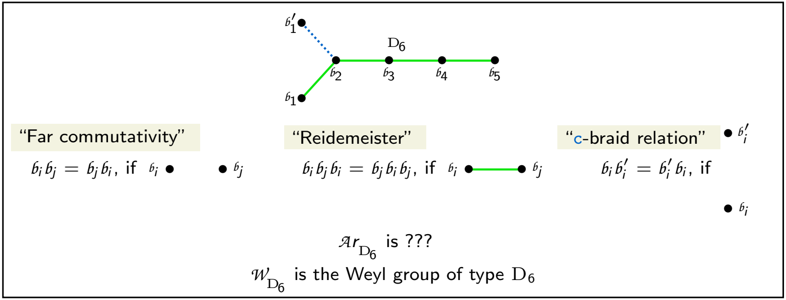


Figure: The Coxeter graphs of finite type.

Example. The type A family is given by the symmetric groups using the simple transpositions as generators.

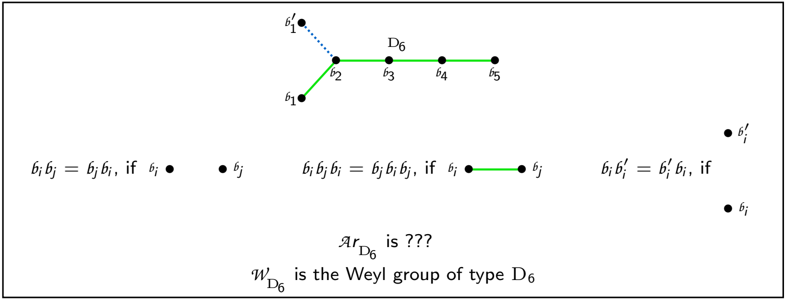
(Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)



Example. The type A family is given by the symmetric groups using the simple transpositions as generators.

(Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)

Back



Example. The type A family is given by the symmetric groups using the simple transpositions a I want to answer ??? in this case, and partially in general.

(Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)

Back

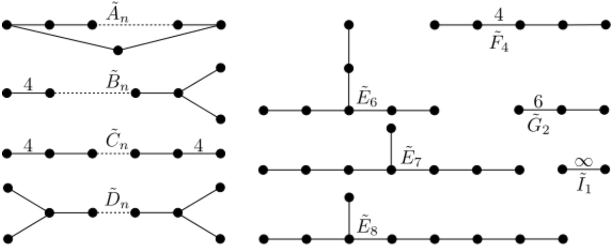
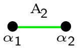


Figure: The Coxeter graphs of affine type.

Example. The type $\widetilde{\mathrm{A}}_n$ corresponds to the affine Weyl group for $\mathfrak{sl}_n$.

(Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)

Back

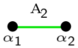


positive root	$\alpha_1 = (1, -1, 0)$	$\alpha_2 = (0, 1, -1)$	$\alpha_1 + \alpha_2 = (1, 0, -1)$
reflection action	$x_1 \leftrightarrow x_2$	$x_2 \leftrightarrow x_3$	$x_1 \leftrightarrow x_3$
$\perp$-hyperplane	$\{(x, x, 0)\}$	$\{(0, y, y)\}$	$\{(z, 0, z)\}$

Hyperplane equations: $\{(x, y, z) \in (\mathbb{R}^2)^3 \mid x = y \text{ or } y = z \text{ or } x = z\}$

This is $\mathfrak{gl}$-notation.

Back

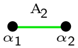


positive root	$\alpha_1 = (1, -1, 0)$	$\alpha_2 = (0, 1, -1)$	$\alpha_1 + \alpha_2 = (1, 0, -1)$
reflection action	$x_1 \leftrightarrow x_2$	$x_2 \leftrightarrow x_3$	$x_1 \leftrightarrow x_3$
$\perp$-hyperplane	$\{(x, x, 0)\}$	$\{(0, y, y)\}$	$\{(z, 0, z)\}$

Hyperplane equations: $\{(x, y, z) \in (\mathbb{R}^2)^3 \mid x = y \text{ or } y = z \text{ or } x = z\}$

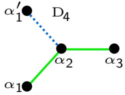
Observe that this matches the diagonal of the configuration space picture.

Back



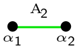
positive root	$\alpha_1 = (1,-1,0)$	$\alpha_2 = (0,1,-1)$	$\alpha_1+\alpha_2 = (1,0,-1)$
reflection action	$x_1 \leftrightarrow x_2$	$x_2 \leftrightarrow x_3$	$x_1 \leftrightarrow x_3$
$\perp$-hyperplane	$\{(x,x,0)\}$	$\{(0,y,y)\}$	$\{(z,0,z)\}$

Hyperplane equations: $\{(x,y,z) \in (\mathbb{R}^2)^3 \mid x = y \text{ or } y = z \text{ or } x = z\}$



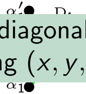
positive root	$\alpha_1' = (1,1,0)$	$\alpha_1 = (1,-1,0)$	more "type A-like"
reflection action	$x_1', x_1 \leftrightarrow -x_1', -x_1$	$x_1 \leftrightarrow x_2$	more "type A-like"
$\perp$-hyperplane	$\{(x,-x,0,0)\}$	$\{(x,x,0,0)\}$	more "type A-like"

Hyperplane equations: $\{(x,y,z,w) \in \mathbb{C}^4 \mid x = \pm y \text{ etc.}\}$



positive root	$\alpha_1 = (1,-1,0)$	$\alpha_2 = (0,1,-1)$	$\alpha_1{+}\alpha_2 = (1,0,-1)$
reflection action	$x_1 \leftrightarrow x_2$	$x_2 \leftrightarrow x_3$	$x_1 \leftrightarrow x_3$
$\perp$-hyperplane	$\{(x,x,0)\}$	$\{(0,y,y)\}$	$\{(z,0,z)\}$

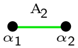
Hyperplane equations: $\{(x,y,z) \in (\mathbb{R}^2)^3 \mid x = y \text{ or } y = z \text{ or } x = z\}$



Observe that this matches the diagonal of the configuration space picture up to a 2-fold covering $(x,y,z,w) \mapsto (x^2,y^2,z^2,w^2)$.

positive root	$\alpha'_1 = (1,1,0)$	$\alpha_1 = (1,-1,0)$	more "type A-like"
reflection action	$x'_1, x_1 \leftrightarrow -x'_1, -x_1$	$x_1 \leftrightarrow x_2$	more "type A-like"
$\perp$-hyperplane	$\{(x,-x,0,0)\}$	$\{(x,x,0,0)\}$	more "type A-like"

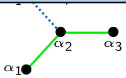
Hyperplane equations: $\{(x,y,z,w) \in \mathbb{C}^4 \mid x = \pm y \text{ etc.}\}$



positive root	$\alpha_1 = (1, -1, 0)$	$\alpha_2 = (0, 1, -1)$	$\alpha_1 + \alpha_2 = (1, 0, -1)$
reflection action	$x_1 \leftrightarrow x_2$	$x_2 \leftrightarrow x_3$	$x_1 \leftrightarrow x_3$
$\perp$-hyperplane	$\{(x, x, 0)\}$	$\{(0, y, y)\}$	$\{(z, 0, z)\}$

Hyperplane equations: $\{(x, y, z) \in (\mathbb{R}^2)^3 \mid x = y \text{ or } y = z \text{ or } x = z\}$

Similarly in (affine) types ABCD.



positive root	$\alpha_1' = (1, 1, 0)$	$\alpha_1 = (1, -1, 0)$	more "type A-like"
reflection action	$x_1', x_1 \leftrightarrow -x_1', -x_1$	$x_1 \leftrightarrow x_2$	more "type A-like"
$\perp$-hyperplane	$\{(x, -x, 0, 0)\}$	$\{(x, x, 0, 0)\}$	more "type A-like"

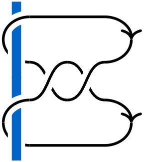
Hyperplane equations: $\{(x, y, z, w) \in \mathbb{C}^4 \mid x = \pm y \text{ etc.}\}$

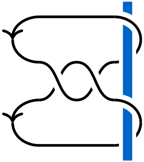
Noumi–Sugitani ∼1994, Letzter ∼1999. Quantum groups have few Hopf subalgebras, but plenty of coideal subalgebras.

$\mathtt{c}\mathcal{U}_{\mathrm{v}}$ is not a Hopf algebra, but rather a right coideal (subalgebra) of $\mathcal{U}_{\mathrm{v}}$:

$$\Delta(\mathtt{B}) = \mathtt{B} \otimes \underbrace{\mathtt{K}^{-1}}_{\notin \mathtt{c}\mathcal{U}_{\mathrm{v}}} + 1 \otimes \mathtt{B} \in \mathtt{c}\mathcal{U}_{\mathrm{v}} \otimes \mathcal{U}_{\mathrm{v}},$$

which gives $\mathcal{R}ep(\mathtt{c}\mathcal{U}_{\mathrm{v}})$ the structure of a right $\mathcal{R}ep(\mathcal{U}_{\mathrm{v}})$-category ⇒ right handedness of diagrams, e.g.:


Ok from this picture


Not ok from this picture

◂ Back

Noumi–Sugitani ∼1994, Letzter ∼1999. Quantum groups have few Hopf subalgebras, but plenty of coideal subalgebras.

$c\mathcal{U}_{\mathrm{v}}$ is not a Hopf algebra, but rather a right coideal (subalgebra) of $\mathcal{U}_{\mathrm{v}}$:

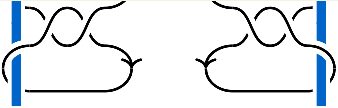
Example. The vector representations of $\mathfrak{gl}_n$, $\mathfrak{so}_n$ and $\mathfrak{sp}_n$ all agree, and indeed
$\mathfrak{so}_n \hookrightarrow \mathfrak{gl}_n$ and $\mathfrak{sp}_n \hookrightarrow \mathfrak{gl}_n$.

But the quantum vector representations do not agree, i.e.
$\mathcal{U}_{\mathrm{v}}(\mathfrak{so}_n) \not\hookrightarrow \mathcal{U}_{\mathrm{v}}(\mathfrak{gl}_n)$ and $\mathcal{U}_{\mathrm{v}}(\mathfrak{sp}_n) \not\hookrightarrow \mathcal{U}_{\mathrm{v}}(\mathfrak{gl}_n)$.

This is bad. Idea: Invent new quantizations such that
$\mathcal{U}'_{\mathrm{v}}(\mathfrak{so}_n) \hookrightarrow \mathcal{U}_{\mathrm{v}}(\mathfrak{gl}_n)$ and $\mathcal{U}'_{\mathrm{v}}(\mathfrak{sp}_n) \hookrightarrow \mathcal{U}_{\mathrm{v}}(\mathfrak{gl}_n)$.

w

h



Ok from this picture

Not ok from this picture

Back

Noumi–Sugitani ∼1994, Letzter ∼1999. Quantum groups have few Hopf subalgebras, but plenty of coideal subalgebras.

---

c$\mathcal{U}_{\mathrm{v}}$ is not a Hopf algebra, but rather a right coideal (subalgebra) of $\mathcal{U}_{\mathrm{v}}$:

Example. The vector representations of $\mathfrak{gl}_n$, $\mathfrak{so}_n$ and $\mathfrak{sp}_n$ all agree, and indeed

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But the quantum vector representations do not agree, i.e.

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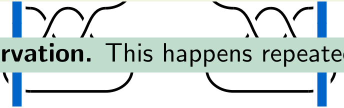
This is bad. Idea: Invent new quantizations such that

$\mathcal{U}'_{\mathrm{v}}(\mathfrak{so}_n) \hookrightarrow \mathcal{U}_{\mathrm{v}}(\mathfrak{gl}_n)$ and $\mathcal{U}'_{\mathrm{v}}(\mathfrak{sp}_n) \hookrightarrow \mathcal{U}_{\mathrm{v}}(\mathfrak{gl}_n)$.



Observation. This happens repeatedly.

Ok from this picture

Not ok from this picture

Back

Noumi–Sugitani ∼1994, Letzter ∼1999. Quantum groups have few Hopf subalgebras, but plenty of coideal subalgebras.

$\mathtt{c}\mathcal{U}_{\mathrm{v}}$ is not a Hopf algebra, but rather a right coideal (subalgebra) of $\mathcal{U}_{\mathrm{v}}$:

$$\Delta(\mathtt{B}) = \mathtt{B} \otimes \underbrace{\mathtt{K}^{-1}}_{\notin \mathtt{c}\mathcal{U}_{\mathrm{v}}} + 1 \otimes \mathtt{B} \in \mathtt{c}\mathcal{U}_{\mathrm{v}} \otimes \mathcal{U}_{\mathrm{v}},$$

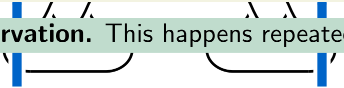
which gives … right

handedness …

This happens really often. In our case we have basically

$$\mathfrak{gl}_1 \hookrightarrow \mathfrak{sl}_2, (t) \mapsto \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$$

which does not quantize properly...



Observation. This happens repeatedly.

Ok from this picture

Not ok from this picture

◂ Back

A version of Schur's remarkable duality.

Plain old $\mathfrak{sl}_2$:
Acts by matrices.

The symmetric group:
Acts by permutation.

$$\mathcal{U}_1(\mathfrak{sl}_2) \circlearrowright \underbrace{\mathtt{V}_1 \otimes \cdots \otimes \mathtt{V}_1}_{d \text{ times}} \circlearrowleft \mathcal{H}_1(\mathrm{A})$$

Schur ∼1901. The natural actions of $\mathcal{U}_1(\mathfrak{sl}_2)$ and $\mathcal{H}_1(\mathrm{A})$ on $\mathtt{V}_1^{\otimes d} = (\mathbb{C}^2)^{\otimes d}$ commute and generate each other's centralizer.

Back

A version of Schur's remarkable duality.

$$\mathcal{U}_1(\mathfrak{sl}_2) \circlearrowright \mathtt{V}_1 \otimes \cdots \otimes \mathtt{V}_1 \circlearrowleft \mathcal{H}_1(\mathrm{A})$$

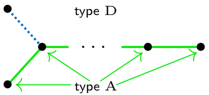
$$||$$

$$\underbrace{\mathtt{V}_1 \otimes \cdots \otimes \mathtt{V}_1}_{d \text{ times}}$$

Back

A version of Schur's remarkable duality.

Weyl group of type D:



$$\mathcal{U}_1(\mathfrak{sl}_2) \circlearrowright \mathtt{V}_1 \otimes \cdots \otimes \mathtt{V}_1 \circlearrowleft \mathcal{H}_1(\mathrm{A})$$

$$\| \qquad\qquad \cap$$

$$\underbrace{\mathtt{V}_1 \otimes \cdots \otimes \mathtt{V}_1}_{d \text{ times}} \qquad \mathcal{H}_1(\mathrm{D}) \rtimes {}^{\mathbb{Z}}/_{2\mathbb{Z}}$$

Ignore the component group ${}^{\mathbb{Z}}/_{2\mathbb{Z}}$.

Back

A version of Schur's remarkable duality.

$$\mathcal{U}_1(\mathfrak{sl}_2) \circlearrowright \underbrace{\mathtt{V}_1 \otimes \cdots \otimes \mathtt{V}_1}_{d \text{ times}} \circlearrowleft \mathcal{H}_1(\mathrm{A})$$

$$\mathtt{V}_1 \otimes \cdots \otimes \mathtt{V}_1 \;\|\; \mathtt{V}_1 \otimes \cdots \otimes \mathtt{V}_1, \qquad \mathcal{H}_1(\mathrm{A}) \cap \mathcal{H}_1(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z}$$

$$\underbrace{\mathtt{V}_1 \otimes \cdots \otimes \mathtt{V}_1}_{d \text{ times}} \circlearrowleft \mathcal{H}_1(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z}$$

Acts by signed permutations.

Back

A version of Schur's remarkable duality.

$$
\begin{array}{ccccc}
\mathcal{U}_1(\mathfrak{sl}_2) & \circlearrowright & \mathtt{V}_1 \otimes \cdots \otimes \mathtt{V}_1 & \circlearrowleft & \mathcal{H}_1(\mathrm{A}) \\
\cup & & \| & & \cap \\
?? & \circlearrowright & \underbrace{\mathtt{V}_1 \otimes \cdots \otimes \mathtt{V}_1}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_1(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z}
\end{array}
$$

◂ Back

A version of Schur's remarkable duality.

The antidiagonal embedding:

$\mathfrak{gl}_1 \hookrightarrow \mathfrak{sl}_2, \ (t) \mapsto \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$

$$\begin{array}{ccccc} \mathcal{U}_1(\mathfrak{sl}_2) & \circlearrowright & \mathtt{V}_1 \otimes \cdots \otimes \mathtt{V}_1 & \circlearrowleft & \mathcal{H}_1(\mathrm{A}) \\ \cup & & \| & & \cap \\ \mathcal{U}_1(\mathfrak{gl}_1) & \circlearrowright & \underbrace{\mathtt{V}_1 \otimes \cdots \otimes \mathtt{V}_1}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_1(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \end{array}$$

Acts by restriction.

Regev ∼1983. The actions of $\mathcal{U}_1(\mathfrak{gl}_1)$ and $\mathcal{H}_1(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z}$ on $\mathtt{V}_1^{\otimes d}$ commute and generate each other's centralizer.

A version of Schur's remarkable duality.

$$\mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2) \circlearrowright \mathtt{V}_{\mathrm{v}} \otimes \cdots \otimes \mathtt{V}_{\mathrm{v}} \circlearrowleft \mathcal{H}_{\mathrm{v}}(\mathrm{A})$$

Jimbo ∼1985. The natural actions of $\mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2)$ and $\mathcal{H}_{\mathrm{v}}(\mathrm{A})$ on $\mathtt{V}_{\mathrm{v}}^{\otimes d} = (\mathbb{C}(\mathrm{v})^2)^{\otimes d}$ commute and generate each other's centralizer.

Back

A version of Schur's remarkable duality.

$$\mathcal{U}_{\mathtt{v}}(\mathfrak{sl}_2) \circlearrowright \mathtt{V}_{\mathtt{v}} \otimes \cdots \otimes \mathtt{V}_{\mathtt{v}} \circlearrowleft \mathcal{H}_{\mathtt{v}}(\mathrm{A})$$

$$\|$$

$$\underbrace{\mathtt{V}_{\mathtt{v}} \otimes \cdots \otimes \mathtt{V}_{\mathtt{v}}}_{d \text{ times}}$$

Back

A version of Schur's remarkable duality.

$$
\begin{array}{ccccc}
\mathcal{U}_{\mathtt{v}}(\mathfrak{sl}_2) \circlearrowright & \mathtt{V}_{\mathtt{v}} \otimes \cdots \otimes \mathtt{V}_{\mathtt{v}} & \circlearrowleft & \mathcal{H}_{\mathtt{v}}(\mathrm{A}) \\
 & \| & & \cap \\
 & \underbrace{\mathtt{V}_{\mathtt{v}} \otimes \cdots \otimes \mathtt{V}_{\mathtt{v}}}_{d \text{ times}} & & \mathcal{H}_{\mathtt{v}}(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z}
\end{array}
$$

Back

A version of Schur's remarkable duality.

$$\begin{array}{ccccc} \mathcal{U}_{\mathtt{v}}(\mathfrak{sl}_2) \circlearrowright & \mathtt{V}_{\mathtt{v}} \otimes \cdots \otimes \mathtt{V}_{\mathtt{v}} & \circlearrowleft & \mathcal{H}_{\mathtt{v}}(\mathrm{A}) \\ & \| & & \cap \\ & \underbrace{\mathtt{V}_{\mathtt{v}} \otimes \cdots \otimes \mathtt{V}_{\mathtt{v}}}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_{\mathtt{v}}(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \end{array}$$

Quantizes nicely.

Back

A version of Schur's remarkable duality.

$$\begin{array}{ccccc} \mathcal{U}_{\mathtt{v}}(\mathfrak{sl}_2) & \circlearrowright & \mathtt{V}_{\mathtt{v}} \otimes \cdots \otimes \mathtt{V}_{\mathtt{v}} & \circlearrowleft & \mathcal{H}_{\mathtt{v}}(\mathrm{A}) \\ \cup & & \| & & \cap \\ ?? & \circlearrowright & \underbrace{\mathtt{V}_{\mathtt{v}} \otimes \cdots \otimes \mathtt{V}_{\mathtt{v}}}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_{\mathtt{v}}(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \end{array}$$

Back

A version of Schur's remarkable duality.

$$\begin{array}{ccccc} \mathcal{U}_{\mathtt{v}}(\mathfrak{sl}_2) & \circlearrowright & \mathtt{V}_{\mathtt{v}} \otimes \cdots \otimes \mathtt{V}_{\mathtt{v}} & \circlearrowleft & \mathcal{H}_{\mathtt{v}}(\mathrm{A}) \\ \cup & & \| & & \cap \\ \mathcal{U}_{\mathtt{v}}(\mathfrak{gl}_1) & \circlearrowright & \underbrace{\mathtt{V}_{\mathtt{v}} \otimes \cdots \otimes \mathtt{V}_{\mathtt{v}}}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_{\mathtt{v}}(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \end{array}$$

Back

A version of Schur's remarkable duality.

$$\begin{array}{ccccc} \mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2) & \circlearrowright & \mathtt{V}_{\mathrm{v}} \otimes \cdots \otimes \mathtt{V}_{\mathrm{v}} & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{A}) \\ \not\cup & & \| & & \cap \\ \mathcal{U}_{\mathrm{v}}(\mathfrak{gl}_1) & \circlearrowright & \underbrace{\mathtt{V}_{\mathrm{v}} \otimes \cdots \otimes \mathtt{V}_{\mathrm{v}}}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \end{array}$$

Does not embed.

Back

A version of Schur's remarkable duality.

$$\begin{array}{ccccc} \mathcal{U}_v(\mathfrak{sl}_2) & \circlearrowright & \mathtt{V}_v \otimes \cdots \otimes \mathtt{V}_v & \circlearrowleft & \mathcal{H}_v(\mathrm{A}) \\ \not\cup & & \| & & \cap \\ \mathcal{U}_v(\mathfrak{gl}_1) & \not\circlearrowright & \underbrace{\mathtt{V}_v \otimes \cdots \otimes \mathtt{V}_v}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_v(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \end{array}$$

No commuting action.

Back

A version of Schur's remarkable duality.

$$
\begin{array}{ccccc}
\mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2) & \circlearrowright & \mathtt{V}_{\mathrm{v}} \otimes \cdots \otimes \mathtt{V}_{\mathrm{v}} & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{A}) \\
\not\cup & & \| & & \cap \\
\cancel{\mathcal{U}_{\mathrm{v}}(\mathfrak{gl}_1)} & \not\circlearrowright & \underbrace{\mathtt{V}_{\mathrm{v}} \otimes \cdots \otimes \mathtt{V}_{\mathrm{v}}}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{D}) \rtimes {}^{\mathbb{Z}}/_{2\mathbb{Z}}
\end{array}
$$

◂ Back

A version of Schur's remarkable duality.

$$\begin{array}{ccccc} \mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2) \circlearrowright & \mathtt{V}_{\mathrm{v}} \otimes \cdots \otimes \mathtt{V}_{\mathrm{v}} & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{A}) \\ & \| & & \cap \\ {\color{blue} \mathrm{c}\,\mathcal{U}_{\mathrm{v}}(\mathfrak{gl}_1)} & \underbrace{\mathtt{V}_{\mathrm{v}} \otimes \cdots \otimes \mathtt{V}_{\mathrm{v}}}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \end{array}$$

◂ Back

A version of Schur's remarkable duality.

$$\begin{array}{ccccc} \mathcal{U}_v(\mathfrak{sl}_2) & \circlearrowright & \mathtt{V}_v \otimes \cdots \otimes \mathtt{V}_v & \circlearrowleft & \mathcal{H}_v(\mathrm{A}) \\ \cup & & \| & & \cap \\ \mathrm{c}\,\mathcal{U}_v(\mathfrak{gl}_1) & & \underbrace{\mathtt{V}_v \otimes \cdots \otimes \mathtt{V}_v}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_v(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \end{array}$$

Is a subalgebra.

Back

A version of Schur's remarkable duality.

$$\mathcal{U}_v(\mathfrak{sl}_2) \circlearrowright \mathtt{V}_v \otimes \cdots \otimes \mathtt{V}_v \circlearrowleft \mathcal{H}_v(\mathrm{A})$$

$$\cup \qquad\qquad || \qquad\qquad \cap$$

$$c\,\mathcal{U}_v(\mathfrak{gl}_1) \circlearrowright \underbrace{\mathtt{V}_v \otimes \cdots \otimes \mathtt{V}_v}_{d \text{ times}} \circlearrowleft \mathcal{H}_v(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z}$$

Act by restriction.

Back

A version of Schur's remarkable duality.

$$\begin{array}{ccc} \mathcal{U}_{\mathtt{v}}(\mathfrak{sl}_2) \circlearrowright & \mathtt{V}_{\mathtt{v}} \otimes \cdots \otimes \mathtt{V}_{\mathtt{v}} & \circlearrowleft \mathcal{H}_{\mathtt{v}}(\mathrm{A}) \\ \cup & \| & \cap \\ \mathtt{c}\,\mathcal{U}_{\mathtt{v}}(\mathfrak{gl}_1) \circlearrowright & \underbrace{\mathtt{V}_{\mathtt{v}} \otimes \cdots \otimes \mathtt{V}_{\mathtt{v}}}_{d \text{ times}} & \circlearrowleft \mathcal{H}_{\mathtt{v}}(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \end{array}$$

Back

Ehrig–Stroppel, Bao–Wang ∼2013. The actions of $\mathtt{c}\,\mathcal{U}_{\mathtt{v}}(\mathfrak{gl}_1)$ and $\mathcal{H}_{\mathtt{v}}(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z}$ on $\mathtt{V}_{\mathtt{v}}^{\otimes d}$ commute and generate each other's centralizer.

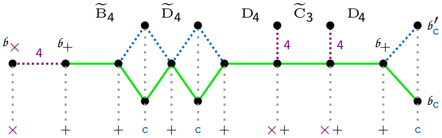
A version of Schur's remarkable duality.

$$\mathcal{U}_{\mathtt{v}}(\mathfrak{sl}_2) \circlearrowright \mathtt{V}_{\mathtt{v}} \otimes \cdots \otimes \mathtt{V}_{\mathtt{v}} \circlearrowleft \mathcal{H}_{\mathtt{v}}(\mathrm{A})$$

Hope.

The same works for the Coxeter diagrams

$$\Gamma = \Gamma((\widetilde{\mathrm{B}}_4, \widetilde{\mathrm{D}}_4, \mathrm{D}_4, \widetilde{\mathrm{C}}_3, \mathrm{D}_4))$$



$$\mathtt{x}(\Gamma) = \times + + \mathtt{c} + \mathtt{c} + \times + \times + + \mathtt{c} +$$

But, again, only in the special case of type ABCD this is known.

A version of Schur's remarkable duality.

$$\begin{array}{ccccc} \mathcal{U}_{\mathtt{v}}(\mathfrak{sl}_2) & \circlearrowright & \mathtt{V}_{\mathtt{v}} \otimes \cdots \otimes \mathtt{V}_{\mathtt{v}} & \circlearrowleft & \mathcal{H}_{\mathtt{v}}(\mathrm{A}) \\ \cup & & \| & & \cap \\ \mathtt{c}\,\mathcal{U}_{\mathtt{v}}(\mathfrak{gl}_1) & \circlearrowright & \underbrace{\mathtt{V}_{\mathtt{v}} \otimes \cdots \otimes \mathtt{V}_{\mathtt{v}}}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_{\mathtt{v}}(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \end{array}$$

◂ Back

Message to take away. Coideal naturally appear in Schur–Weyl-like games.

And these pull the strings from the background for tangle and link invariants.