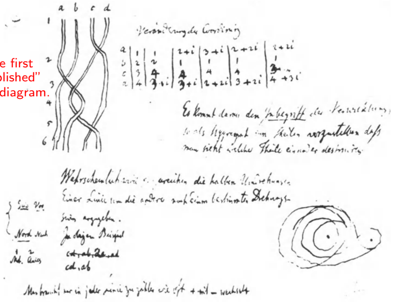
All I know about Artin–Tits groups

Or: Why type A is so much easier...

Daniel Tubbenhauer

The first "published" braid diagram.



(Page 283 from Gauß' handwritten notes, volume seven, $\leq$1830).

Joint with David Rose

March 2020

Let Γ be a Coxeter graph.

Artin ∼1925, Tits ∼1961++. The (Gauß–)Artin–Tits group and its Coxeter group quotient are given by generators-relations:

$$\mathrm{AT}(\Gamma) = \langle b_i \mid \underbrace{\cdots b_i b_j b_i}_{m_{ij}\text{ factors}} = \underbrace{\cdots b_j b_i b_j}_{m_{ij}\text{ factors}} \rangle$$

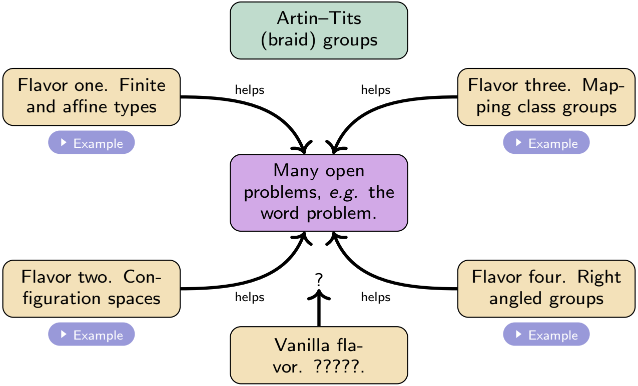
$$\downarrow$$

$$\mathrm{W}(\Gamma) = \langle \sigma_i \mid \sigma_i^2 = 1, \underbrace{\cdots \sigma_i \sigma_j \sigma_i}_{m_{ij}\text{ factors}} = \underbrace{\cdots \sigma_j \sigma_i \sigma_j}_{m_{ij}\text{ factors}} \rangle$$

Artin–Tits groups generalize classical braid groups, Coxeter groups generalize polyhedron groups.

My failure. What I would like to understand, but I do not.

Artin–Tits groups come in four main flavors.
Question: What happens in general type?



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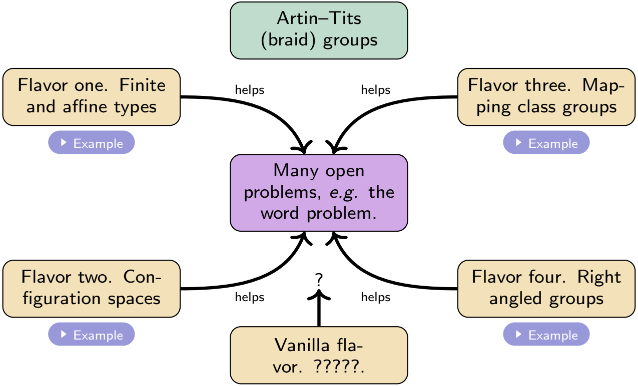

Maybe some categorical considerations help?
In particular, what can Artin–Tits groups tell you about flavor two?

Please stop!

Let Γ be a Coxeter graph. The following commuting diagram exists in any type:

$$
\begin{array}{ccc}
\mathrm{AT}(\Gamma) & \xlongequal{\quad\quad} & \mathrm{AT}(\Gamma) \\
[\![-]\!] \circlearrowright & & \circlearrowright [-] \\
\mathcal{K}^b(\mathscr{S}^{\mathbf{q}}(\Gamma)) & \xrightarrow{\text{decat.}} & \mathcal{H}^{\mathbf{q}}(\Gamma) \\
\downarrow & & \downarrow \\
\mathcal{K}^b(\mathscr{Z}^{\mathbf{q}}(\Gamma)) & \xrightarrow{\text{decat.}} & \mathcal{B}^{\mathbf{q}}(\Gamma)
\end{array}
$$

Question. How does this help to study Artin–Tits groups?

Here (killing idempotents for the last row):

- Hecke algebra $\mathcal{H}^{\mathbf{q}}(\Gamma)$, homotopy category of Soergel bimodules $\mathcal{K}^b(\mathscr{S}^{\mathbf{q}}(\Gamma))$.
- Hecke action $[\_]$, Rouquier complex $[\![\_]\!]$.
- Burau representation $\mathcal{B}^{\mathbf{q}}(\Gamma)$, homotopy category of representations of zigzag algebras $\mathcal{K}^b(\mathscr{Z}^{\mathbf{q}}(\Gamma))$.

Let Γ be a Coxeter graph. The following commuting diagram exists in any type:

$$\mathrm{AT}(\Gamma) = \mathrm{AT}(\Gamma)$$

Faithfulness?

The Hecke action is known to be faithful in very few cases, e.g. for Γ of rank $1, 2$. But there is "no way" to prove faithfulness in general.

Example (seems to work). Hecke distinguishes the braids where Burau failed:

```
sage: R.<q> = LaurentPolynomialRing(ZZ)
sage: H = IwahoriHeckeAlgebra('A5', q, -q^-1)
sage: T=H.T();
sage: psi1 = T[4] * T[5]^(-1) * T[2]^(-1) * T[1]
sage: psi2 = T[4]^(-1) * T[5]^(2) * T[2] * T[1]^(-2)
sage: Psi1 = T[1]^(-1) * T[2] * T[5] * T[4]^(-1)
sage: Psi2 = T[1]^(2) * T[2]^(-1) * T[5]^(-2) * T[4]
sage: w1 = Psi1 * T[3] * psi1
sage: w2 = Psi2 * T[3] * psi2
sage: W1 = Psi1 * T[3]^(-1) * psi1
sage: W2 = Psi2 * T[3]^(-1) * psi2
sage: w1 * w2 * W1 * W2
```

evaluate

```
WARNING: Output truncated!
full_output.txt

-(q^-21-10*q^-19+50*q^-17-168*q^-15+428*q^-13-882*q^-11+1531*q^-9-2303*q\
^-7+3067*q^-5-3676*q^-3+4012*q^-1-4012*q+3676*q^3-3067*q^5+2303*q^7-1531\
*q^9+882*q^11-428*q^13+168*q^15-50*q^17+10*q^19-q^21)*T[1,2,3,4,5,1,2,3,\
4,1,2,3,1,2,1] +
```

Q

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algebras $\mathcal{K}^{\oplus}(\mathcal{Z}^{\cdot}(\Gamma))$.

Let Γ be a Coxeter graph. The following commuting diagram exists in any type:

$$\mathrm{AT}(\Gamma) = \mathrm{AT}(\Gamma)$$

Faithfulness?

Rouquier's action is known to be faithful in quite a few cases:
finite type (Khovanov–Seidel, Brav–Thomas),
affine type A (Gadbled–Thiel–Wagner), affine type C (handlebody).

Example (the whole point). Zigzag already distinguishes braids:

```
sage: R.<t,q> = LaurentPolynomialRing(ZZ);
sage: psi1 = z4 * z5^(-1) * z2^(-1) * z1
sage: psi2 = z4^(-1) * z5^(2) * z2 *z1^(-2)
sage: w1 = psi1^(-1) * z3 * psi1
sage: w2 = psi2^(-1) * z3 * psi2
sage: (w1 * w2 * w1^(-1) * w2^(-1)).substitute(t=-1), (w1 * w2 * w1^(-1) * w2^(-1)).substitute(t=1,q=-2)
evaluate
(
[1 0 0 0 0]  [ -6900766331/4782969 119949646700/4782969 -27606410000/1594323   -1446875300/59049   10123227400/177147]
[0 1 0 0 0]  [ -6008522000/1594323 104398156073/1594323  -24028111250/531441   -1259219000/19683     8810639500/59049]
[0 0 1 0 0]  [  3077274850/1594323 -53464229650/1594323  12305843941/531441      644883850/19683    -4512158300/59049]
[0 0 0 1 0]  [  2639191750/4782969 -45868537000/4782969  10557771250/1594323     553296799/59049   -3871127000/177147]
[0 0 0 0 1], [ -6175410800/4782969 107290158950/4782969 -24693841250/1594323   -1294131800/59049    9055019047/177147]
)
```

Question

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- Hec … $\mathscr{S}^{\mathsf{q}}(\Gamma))$.
- Hec
- Burau representation $\mathcal{B}^{\mathsf{q}}(\Gamma)$, homotopy category of representations of zigzag algebras $\mathcal{K}^b(\mathscr{Z}^{\mathsf{q}}(\Gamma))$.

Let Γ be a Coxeter graph. The following commuting diagram exists in any type:

$$\mathrm{AT}(\Gamma) \;=\!=\!=\!=\; \mathrm{AT}(\Gamma)$$

Theorem (handlebody faithfulness).

For all g, n, Rouquier's action $[\![\_]\!]$ gives rise to a family of faithful actions

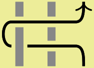
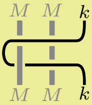
$$\mathscr{B}\mathrm{r}(g,n) \curvearrowright \mathcal{K}^b(\mathscr{S}^{\mathbf{q}}(\Gamma)), \mathscr{b} \mapsto [\![\mathscr{b}]\!]_{\mathtt{M}}.$$

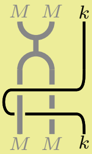
$$\mathcal{K}^b(\mathscr{Z}^{\mathbf{q}}(\Gamma)) \xrightarrow{\text{decat.}} \mathcal{B}^{\mathbf{q}}(\Gamma)$$

Questio

Theorem (handlebody HOMFLYPT homology).

This action extends to a HOMFLYPT invariant for handlebody links.
Mnemonic:

$\mathscr{b} =$  & $[\![\mathscr{b}]\!]_{\mathtt{M}} =$ 

& $[\![\mathscr{b}]\!]_{\mathscr{H}_2} =$ 

Please stop!

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- Hec
- Hec $\mathscr{S}^{\mathbf{q}}(\Gamma))$.
- Bur of zigzag alge

Rouquier ∼2004. The 2-braid group $\mathcal{AT}(\Gamma)$ is $\mathsf{im}(\llbracket \_ \rrbracket) \subset \mathcal{K}^b(\mathscr{S}_{\mathrm{s}}^{\mathbf{q}}(\Gamma))$.

$\Gamma = \mathsf{A}, \mathsf{C}, \tilde{\mathsf{C}} \rightsquigarrow$ category of braid cobordisms $\mathscr{B}_{\mathrm{cob}}(\Gamma)$ in four space.

Fact (well-known?). For Γ of type A, B = C or affine type C we have

$$\mathcal{AT}(\Gamma) = \mathsf{inv}(\mathscr{B}_{\mathrm{cob}}(\Gamma)).$$

Corollary (strictness). We have a categorical action

$$\mathsf{inv}(\mathscr{B}_{\mathrm{cob}}(g,n)) \curvearrowright \mathcal{K}^b(\mathscr{S}^{\mathbf{q}}(\Gamma)), \mathscr{b} \mapsto \llbracket \mathscr{b} \rrbracket, \mathscr{b}_{\mathrm{cob}} \mapsto \llbracket \mathscr{b}_{\mathrm{cob}} \rrbracket.$$

Question (functoriality). Can we lift $\llbracket \_ \rrbracket$ to a categorical action

$$\mathscr{B}_{\mathrm{cob}}(g,n) \curvearrowright \mathcal{K}^b(\mathscr{S}^{\mathbf{q}}(\Gamma))?$$

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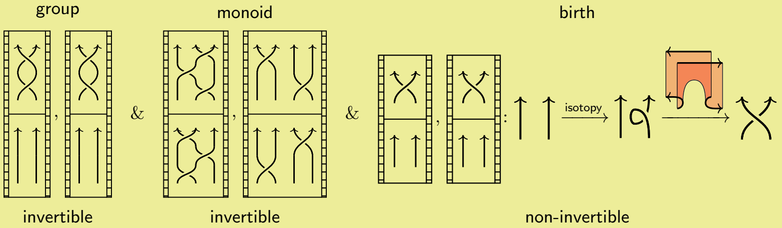
Example (type A).

Braid cobordisms are movies of braids. E.g. some generators are



Invertible ones encode isotopies, non-invertible ones "more interesting" topology.

Question (functoriality). Can we lift $[\![\_]\!]$ to a categorical action

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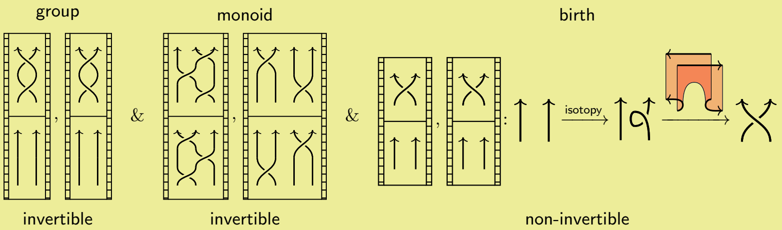

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Theorem (well-known?).

The Rouquier complex is functorial in types A, B = C and affine C.

Rouquier ∼2004. The 2-braid group $\mathcal{AT}(\Gamma)$ is $\mathrm{im}([\![\_]\!]) \subset \mathcal{K}^b(\mathscr{S}_s^{\mathbf{q}}(\Gamma))$.

$\Gamma =$ /

Fact

Theorem (handlebody functoriality).

For all g, n, Rouquier's action $[\![\_]\!]$ gives rise to a family of functorial actions

$$\mathscr{B}_{\mathrm{cob}}(g,n) \curvearrowright \mathcal{K}^b(\mathscr{S}^{\mathbf{q}}(\Gamma)), \mathscr{b} \mapsto [\![\mathscr{b}]\!]_{\mathtt{M}}, \mathscr{b}_{\mathrm{cob}} \mapsto [\![\mathscr{b}_{\mathrm{cob}}]\!]_{\mathtt{M}}.$$

($\mathscr{B}_{\mathrm{cob}}(g,n)$ is the 2-category of handlebody braid cobordisms.)

Coro

$$\mathrm{inv}(\mathscr{B}_{\mathrm{cob}}(g,n)) \curvearrowright \mathcal{K}^b(\mathscr{S}^{\mathbf{q}}(\Gamma)), \mathscr{b} \mapsto [\![\mathscr{b}]\!], \mathscr{b}_{\mathrm{cob}} \mapsto [\![\mathscr{b}_{\mathrm{cob}}]\!].$$

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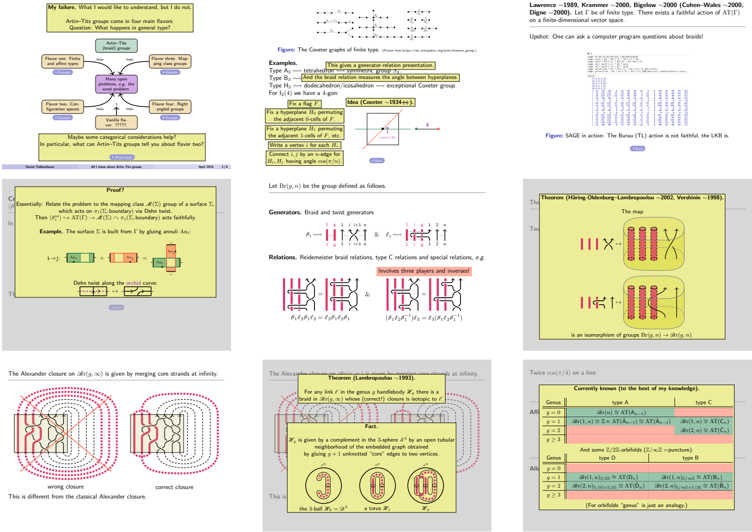
Question (functoriality). ...ical action

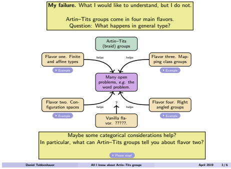
Final observation.

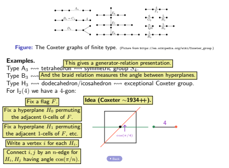
In all (non-trivial) cases I know "faithful ⇔ functorial".

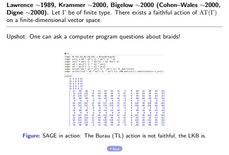
Is there a general statement?

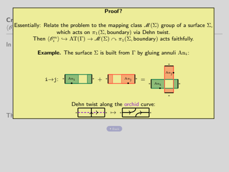


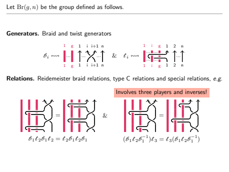
There is still much to do...

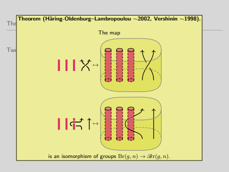


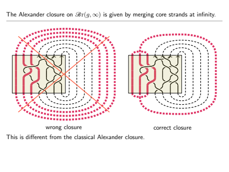


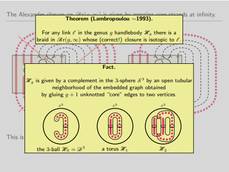


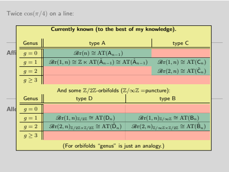












Thanks for your attention!

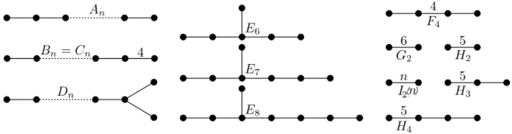
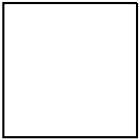
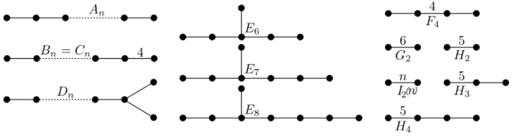


Figure: The Coxeter graphs of finite type. (Picture from https://en.wikipedia.org/wiki/Coxeter_group.)

Examples.
Type A_3 $\leftrightsquigarrow$ tetrahedron $\leftrightsquigarrow$ symmetric group S_4.
Type B_3 $\leftrightsquigarrow$ cube/octahedron $\leftrightsquigarrow$ Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \ltimes S_3$.
Type H_3 $\leftrightsquigarrow$ dodecahedron/icosahedron $\leftrightsquigarrow$ exceptional Coxeter group.
For $I_2(4)$ we have a 4-gon:

Idea (Coxeter ∼1934++).





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Examples.

Type A_3 ⇝ tetra

Fact. The symmetries are given by exchanging flags.

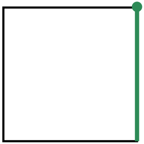
Type B_3 ⇝ cube/octahedron ⇝ Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \ltimes S_3$.

Type H_3 ⇝ dodecahedron/icosahedron ⇝ exceptional Coxeter group.

For $I_2(4)$ we have a 4-gon:

Fix a flag F.

Idea (Coxeter ∼1934++).



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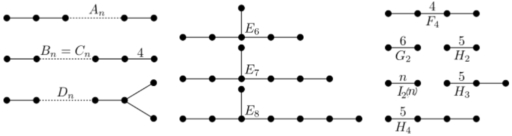

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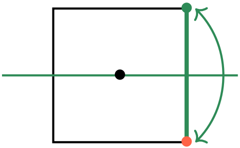
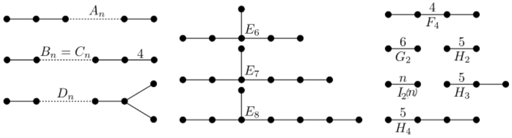
For $\mathrm{I}_2(4)$ we have a 4-gon:

Fix a flag F.

Fix a hyperplane H_0 permuting the adjacent 0-cells of F.

Idea (Coxeter ∼1934++).





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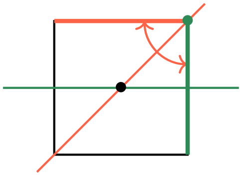

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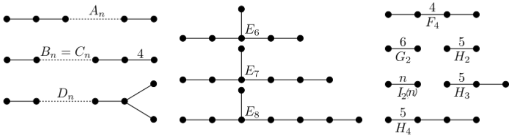

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Fix a flag F.

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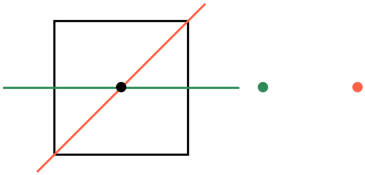
Fix a hyperplane H_1 permuting the adjacent 1-cells of F, etc.

Write a vertex i for each H_i.

Idea (Coxeter ∼1934++).



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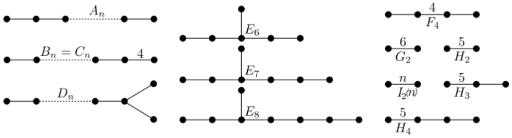

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Examples.

This gives a generator-relation presentation.

Type A_3 ⇝ tetrahedron ⇝ symmetric group S_4.

Type B_3 ⇝

And the braid relation measures the angle between hyperplanes.

Type H_3 ⇝ dodecahedron/icosahedron ⇝ exceptional Coxeter group.

For $I_2(4)$ we have a 4-gon:

Fix a flag F.

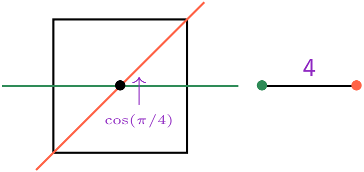
Fix a hyperplane H_0 permuting the adjacent 0-cells of F.

Fix a hyperplane H_1 permuting the adjacent 1-cells of F, etc.

Write a vertex i for each H_i.

Connect i, j by an n-edge for H_i, H_j having angle $\cos(\pi/n)$.

Idea (Coxeter ∼1934++).



Back

Lawrence ∼1989, Krammer ∼2000, Bigelow ∼2000 (Cohen–Wales ∼2000, Digne ∼2000). Let Γ be of finite type. There exists a faithful action of $\mathrm{AT}(\Gamma)$ on a finite-dimensional vector space.

Upshot: One can ask a computer program questions about braids!

```
sage: B.<b1,b2,b3,b4,b5> = BraidGroup(6)
sage: psi1 = b4 * b5^(-1) * b2^(-1) * b1
sage: psi2 = b4^(-1) * b5^(2) * b2 *b1^(-2)
sage: w1 = psi1^(-1) * b3 * psi1
sage: w2 = psi2^(-1) * b3 * psi2
sage: print((w1 * w2 * w1^(-1) * w2^(-1)).TL_matrix(4))
sage: print(((w1 * w2 * w1^(-1) * w2^(-1)).LKB_matrix()).substitute(x=-1,y=1))
```

evaluate

```
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]
[ -15  -80  -80    0  -16  -64  -64   16    0    0   80   64   80   64  -16]
[  32  129  128   32   64   96   96    0   32    0  -96  -64  -96  -64   32]
[ -32 -128 -127  -32  -64  -96  -96    0  -32    0   96   64   96   64  -32]
[  16   80   80    1   16   64   64  -16    0    0  -80  -64  -80  -64   16]
[   0    0    0    0    1    0    0    0    0    0    0    0    0    0    0]
[ -64 -192 -192  -32  -96 -127 -128   32  -32    0  160   96  160   96  -64]
[  64  192  192   32   96  128  129  -32   32    0 -160  -96 -160  -96   64]
[   0    0    0    0    0    0    0    1    0    0    0    0    0    0    0]
[ -16  -80  -80    0  -16  -64  -64   16    1    0   80   64   80   64  -16]
[   0    0    0    0    0    0    0    0    0    1    0    0    0    0    0]
[ -64 -192 -192  -32  -96 -128 -128   32  -32    0  161   96  160   96  -64]
[  32  128  128   32   64   96   96    0   32    0  -96  -63  -96  -64   32]
[  64  192  192   32   96  128  128  -32   32    0 -160  -96 -159  -96   64]
[ -32 -128 -128  -32  -64  -96  -96    0  -32    0   96   64   96   65  -32]
[  16   80   80    0   16   64   64  -16    0    0  -80  -64  -80  -64   17]
```

Figure: SAGE in action: The Burau (TL) action is not faithful, the LKB is.

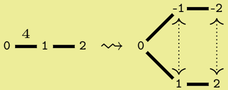
Back

Lawrence ∼1989, Krammer ∼2000, Bigelow ∼2000 (Cohen–Wales ∼2000, Digne ∼2000). Let Γ be of finite type. There exists a faithful action of $\mathrm{AT}(\Gamma)$ on a finite-dimensional vector space.

Proof?

Uses root combinatorics of ADE diagrams
and the fact that each $\mathrm{AT}(\Gamma)$ of finite
type can be embedded in types ADE.

Example. Type B "unfolds" into type A:



$b_0 \mapsto$  and $b_1 \mapsto$  and $b_2 \mapsto$ 

But there is also a different way, discussed later.

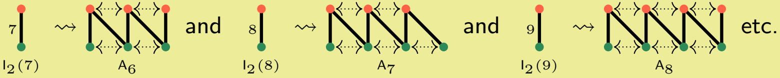
Figure: SAGE in action: The Burau (TL) action is not faithful, the LKB is.

Back

Lawrence ∼1989, Krammer ∼2000, Bigelow ∼2000 (Cohen–Wales ∼2000, Digne ∼2000). Let Γ be of finite type. There exists a faithful action of $\mathrm{AT}(\Gamma)$ on a

Upsh

Example. In the dihedral case these (un)foldings correspond to bicolorings:



Fact.

This gives $\mathrm{AT}(\mathsf{I}_2(n)) \hookrightarrow \mathrm{AT}(\Gamma)$

$\Leftrightarrow$

$\Gamma =$ ADE for $n =$Coxeter number.

```
[0 0 0 1 0]
[0 0 0 0 1]
[ -15  -80  -80    0  -16  -64  -64   16    0    0   80   64   80   64  -16]
[  32  129  128   32   64   96   96    0   32    0  -96  -64  -96  -64   32]
[ -32 -128 -127  -32  -64  -96  -96    0  -32    0   96   64   96   64  -32]
[  16   80   80    1   16   64   64  -16    0    0  -80  -64  -80  -64   16]
[   0    0    0    0    1    0    0    0    0    0    0    0    0    0    0]
[ -64 -192 -192  -32  -96 -127 -128   32  -32    0  160   96  160   96  -64]
[  64  192  192   32   96  128  129  -32   32    0 -160  -96 -160  -96   64]
[   0    0    0    0    0    0    0    1    0    0    0    0    0    0    0]
[ -16  -80  -80    0  -16  -64  -64   16    1    0   80   64   80   64  -16]
[   0    0    0    0    0    0    0    0    0    1    0    0    0    0    0]
[ -64 -192 -192  -32  -96 -128 -128   32  -32    0  161   96  160   96  -64]
[  32  128  128   32   64   96   96    0   32    0  -96  -63  -96  -64   32]
[  64  192  192   32   96  128  128  -32   32    0 -160  -96 -159  -96   64]
[ -32 -128 -128  -32  -64  -96  -96    0  -32    0   96   64   96   65  -32]
[  16   80   80    0   16   64   64  -16    0    0  -80  -64  -80  -64   17]
```

Figure: SAGE in action: The Burau (TL) action is not faithful, the LKB is.

Back

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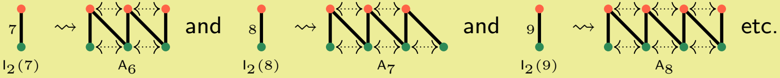


Fact.

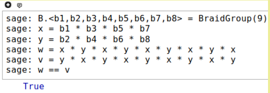
This gives $\mathrm{AT}(\mathrm{I}_2(n)) \hookrightarrow \mathrm{AT}(\Gamma)$

$\Leftrightarrow$

$\Gamma =$ ADE for $n =$Coxeter number.

[0 0 0 1 0]
[0 0 0 0 1]
[-15 -80 -80 0 -16 -64 -64 16 0 0 80 64 80 64 -16]

Example (SAGE; $n = 9$). LKB says it is true:



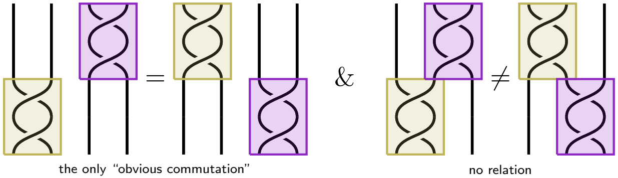
```
sage: B.<b1,b2,b3,b4,b5,b6,b7,b8> = BraidGroup(9)
sage: x = b1 * b3 * b5 * b7
sage: y = b2 * b4 * b6 * b8
sage: w = x * y * x * y * x * y * x * y * x
sage: v = y * x * y * x * y * x * y * x * y
sage: w == v
    True
```

Figure: SAGE in action: The Burau (TL) action is not faithful, the LKB is.

Back

Crisp–Paris ∼2000 (Tits conjecture). For all $m > 1$, the subgroup $\langle \theta_i^m \rangle \subset \mathrm{AT}(\Gamma)$ is free (up to "obvious commutation").

In finite type this is a consequence of LKB; in type A it is clear:



This should have told me something: I will come back to this later.

Back

Cr

In

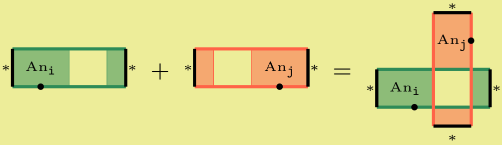
Proof?

Essentially: Relate the problem to the mapping class $\mathscr{M}(\Sigma)$ group of a surface Σ, which acts on $\pi_1(\Sigma, \text{boundary})$ via Dehn twist.

Then $\langle \mathscr{b}_i^m \rangle \hookrightarrow \mathrm{AT}(\Gamma) \to \mathscr{M}(\Sigma) \curvearrowright \pi_1(\Sigma, \text{boundary})$ acts faithfully.

Example. The surface Σ is built from Γ by gluing annuli $\mathrm{An}_\mathtt{i}$:

i→j:



Dehn twist along the orchid curve:

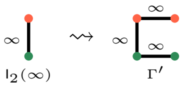
This should have told me something. I will come back to this later.

Back

Recall. Right-angled means $m_{ij} \in \{2, \infty\}$.

Fact (well-known?). Let Γ be of right-angled type. There exists a faithful action of $\mathrm{AT}(\Gamma)$ on a finite-dimensional $\mathbb{R}$-vector space.

Example. $\Gamma = \mathsf{I}_2(\infty)$, the infinite dihedral group.



Define a map

$$\mathrm{AT}(\Gamma) \to \mathrm{W}(\Gamma'), s \mapsto ss, t \mapsto tt.$$

Crazy fact: This is an embedding, and actually

$$\mathrm{W}(\Gamma') \cong \mathrm{AT}(\Gamma) \rtimes (\mathbb{Z}/2\mathbb{Z})^2.$$

Thus, via Tits' reflection representation, it follows that $\mathrm{AT}(\Gamma)$ is linear.

Back

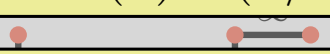
Recall. Right-angled means $m_{ij} \in \{2, \infty\}$.

Fact (well-known?). Let Γ be of right-angled type. There exists a faithful action of $\mathrm{AT}(\Gamma)$ on a fi

Proof?

This works in general:
For each right-angled Γ there exists a Γ' such that
$\mathrm{W}(\Gamma') \cong \mathrm{AT}(\Gamma) \rtimes (\mathbb{Z}/2\mathbb{Z})^i$.

Example. $\Gamma = \mathrm{I}_2$



Corollary.

Tits' reflection representation gives a faithful action on a finite-dimensional $\mathbb{R}$-vector space.

Define a map

$$\mathrm{AT}(\Gamma) \to \mathrm{W}(\Gamma'), s \mapsto ss, t \mapsto tt.$$

Crazy fact: This

This is the only case where I know that the Artin–Tits group embeds into a Coxeter group.

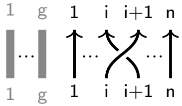
$$\mathrm{W}(\Gamma') \cong \mathrm{AT}(\Gamma) \rtimes (\mathbb{Z}/2\mathbb{Z})^2.$$

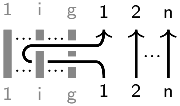
Thus, via Tits' reflection representation, it follows that $\mathrm{AT}(\Gamma)$ is linear.

Back

Let $\mathrm{Br}(g,n)$ be the group defined as follows.

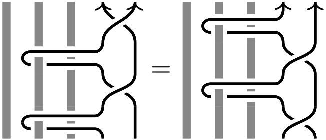
Generators. Braid and twist generators

$b_i \leftrightsquigarrow$ 

& $t_i \leftrightsquigarrow$ 

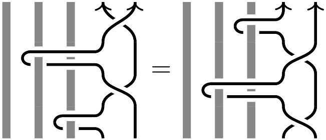
Relations. Reidemeister braid relations, type C relations and special relations, *e.g.*

Involves three players and inverses!



$b_1 t_2 b_1 t_2 = t_2 b_1 t_2 b_1$

&

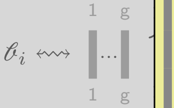


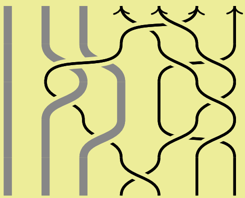
$(b_1 t_2 b_1^{-1}) t_3 = t_3 (b_1 t_2 b_1^{-1})$

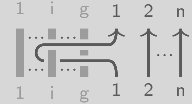
Let $\mathrm{Br}(g, n)$ be the group defined as follows.

Example.

Generators. Braid and twis

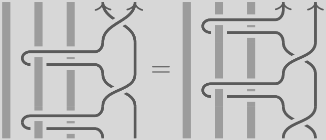






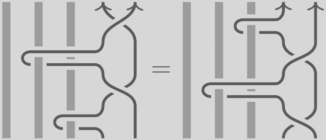
Relations. Reidemeister bra ations and special relations, *e.g.*

Involves three players and inverses!



$b_1 t_2 b_1 t_2 = t_2 b_1 t_2 b_1$

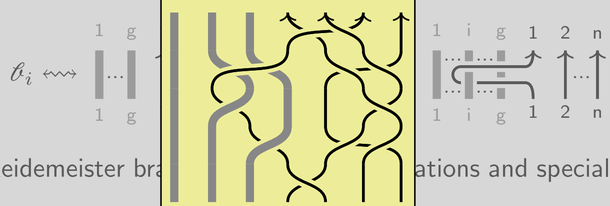
&



$(b_1 t_2 b_1^{-1}) t_3 = t_3 (b_1 t_2 b_1^{-1})$

Let $\mathrm{Br}(g, n)$ be the group defined as follows.

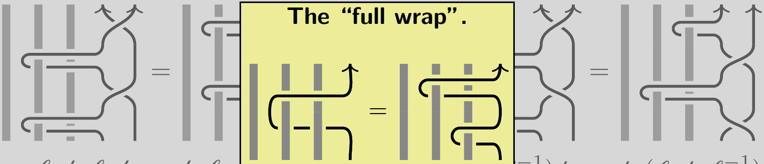
Generators. Braid and twis



Example.

Relations. Reidemeister bra ations and special relations, *e.g.*

Involves three players and inverses!



The “full wrap”.

$b_1 t_2 b_1 t_2 = t_2 b_1 t$

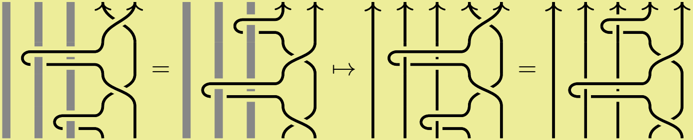
$b_1^{-1}) t_3 = t_3 (b_1 t_2 b_1^{-1})$

Let $\mathrm{Br}(g,n)$ be the group defined as follows.

Generators. Braid and twist generators

Fact (type A embedding).

$\mathrm{Br}(g,n)$ is a subgroup of the usual braid group $\mathscr{B}\mathrm{r}(g{+}n)$.



Proof? A visualization exercise.

Relatio... ...lations, *e.g.*

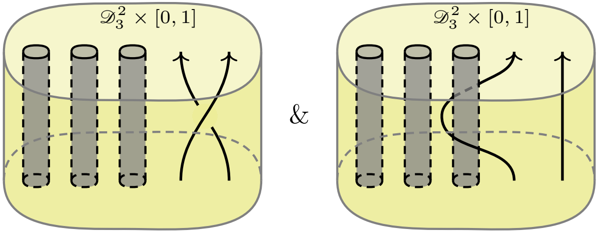
...erses!

$\mathscr{b}_1 t_2 \mathscr{b}_1 t_2 = t_2 \mathscr{b}_1 t_2 \mathscr{b}_1$

$(\mathscr{b}_1 t_2 \mathscr{b}_1^{-1}) t_3 = t_3 (\mathscr{b}_1 t_2 \mathscr{b}_1^{-1})$

The group $\mathscr{B}\mathrm{r}(g,n)$ of braid in a g-times punctures disk $\mathscr{D}^2_g \times [0,1]$:

Two types of braidings, the usual ones and “winding around cores”, *e.g.*

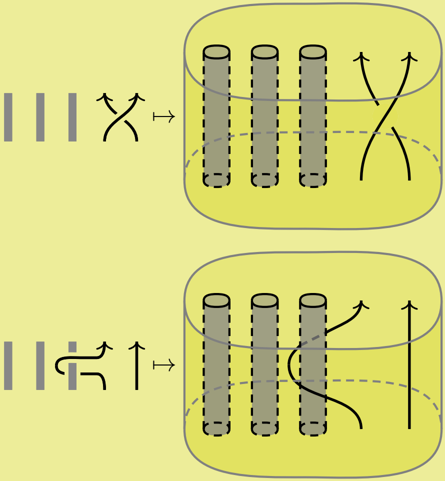


The

Two

Theorem (Häring-Oldenburg–Lambropoulou ∼2002, Vershinin ∼1998).

The map

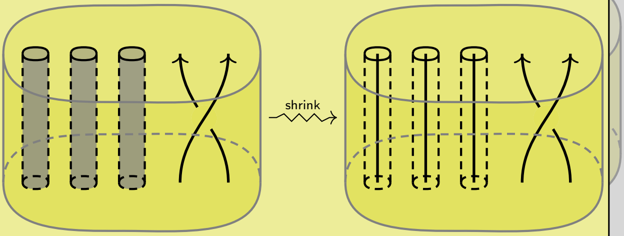


is an isomorphism of groups $\mathrm{Br}(g,n) \to \mathscr{B}\mathrm{r}(g,n)$.

The group $\mathscr{B}\mathrm{r}(g,n)$ of braid in a g-times punctures disk $\mathscr{D}_g^2 \times [0,1]$:

Two types ... , *e.g.*

From this perspective the type A embedding
is just shrinking holes to points!

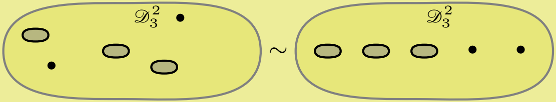


The group $\mathscr{B}\mathrm{r}(g,n)$ of braid in a g-times punctures disk $\mathscr{D}_g^2 \times [0,1]$:

Two types of braidings, the usual ones and "winding around cores", e.g.

Note.

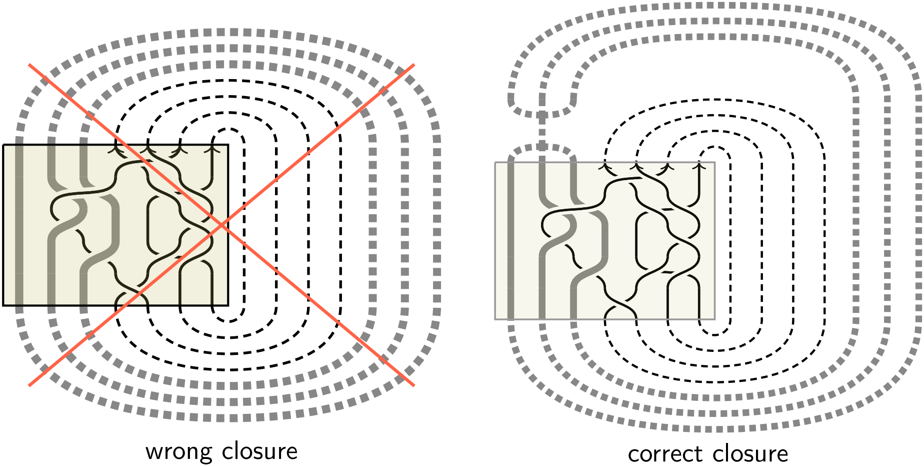
For the proof it is crucial that $\mathscr{D}_g^2$ and the boundary points of the braids • are only defined up to isotopy, *e.g.*



$\Rightarrow$ one can always "conjugate cores to the left".

This is useful to define $\mathscr{B}\mathrm{r}(g,\infty)$.

The Alexander closure on $\mathscr{B}\mathrm{r}(g,\infty)$ is given by merging core strands at infinity.

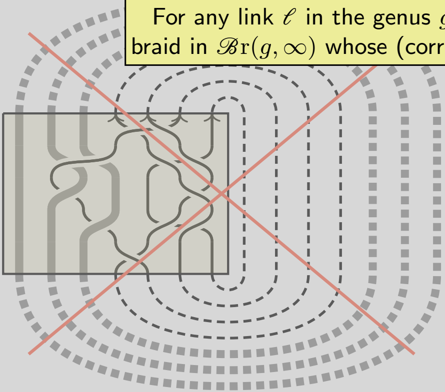


This is different from the classical Alexander closure.

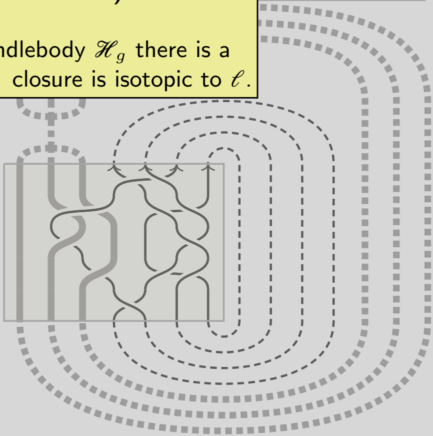
The Alexander closure on $\mathscr{B}\mathrm{r}(g,\infty)$ is given by merging core strands at infinity.

Theorem (Lambropoulou ∼1993).

For any link ℓ in the genus g handlebody $\mathscr{H}_g$ there is a braid in $\mathscr{B}\mathrm{r}(g,\infty)$ whose (correct!) closure is isotopic to ℓ.



wrong closure



correct closure

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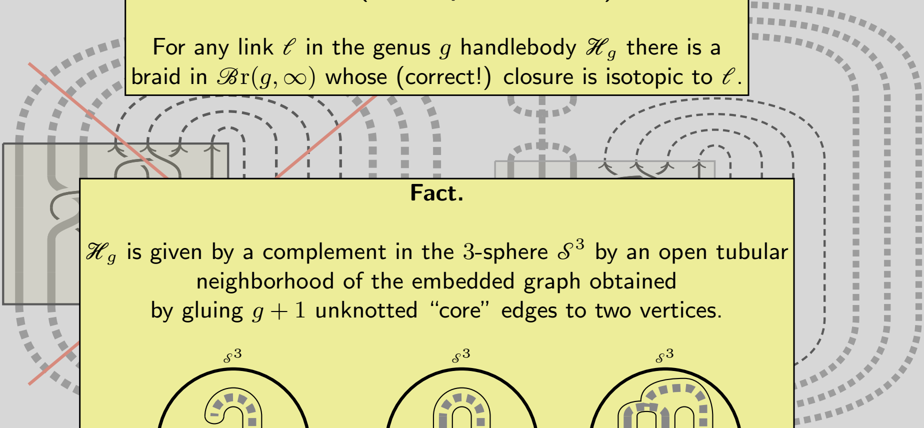
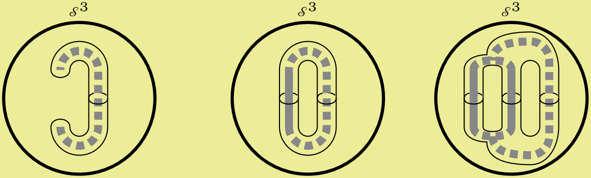


Theorem (Lambropoulou ∼1993).

For any link ℓ in the genus g handlebody $\mathscr{H}_g$ there is a braid in $\mathscr{B}\mathrm{r}(g,\infty)$ whose (correct!) closure is isotopic to ℓ.

Fact.

$\mathscr{H}_g$ is given by a complement in the 3-sphere $\mathscr{S}^3$ by an open tubular neighborhood of the embedded graph obtained by gluing $g+1$ unknotted "core" edges to two vertices.



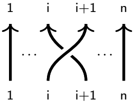
the 3-ball $\mathscr{H}_0 = \mathscr{D}^3$ a torus $\mathscr{H}_1$ $\mathscr{H}_2$

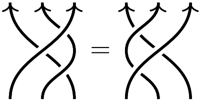
This is

$\cos(\pi/3)$ on a line:

$$\text{type } \mathsf{A}_{n-1}\colon\quad \mathtt{1} - \mathtt{2} - \ldots - \mathtt{n{-}2} - \mathtt{n{-}1}$$

The classical case. Consider the map

$\beta_i \mapsto$ 

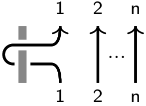
braid rel.: 

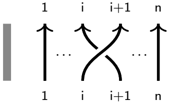
Artin ∼1925. This gives an isomorphism of groups $\mathrm{AT}(\mathsf{A}_{n-1}) \xrightarrow{\cong} \mathscr{B}\mathrm{r}(0,n)$.

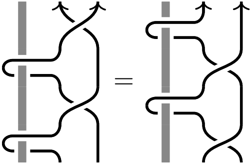
$\cos(\pi/4)$ on a line:

$$\text{type } C_n: \quad 0 \overset{4}{\textbf{—}} 1 \textbf{—} 2 \textbf{—} \ldots \textbf{—} n{-}1 \textbf{—} n$$

The semi-classical case. Consider the map

$\beta_0 \mapsto$ 

$\&$ $\beta_i \mapsto$ 

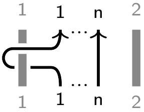
braid rel.: 

Brieskorn ∼1973. This gives an isomorphism of groups $\mathrm{AT}(\mathsf{C}_n) \xrightarrow{\cong} \mathscr{B}\mathrm{r}(1,n)$.

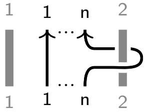
Twice $\cos(\pi/4)$ on a line:

$$\text{type } \tilde{\mathsf{C}}_n: \quad \mathtt{0}^1 \overset{4}{\textbf{---}} \mathtt{1} \textbf{---} \mathtt{2} \textbf{---} \dots \textbf{---} \mathtt{n}-\mathtt{1} \textbf{---} \mathtt{n} \overset{4}{\textbf{---}} \mathtt{0}^2$$

Affine adds genus. Consider the map

$\beta_{0^1} \mapsto$ 

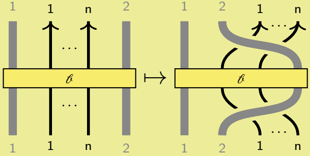
& $\beta_i \mapsto$ 

& $\beta_{0^2} \mapsto$ 

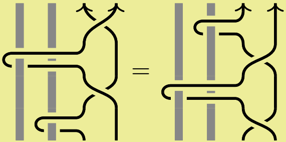
Allcock ~1999. This gives an isomorphism of groups $\mathrm{AT}(\tilde{\mathsf{C}}_n) \xrightarrow{\cong} \mathscr{B}\mathrm{r}(2,n)$.

Twice $\cos(\pi/4)$

This case is strange – it only arises under conjugation:



By a miracle, one can avoid the special relation



This relation involves three players and inverses. Bad!

Affine adds ge

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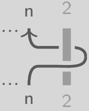

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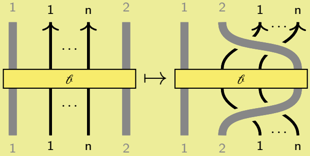

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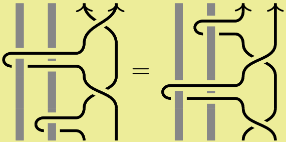

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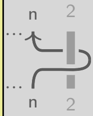

Currently, not much seems to be known, but I think the same story works.

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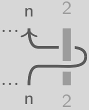

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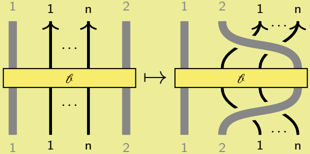

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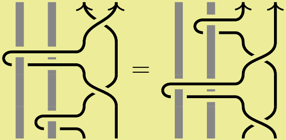
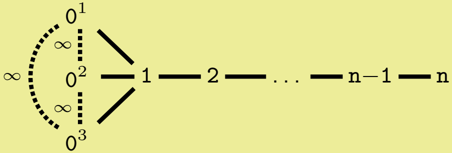


This relation involves three players and inverses. Bad!

Currently, not much seems to be known, but I think the same story works.

Allcock ~1999. This gives an isomorphism of groups $\mathrm{AT}(\tilde{\mathrm{C}}_n) \xrightarrow{\cong} \mathscr{P}r(2, n)$.

However, this is where it seems to end, *e.g.* genus $g = 3$ wants to be



In some sense this can not work; remember Tits' conjecture.

Twice $\cos(\pi/4)$ on a line:

Affi

Allo

Currently known (to the best of my knowledge).

Genus	type A	type C
$g=0$	$\mathscr{B}\mathrm{r}(n) \cong \mathrm{AT}(\mathsf{A}_{n-1})$	
$g=1$	$\mathscr{B}\mathrm{r}(1,n) \cong \mathbb{Z} \ltimes \mathrm{AT}(\tilde{\mathsf{A}}_{n-1}) \cong \mathrm{AT}(\hat{\mathsf{A}}_{n-1})$	$\mathscr{B}\mathrm{r}(1,n) \cong \mathrm{AT}(\mathsf{C}_n)$
$g=2$		$\mathscr{B}\mathrm{r}(2,n) \cong \mathrm{AT}(\tilde{\mathsf{C}}_n)$
$g\geq 3$		

And some $\mathbb{Z}/2\mathbb{Z}$-orbifolds ($\mathbb{Z}/\infty\mathbb{Z}$ =puncture):

Genus	type D	type B
$g=0$		
$g=1$	$\mathscr{B}\mathrm{r}(1,n)_{\mathbb{Z}/2\mathbb{Z}} \cong \mathrm{AT}(\mathsf{D}_n)$	$\mathscr{B}\mathrm{r}(1,n)_{\mathbb{Z}/\infty\mathbb{Z}} \cong \mathrm{AT}(\mathsf{B}_n)$
$g=2$	$\mathscr{B}\mathrm{r}(2,n)_{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}} \cong \mathrm{AT}(\tilde{\mathsf{D}}_n)$	$\mathscr{B}\mathrm{r}(2,n)_{\mathbb{Z}/\infty\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}} \cong \mathrm{AT}(\tilde{\mathsf{B}}_n)$
$g\geq 3$		

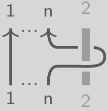
(For orbifolds "genus" is just an analogy.)

Twice $\cos(\pi/4)$ on a line:

type ... 0^2

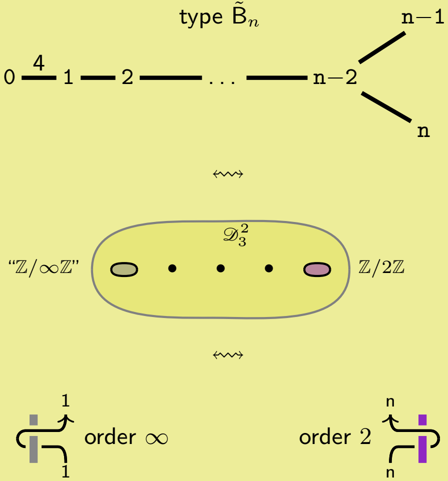
Affine adds genus

$\beta_{0^1} \mapsto$ 



Allcock ∼1999. T ... $\xrightarrow{\cong} \mathscr{B}\mathrm{r}(2,n)$.

Example.



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