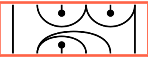
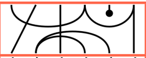
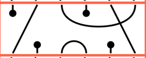
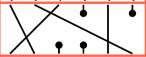
Diagrammatics and cryptography

Or: Not too small, please!

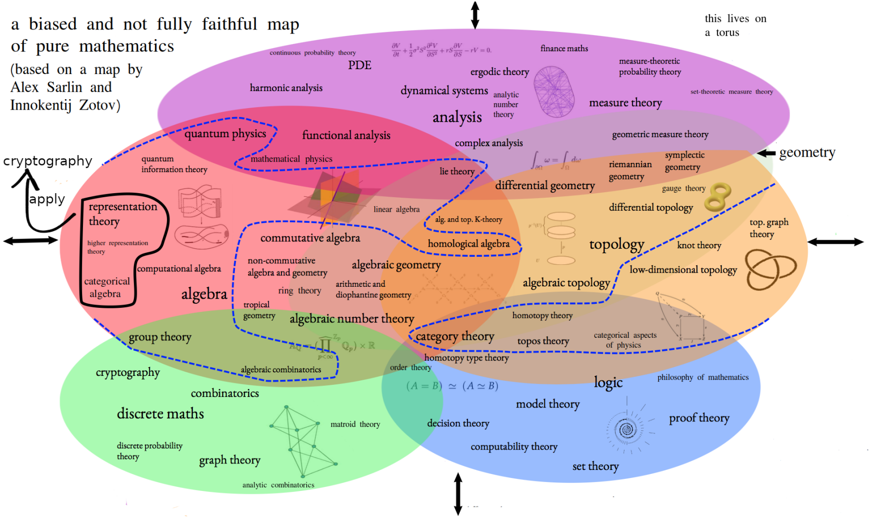
Daniel Tubbenhauer

Symbol	Diagrams	Useful?	Symbol	Diagrams	Useful?
pPa_n		YES*	Pa_n		YES*
Mo_n		YES	$RoBr_n$		YES*
TL_n		YES	Br_n		YES*
pRo_n		YES*	Ro_n		YES*
pS_n		EX	S_n		NO

Joint with Mikhail Khovanov and Maithreya Sitaraman

December 2021

Where are we?



The six main fields of pure mathematics algebra, analysis, geometry, topology, logic, discrete mathematics

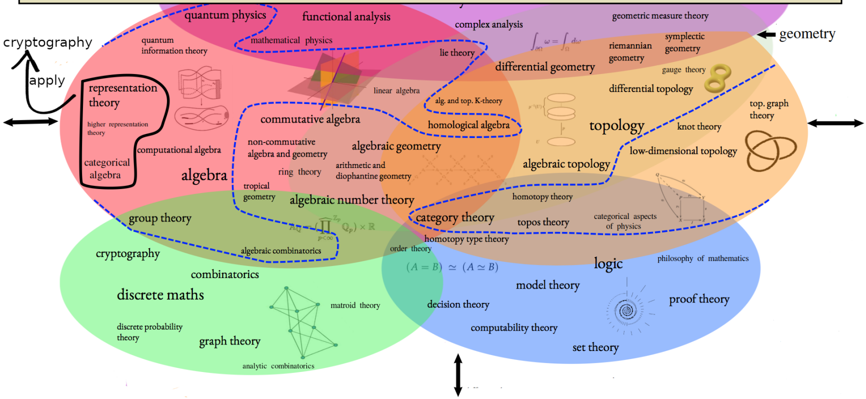
Wh

The map of (pure) mathematics

Black box. Representation theory and its categorical analog My research area

Dashed box. Where I usually apply them My research outreach

Applications beyond my current research? The future (within NU?)



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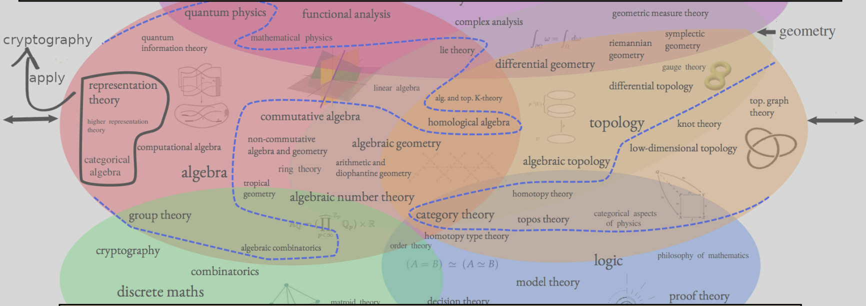


Today

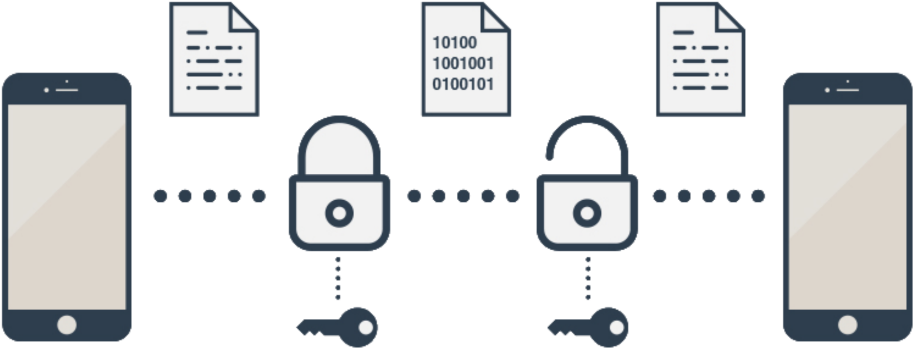
An applications of my work to cryptography **The future (within NU?)**

Why? Because it is neat (judge yourself!)
summarizes my research areas
and fits well to NU

The six

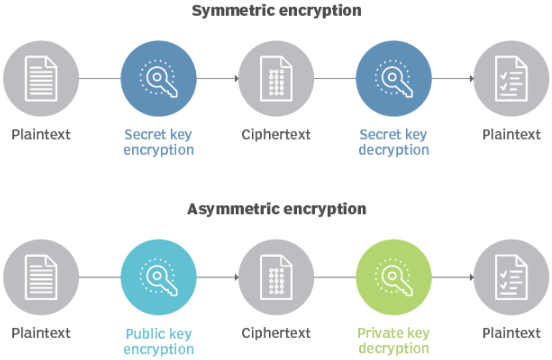
topolog

End-to-end encryption



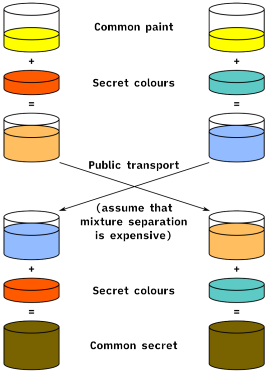
- E2EE Only the two communicating parties should decrypt the message
- Problem How to transfer the encryption key?
- Diffie–Hellman (DH) Addresses this problem

End-to-end encryption



- Symmetric Both parties us the same secret key
- Problem (still) How to transfer the encryption key?
- Asymmetric Both parties have a public and a private key, no sharing needed

End-to-end encryption



- DH Two secrets a, b, public g, send g^a or g^b and get $(g^b)^a = g^{ab} = (g^a)^b$
- Catch Relies on the mixtures to be hard ot decompose (discrete log problem)
- BTW Using colors is not very practical ;-), so usually take $a, b, g \in (\mathbb{Z}/p\mathbb{Z})^{x}$

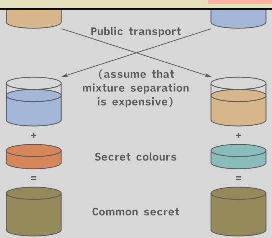
End-to-end encryption

Colors!

The color picture makes it clear that this can easily be generalized

For example, one could take a different group

Varying the protocol and one can even allow arbitrary monoids



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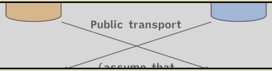


Example (Shpilrain–Ushakov (SU) key exchange protocol)

The public data is a monoid S, and two sets $A, B \subset S$ of commuting elements and $g \in S$

Party A chooses privately $a, a' \in A$ and party B chooses privately $b, b' \in A$

Party A communicates aga', B sends bgb' and the common secret is $abgb'a' = baga'b'$

Note that S can be an arbitrary monoid in this protocol

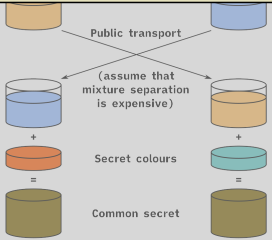
The complexity of S determines how difficult it
is to find the common secret from the public data.

End-to-end en

Linear attack (Myasnikov–Roman'kov ∼2015)

"All" protocol involving monoids can be attacked
if the monoid admits a small non-trivial representation

Enter representation theory



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Linear attack (Myasnikov–Roman'kov ∼2015)

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Enter representation theory

Our idea

Systematically study and construct monoids with no small non-trivial representations

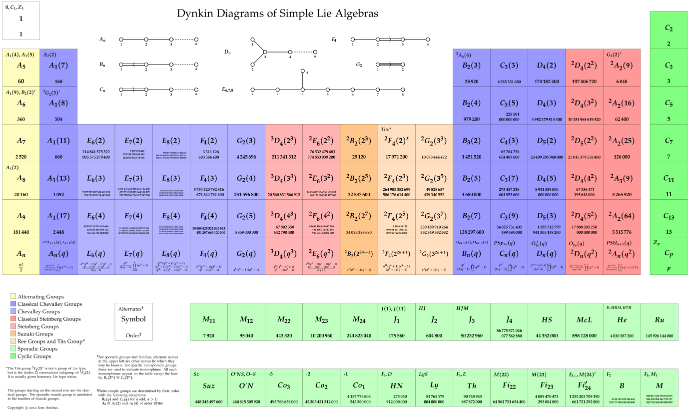
The abstract theory is governed by Green's theory of cells (Green's relations)

The good finite examples come from quantum topology and monoidal categories
The representation theory group at NU knows these very well!

The good infinite examples are Artin, Thompson, Grigorchuk groups and alike
The geometric group theory group at NU knows these very well!

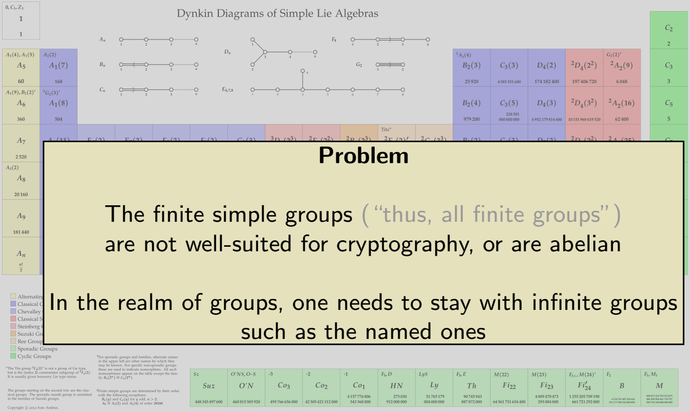
Other examples we know come from 2-representation theory and fusion categories
Vaguely related to the operator algebras group at NU

Examples and non-examples



- ▶ **Classical examples** Cyclic groups have only big representations over $\mathbb{F}_p$
- ▶ **Non-examples** Groups of Lie type have all very small representations
- ▶ **Non-examples** Sporadic groups are too small to be useful

Examples and non-examples



Problem

The finite simple groups (“thus, all finite groups”)
are not well-suited for cryptography, or are abelian

In the realm of groups, one needs to stay with infinite groups
such as the named ones

- Classical examples Cyclic groups have only big representations over $\mathbb{F}_p$
- Non-examples Groups of Lie type have all very small representations
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Examples and non-examples

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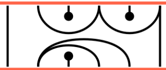
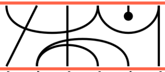
pPa_n		YES*	Pa_n		YES*
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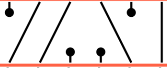
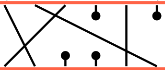
pRo_n		YES*	Ro_n		YES*
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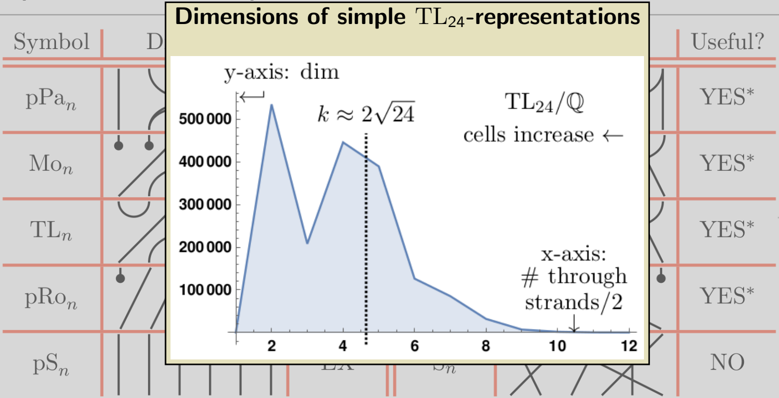
- New examples Finite monoids coming from quantum topology
- More specific Submonoids of the partition monoid above
- Completely open I claim your favorite example from quantum topology and geometric group theory will also work - lets work on this at NU!

Examples and non-examples



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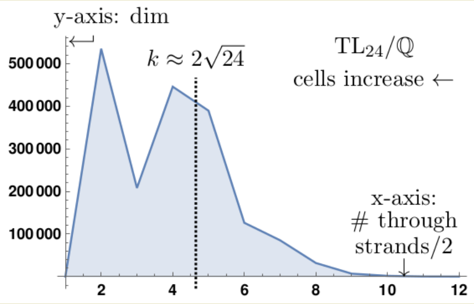
pS_n	… S_n	NO

Dimensions of simple TL_{24}-representations

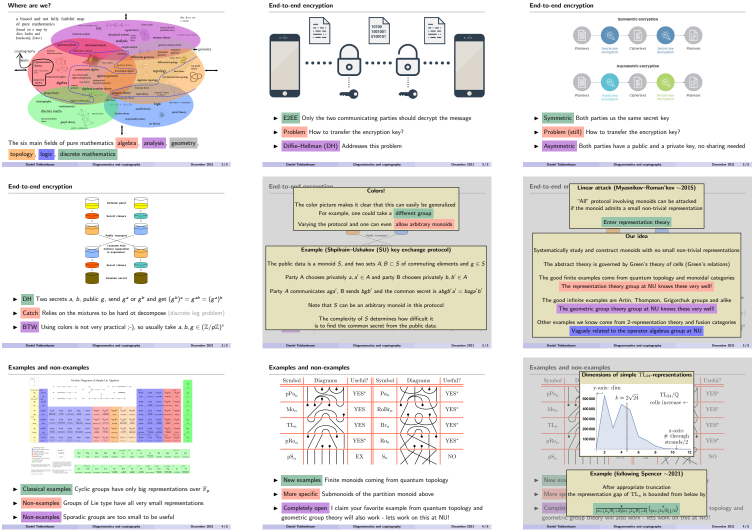


Example (following Spencer ∼2021)

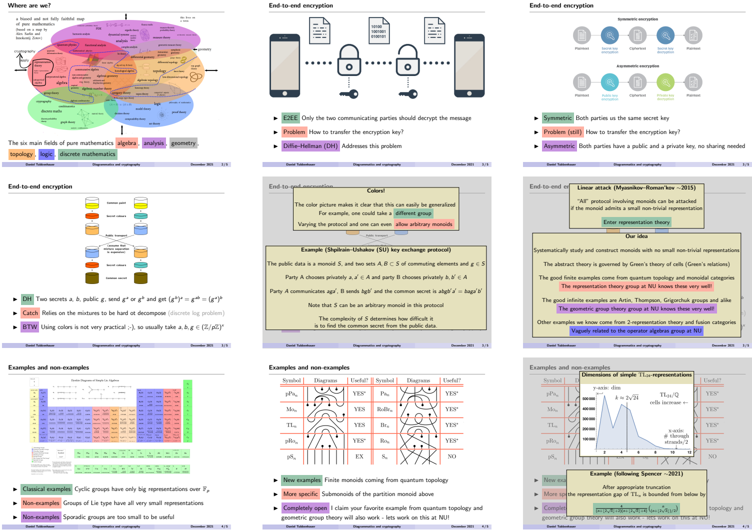
After appropriate truncation
the representation gap of TL_n is bounded from below by

$$\frac{4}{(n+\lfloor 2\sqrt{n}\rfloor+2)(n+\lfloor 2\sqrt{n}\rfloor+4)}\binom{n}{(n+\lfloor 2\sqrt{n}\rfloor)/2}$$

- New exa… …y
- More spe…
- Complet… topology and geometric group theory will also work - lets work on this at NU!



There is still much to do...



Thanks for your attention!