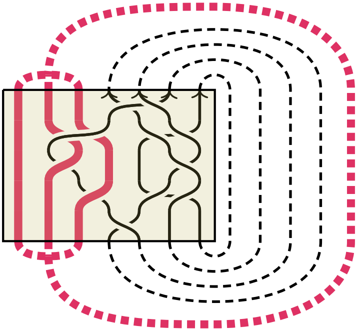
Handlebodies, Artin–Tits and HOMFLYPT

Or: All I know about Artin–Tits groups; and a filler for the remaining 59 minutes

Daniel Tubbenhauer

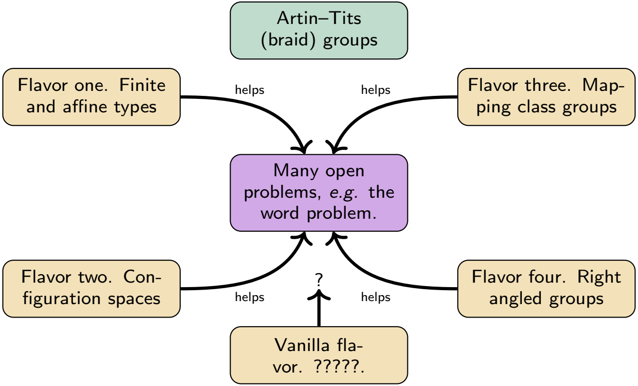


Joint with David Rose

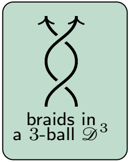
March 2019

My failure. What I would like to understand, but I do not.

Artin–Tits groups come in four main flavors.
Question: Why are these special? What happens in general type?



A different idea for today:
What can Artin–Tits groups tell you about flavor two?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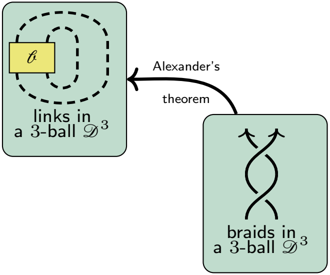

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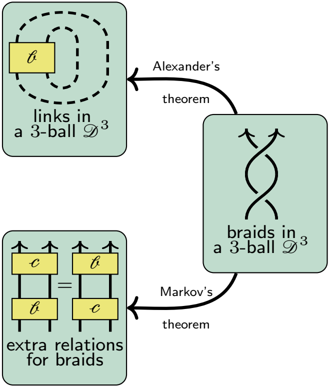
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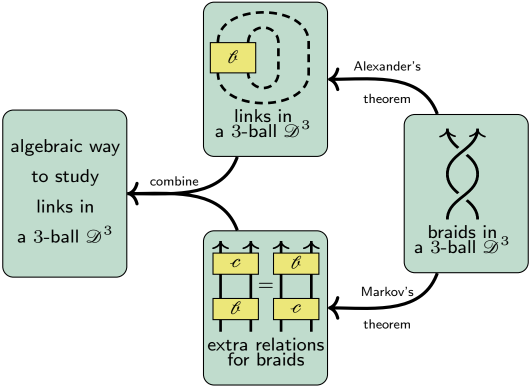

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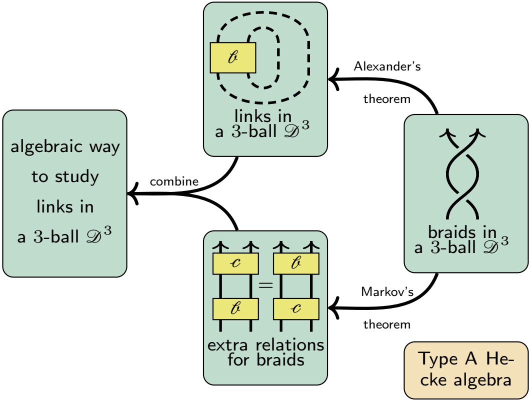
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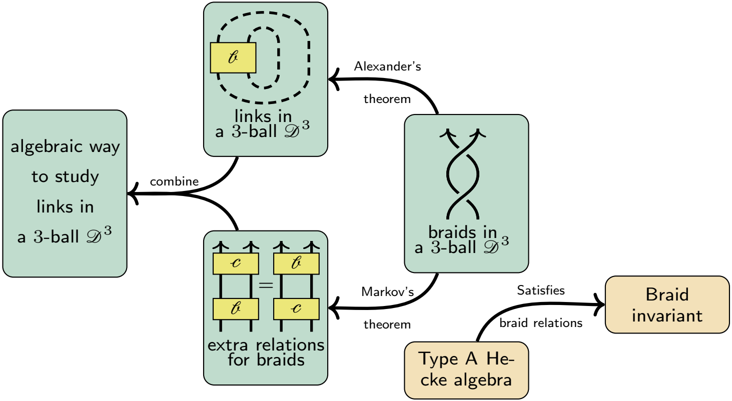
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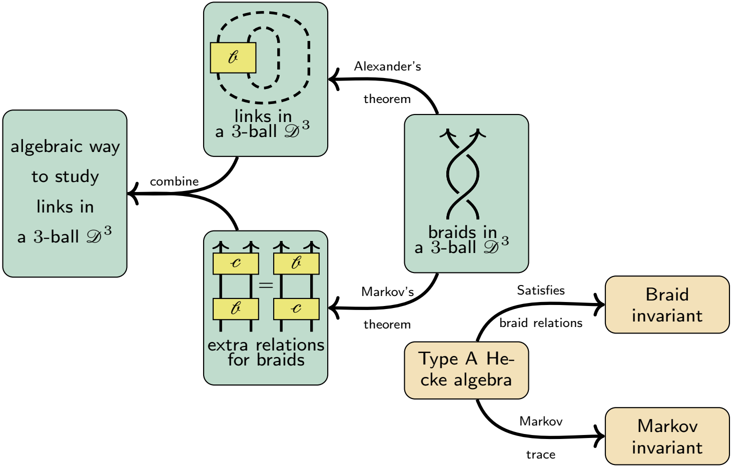
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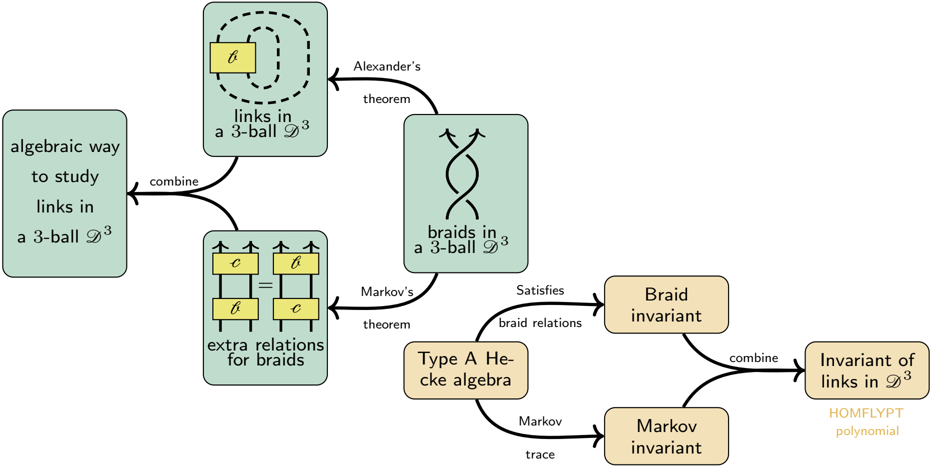
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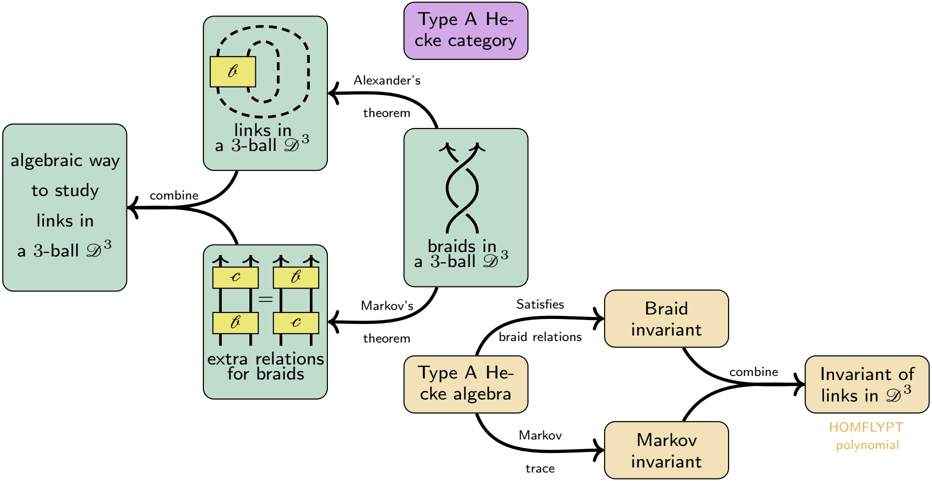

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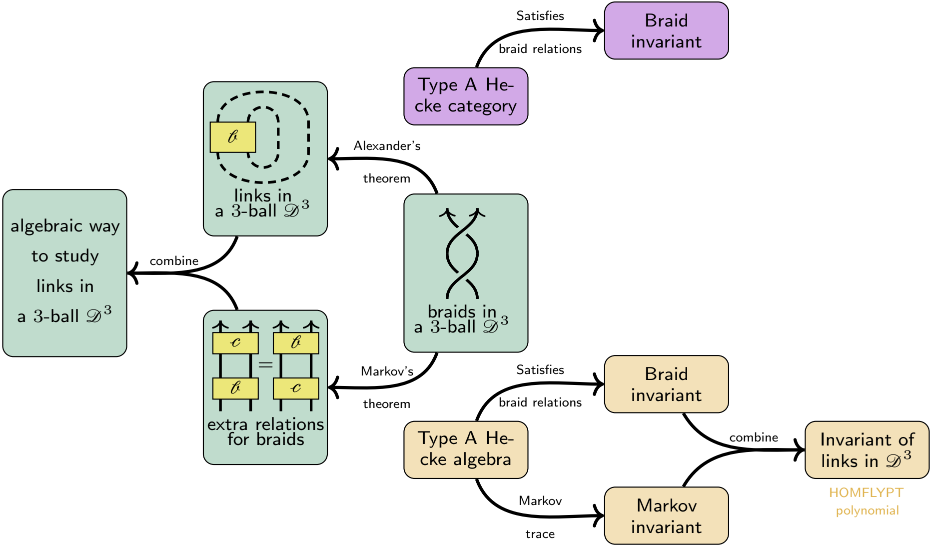
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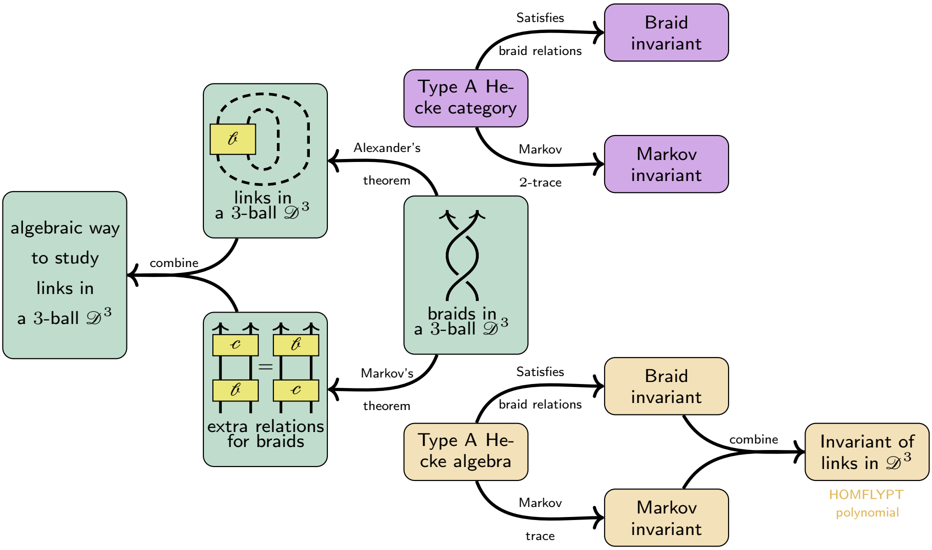
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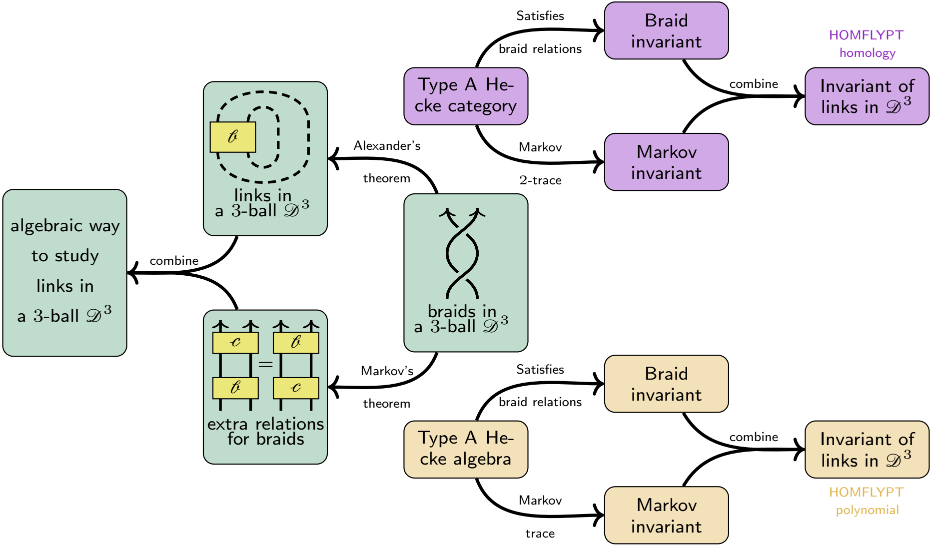
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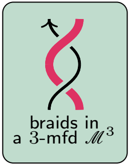
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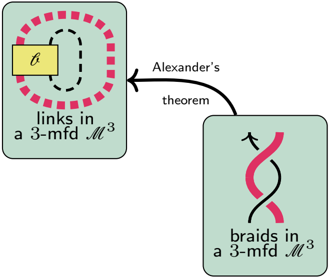
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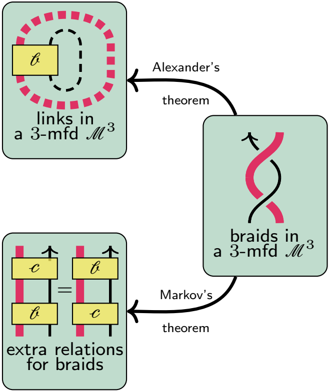

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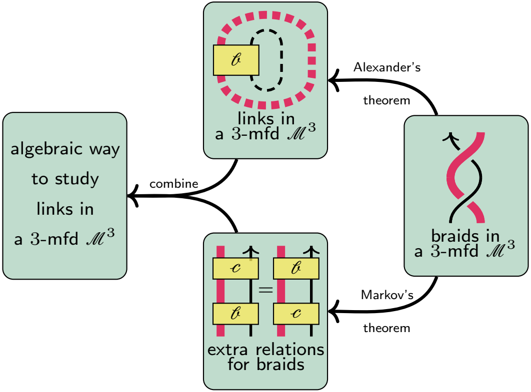

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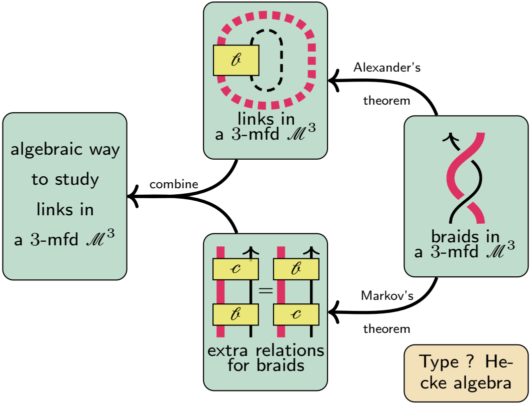
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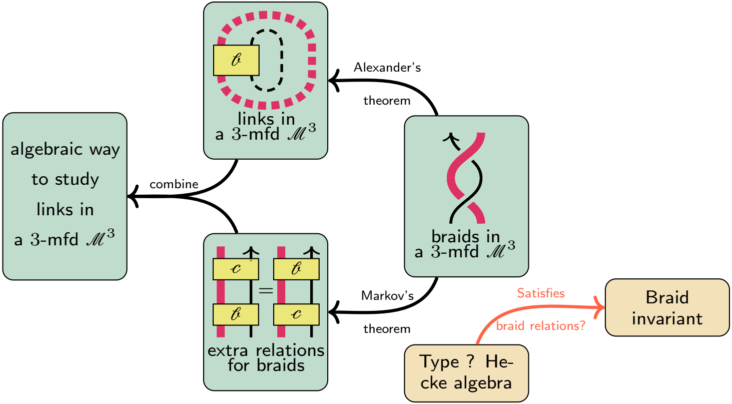

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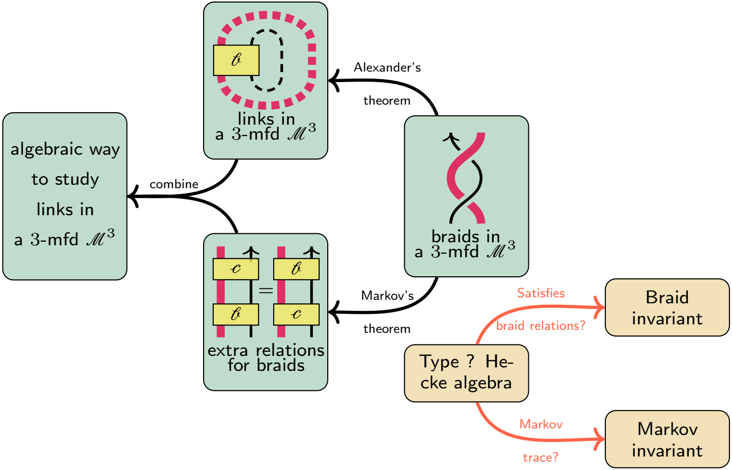
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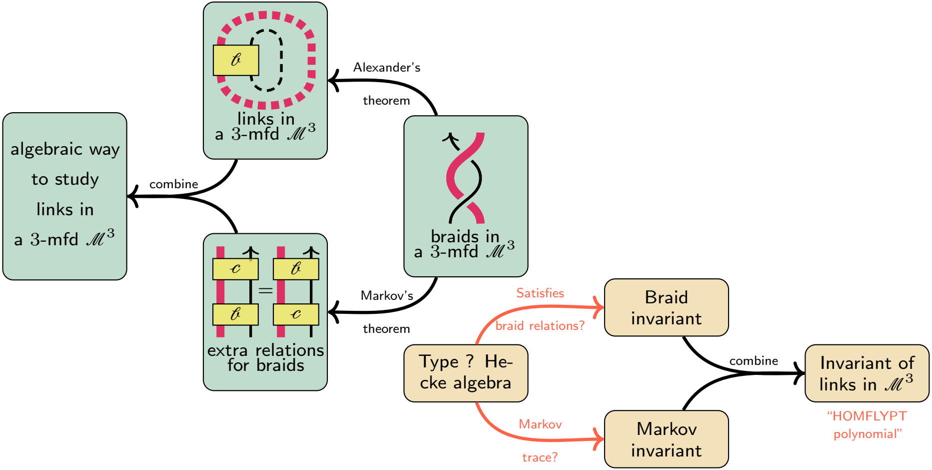

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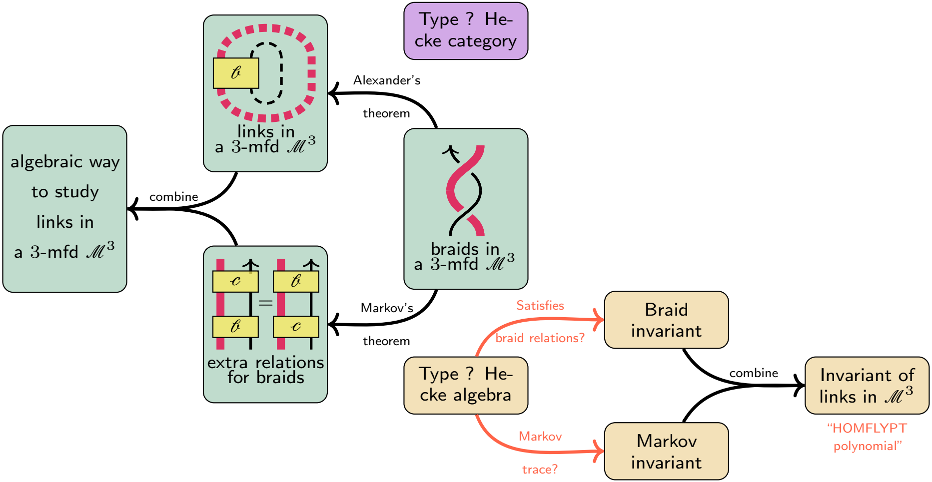
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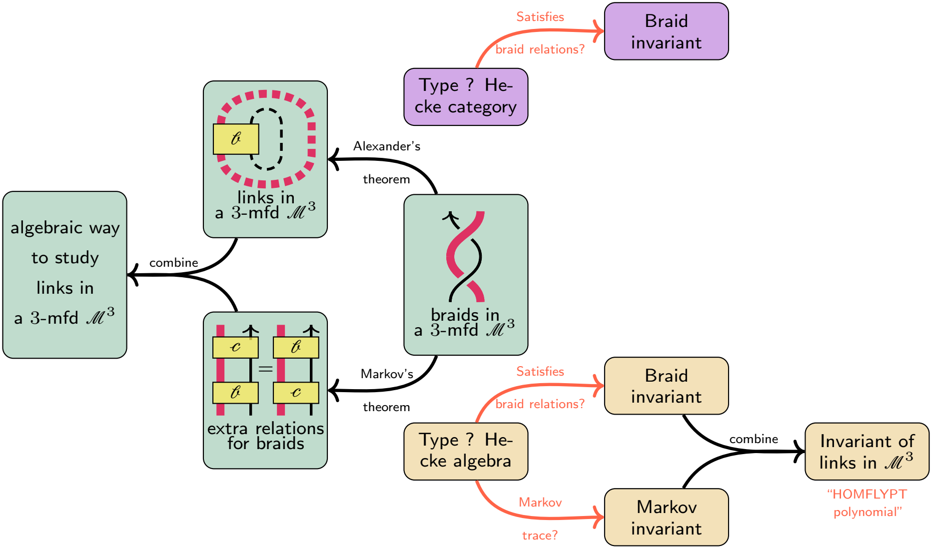

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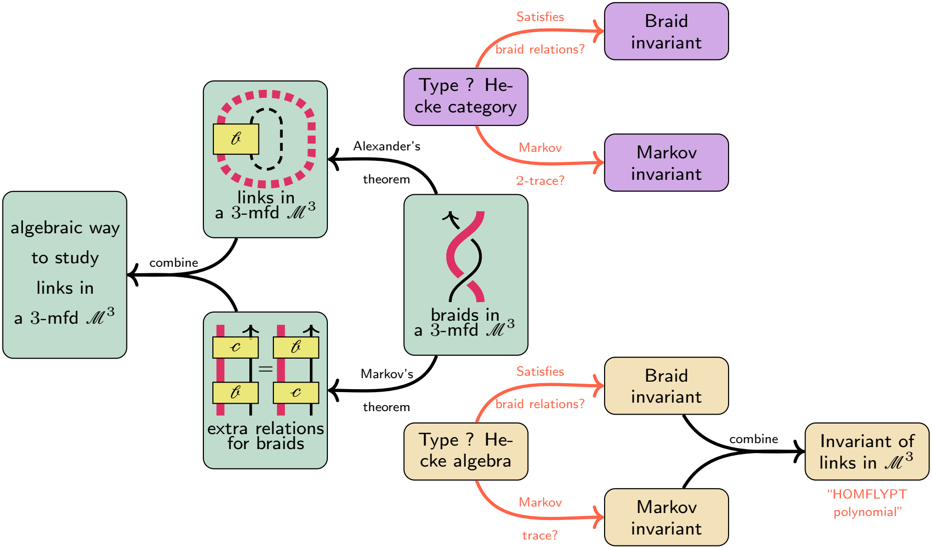
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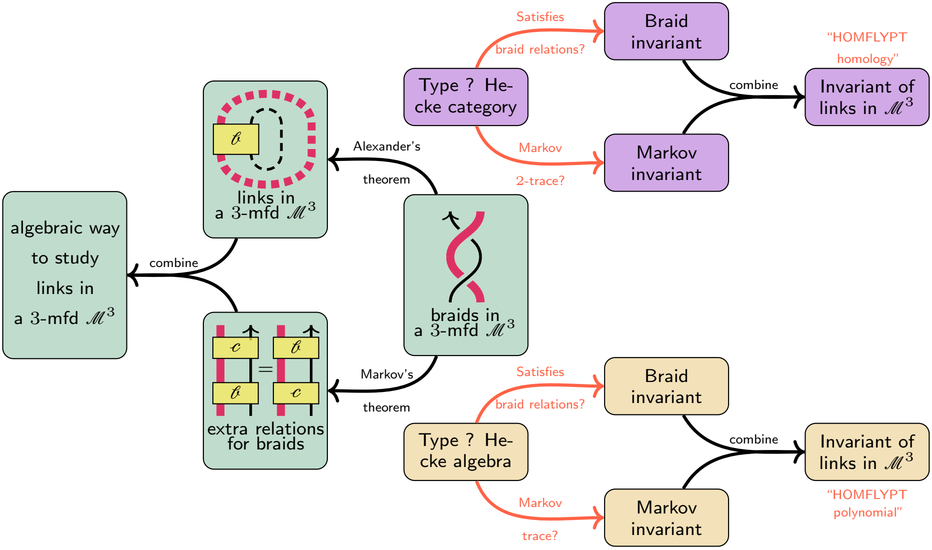
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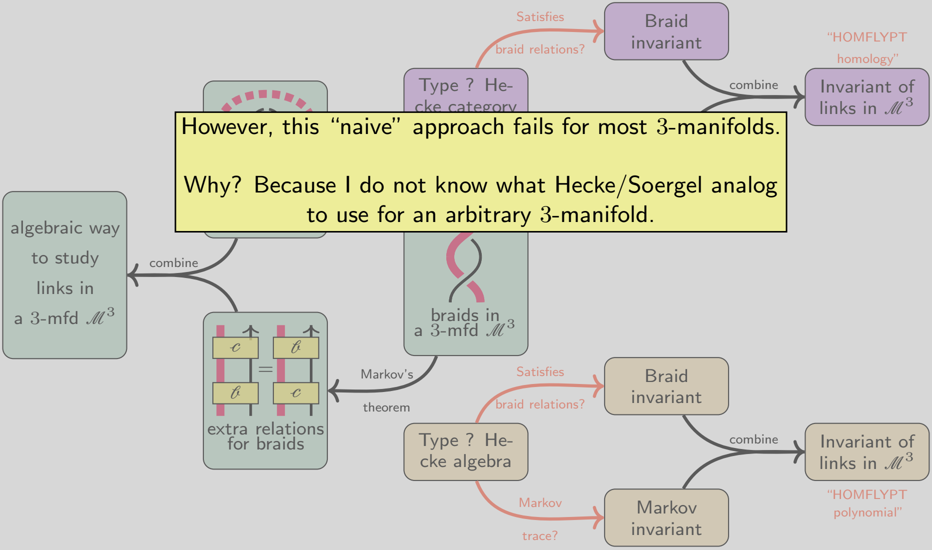
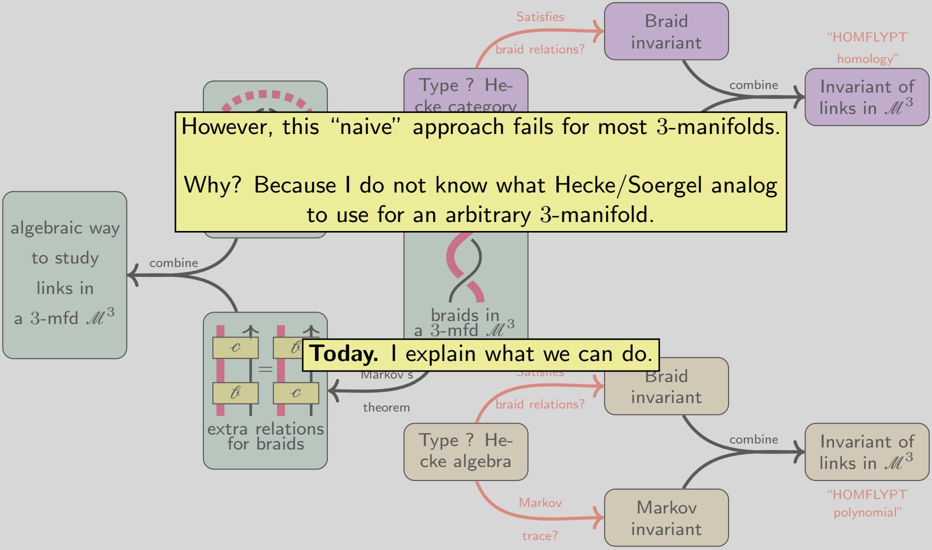


However, this "naive" approach fails for most 3-manifolds.

Why? Because I do not know what Hecke/Soergel analog to use for an arbitrary 3-manifold.



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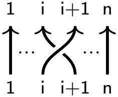
Today. I explain what we can do.

Let $\mathrm{Br}(n)$ be the group defined as follows.

Generators. Braid generators

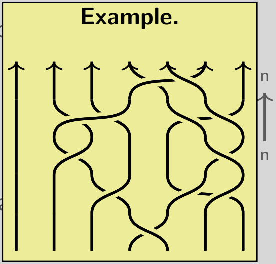
$\mathscr{b}_i \leftrightsquigarrow \dots$ 

Relations. Reidemeister braid relations, *e.g.*

$\mathscr{b}_i\mathscr{b}_i^{-1} = 1 = \mathscr{b}_i^{-1}\mathscr{b}_i$ $\qquad$ $\mathscr{b}_{i+1}\mathscr{b}_i\mathscr{b}_{i+1} = \mathscr{b}_i\mathscr{b}_{i+1}\mathscr{b}_i$

Let $\mathrm{Br}(n)$ be the group defined as follows.

Generators. Braid generato



Relations. Reidemeister bra

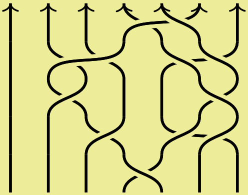
$$b_i b_i^{-1} = 1 = b_i^{-1} b_i \qquad b_{i+1} b_i b_{i+1} = b_i b_{i+1} b_i$$

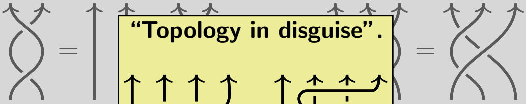
Let $\mathrm{Br}(n)$ be the group defined as follows.

Generators. Braid generato

Relations. Reidemeister bra

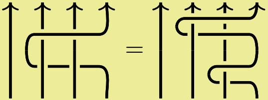
Example.





$b_i b_i^{-1} = 1$ ... $_{+1} = b_i b_{i+1} b_i$

"Topology in disguise".



Let $\mathrm{Br}(n)$ be the group defined as follows.

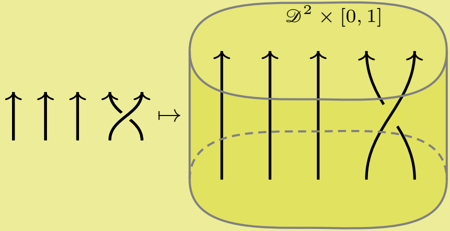
Generators.

Relations. R

Theorem (Gauß ≤1830 ▸ ?, Artin ∼1925.)

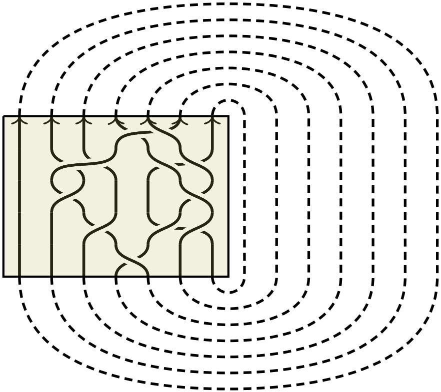
Let $\mathscr{B}\mathrm{r}(n)$ be the group of braids in a disk $\mathscr{D}^2 \times [0,1]$.

The map



is an isomorphism of groups $\mathrm{Br}(n) \to \mathscr{B}\mathrm{r}(n)$.

The Alexander closure on $\mathscr{B}\mathrm{r}(\infty)$ is given by:



This is the classical Alexander closure.

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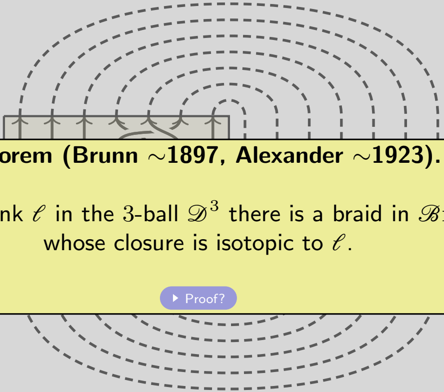


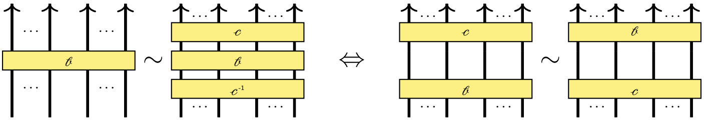
Theorem (Brunn ∼1897, Alexander ∼1923).

For any link ℓ in the 3-ball $\mathscr{D}^3$ there is a braid in $\mathscr{B}\mathrm{r}(\infty)$ whose closure is isotopic to ℓ.

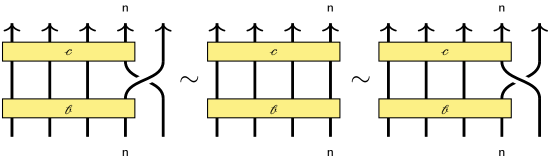
▸ Proof?

This is the classical Alexander closure.

Conjugation.

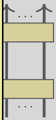


Stabilization.



These are the classical Markov moves.

Conjugation.

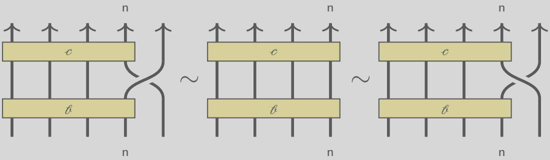


Theorem (Markov ∼1936).

Two links in $\mathscr{D}^3$ are equivalent if and only if they are equal in $\mathscr{B}\mathrm{r}(\infty)$ up to conjugation and stabilization.

Proof?

Stabilization.



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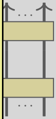
Conjugation.



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▸ Proof?

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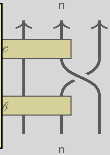


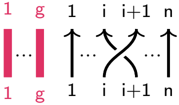
The upshot.

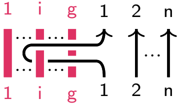
Together with Alexander's theorem, this gives a way to ▸ algebraically study links in $\mathscr{D}^3$.

These are the classical Markov moves.

Let $\mathrm{Br}(g,n)$ be the group defined as follows.

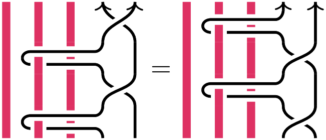
Generators. Braid and twist generators

$\mathscr{b}_i \rightsquigarrow$ 

& $\mathscr{t}_i \rightsquigarrow$ 

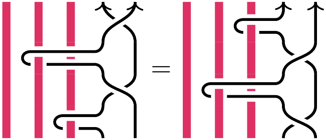
Relations. ▸ Reidemeister braid relations, type C relations and special relations, *e.g.*

Involves three players and inverses!



$\mathscr{b}_1\mathscr{t}_2\mathscr{b}_1\mathscr{t}_2 = \mathscr{t}_2\mathscr{b}_1\mathscr{t}_2\mathscr{b}_1$

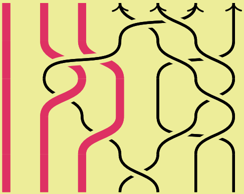
&



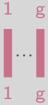
$(\mathscr{b}_1\mathscr{t}_2\mathscr{b}_1^{-1})\mathscr{t}_3 = \mathscr{t}_3(\mathscr{b}_1\mathscr{t}_2\mathscr{b}_1^{-1})$

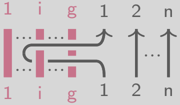
Let $\mathrm{Br}(g,n)$ be the group defined as follows.

Example.



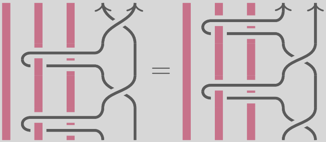
Generators. Braid and twis

$b_i \leftrightsquigarrow$ 

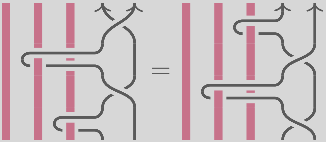


Relations. ▸ Reidemeister braid relatio d special relations, *e.g.*

Involves three players and inverses!



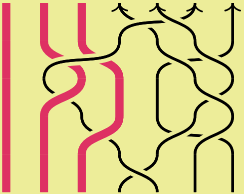
$b_1 t_2 b_1 t_2 = t_2 b_1 t_2 b_1$



$(b_1 t_2 b_1^{-1}) t_3 = t_3 (b_1 t_2 b_1^{-1})$

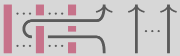
Let $\mathrm{Br}(g, n)$ be the group defined as follows.

Example.



Generators. Braid and twis

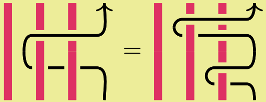
$b_i \leftrightsquigarrow$ 

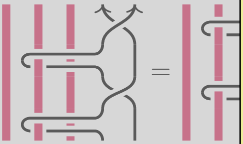


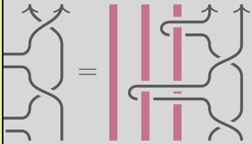
Relations. Reidemeister braid relatio ... d special relations, *e.g.*

Involves three players and inverses!

The "full wrap".







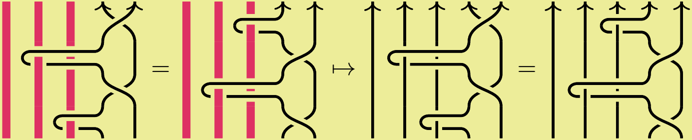
$b_1 t_2 b_1 t_2 = t_2 b_1 t_2 b_1$ $\qquad$ $(b_1 t_2 b_1^{-1}) t_3 = t_3 (b_1 t_2 b_1^{-1})$

Let $\mathrm{Br}(g,n)$ be the group defined as follows.

Generators. Braid and twist generators

Fact (type A embedding).

$\mathrm{Br}(g,n)$ is a subgroup of the usual braid group $\mathscr{B}\mathrm{r}(g{+}n)$.



A visualization exercise.

Relatio

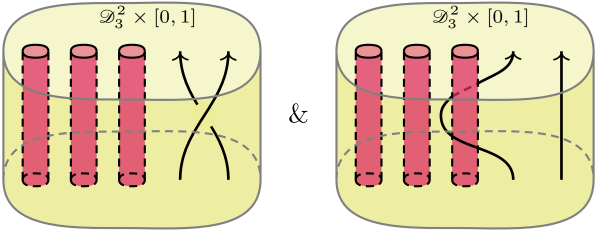
.g.

erses!

$b_1 t_2 b_1 t_2 = t_2 b_1 t_2 b_1$ $\qquad$ $(b_1 t_2 b_1^{-1}) t_3 = t_3 (b_1 t_2 b_1^{-1})$

The group $\mathscr{B}\mathrm{r}(g,n)$ of braid in a g-times punctures disk $\mathscr{D}^2_g \times [0,1]$:

Two types of braidings, the usual ones and "winding around cores", *e.g.*

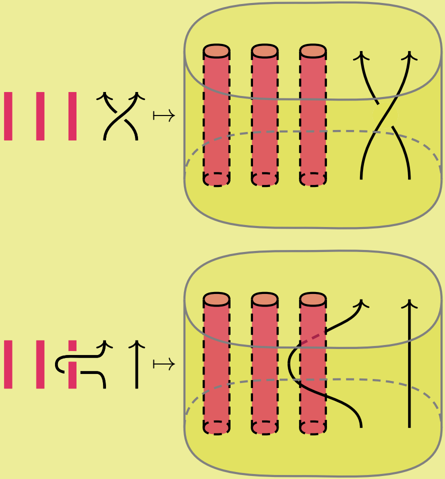


The

Two

Theorem (Häring-Oldenburg–Lambropoulou ∼2002, Vershinin ∼1998).

The map

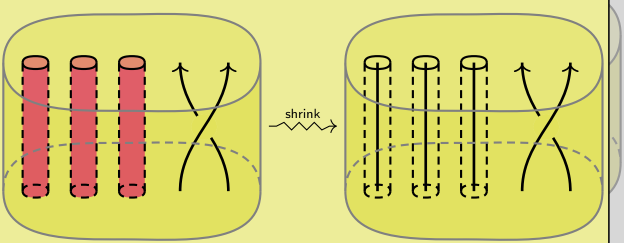


is an isomorphism of groups $\mathrm{Br}(g,n) \to \mathscr{B}\mathrm{r}(g,n)$.

The group $\mathscr{B}\mathrm{r}(g,n)$ of braid in a g-times punctures disk $\mathscr{D}_g^2 \times [0,1]$:

Two types o[...], *e.g.*

From this perspective the type A embedding is just shrinking holes to points!



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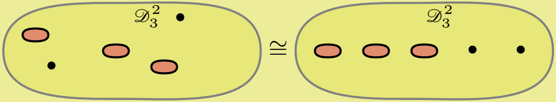
Two types of braidings, the usual ones and "winding around cores", e.g.

Note.

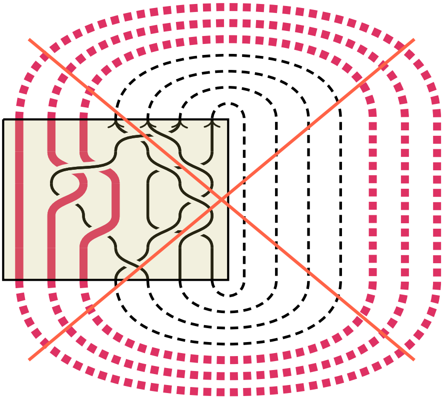
For the proof it is crucial that $\mathscr{D}_g^2$ and the boundary points of the braids $\bullet$ are only defined up to isotopy, *e.g.*



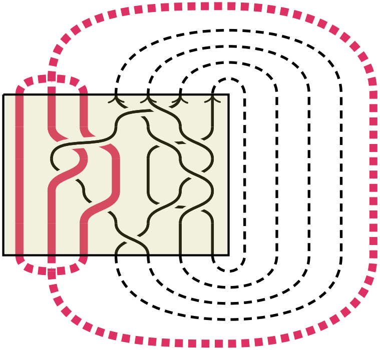
$\Rightarrow$ one can always "conjugate cores to the left".

This is useful to define $\mathscr{B}\mathrm{r}(g,\infty)$.

The Alexander closure on $\mathscr{B}\mathrm{r}(g,\infty)$ is given by merging core strands at infinity.



wrong closure



correct closure

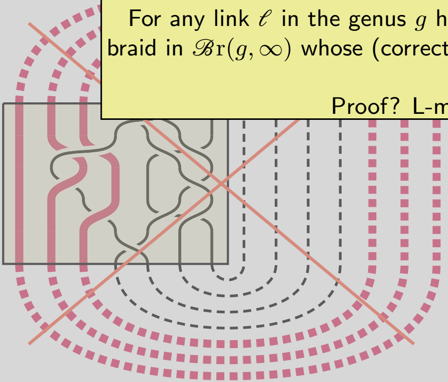
This is different from the classical Alexander closure.

The Alexander closure on $\mathscr{B}\mathrm{r}(g,\infty)$ is given by merging core strands at infinity.

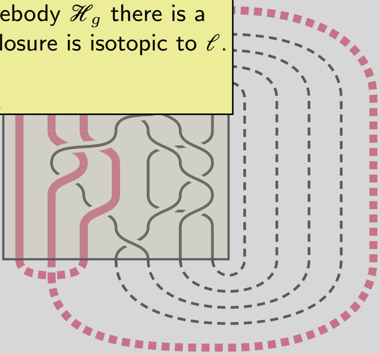
Theorem (Lambropoulou ∼1993).

For any link ℓ in the genus g handlebody $\mathscr{H}_g$ there is a braid in $\mathscr{B}\mathrm{r}(g,\infty)$ whose (correct!) closure is isotopic to ℓ.

Proof? L-move.



wrong closure



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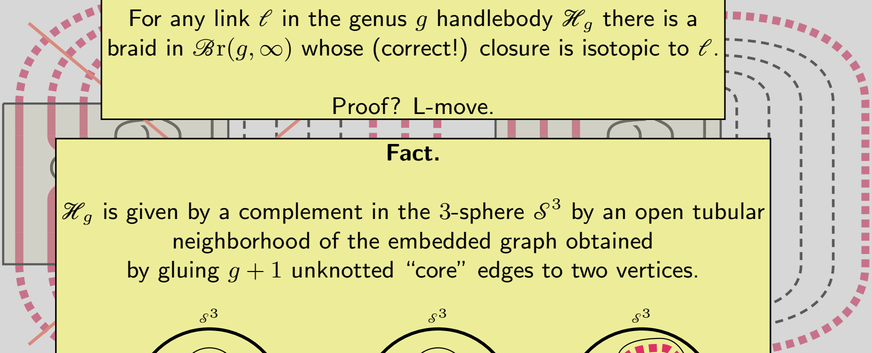

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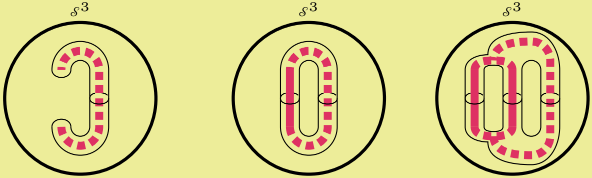
Proof? L-move.

Fact.

$\mathscr{H}_g$ is given by a complement in the 3-sphere $\mathcal{S}^3$ by an open tubular neighborhood of the embedded graph obtained by gluing $g+1$ unknotted "core" edges to two vertices.



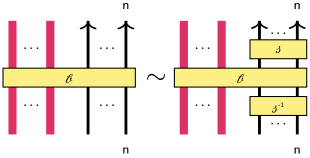
the 3-ball $\mathscr{H}_0 = \mathscr{D}^3$ a torus $\mathscr{H}_1$ $\mathscr{H}_2$

This is

The Markov moves on $\mathscr{B}\mathrm{r}(g,\infty)$ are conjugation and stabilization.

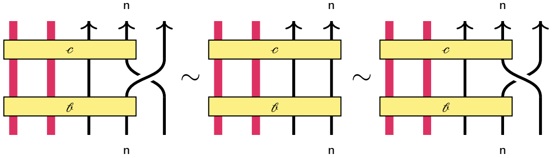
Conjugation.

$$\mathscr{b} \sim \mathscr{s}\mathscr{b}\mathscr{s}^{-1}$$
for $\mathscr{b} \in \mathscr{B}\mathrm{r}(g,n), \mathscr{s} \in \langle \mathscr{b}_1, \dots, \mathscr{b}_{n-1} \rangle$ $\iff$



Stabilization.

$$(\mathscr{c}\uparrow)\mathscr{b}_n(\mathscr{b}\uparrow)$$
$$\sim \mathscr{c}\mathscr{b} \sim (\mathscr{c}\uparrow)\mathscr{b}_n^{-1}(\mathscr{b}\uparrow)$$
for $\mathscr{b}, \mathscr{c} \in \mathscr{B}\mathrm{r}(g,n)$, $\iff$



They are weaker than the classical Markov moves.

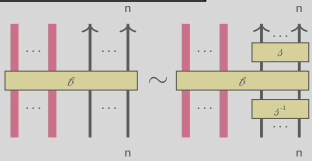
Theorem (Häring-Oldenburg–Lambropoulou ∼2002).

Two links in $\mathscr{H}_g$ are equivalent if and only if they are equal in $\mathscr{B}\mathrm{r}(g,\infty)$ up to conjugation and stabilization.

Proof? L-move.

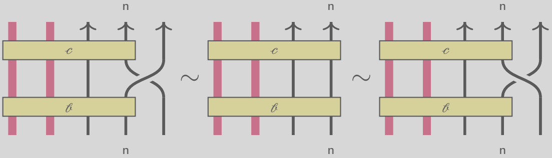
Conjugat

$$b \sim s b s^{-1}$$
for $b \in \mathscr{B}\mathrm{r}(g,n), s \in \langle b_1,\dots,b_{n-1}\rangle$ $\Longleftrightarrow$



Stabilization.

$$(c\uparrow) b_n (b\uparrow) \sim cb \sim (c\uparrow) b_n^{-1} (b\uparrow)$$
for $b, c \in \mathscr{B}\mathrm{r}(g,n)$, $\Longleftrightarrow$



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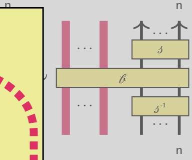
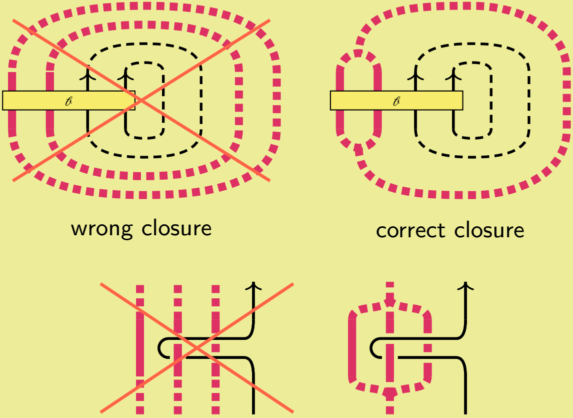
Proof? L-move.

Conjugat

for $\mathscr{b} \in \mathscr{B}$



Example.

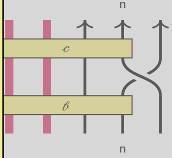


Stabilization

$(c\uparrow)b$

$\sim cb \sim (c$

for $b, c \in$



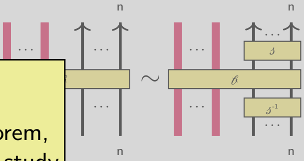
They are wea

The Markov moves on $\mathscr{B}\mathrm{r}(g,\infty)$ are conjugation and stabilization.

Conjugation.

$$b \sim sbs^{-1}$$

for $b \in \mathscr{B}\mathrm{r}(g,n)$,



The upshot.

Together with Alexander's theorem, this gives a way to algebraically study links in $\mathscr{H}_g$.

Stabilization.

$$(c\uparrow)b_n(b\uparrow) \sim cb \sim (c\uparrow)b_n^{-1}(b\uparrow) \iff$$

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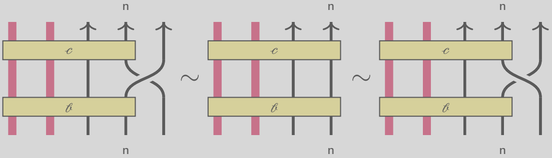

They are weaker than the classical Markov moves.

Conjugation.

$$\mathcal{b} \sim \mathcal{s}\mathcal{b}\mathcal{s}^{-1}$$

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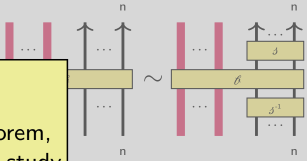


The upshot.

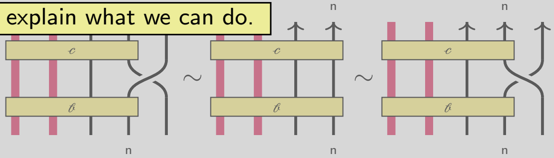
Together with Alexander's theorem, this gives a way to algebraically study links in $\mathscr{H}_g$.

Stabilization.

Let me explain what we can do.

$$(\mathcal{c}\uparrow)\mathcal{b}_n(\mathcal{b}\uparrow) \sim \mathcal{c}\mathcal{b} \sim (\mathcal{c}\uparrow)\mathcal{b}_n^{-1}(\mathcal{b}\uparrow) \iff$$

for $\mathcal{b}, \mathcal{c} \in \mathscr{B}\mathrm{r}(g,n)$,



They are weaker than the classical Markov moves.

Let Γ be a Coxeter graph.

Artin ∼1925, Tits ∼1961++. The Artin–Tits group and its Coxeter group quotient are given by generators-relations:

$$\mathrm{AT}(\Gamma) = \langle \mathcal{b}_i \mid \underbrace{\cdots \mathcal{b}_i \mathcal{b}_j \mathcal{b}_i}_{m_{ij} \text{ factors}} = \underbrace{\cdots \mathcal{b}_j \mathcal{b}_i \mathcal{b}_j}_{m_{ij} \text{ factors}} \rangle$$

$$\downarrow$$

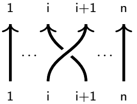
$$\mathrm{W}(\Gamma) = \langle \sigma_i \mid \sigma_i^2 = 1, \underbrace{\cdots \sigma_i \sigma_j \sigma_i}_{m_{ij} \text{ factors}} = \underbrace{\cdots \sigma_j \sigma_i \sigma_j}_{m_{ij} \text{ factors}} \rangle$$

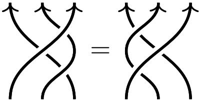
Artin–Tits groups generalize classical braid groups, Coxeter groups ▸ generalize polyhedron groups.

$\cos(\pi/3)$ on a line:

$$\text{type } \mathsf{A}_{n-1}\colon \quad 1 — 2 — \ldots — n{-}2 — n{-}1$$

The classical case. Consider the map

$\beta_i \mapsto$ 

braid rel.: 

Artin ∼1925. This gives an isomorphism of groups $\mathrm{AT}(\mathsf{A}_{n-1}) \xrightarrow{\cong} \mathscr{B}\mathrm{r}(0,n)$.

$\cos(\pi/3)$ on a line:

Jones ~1987.

Markov trace on the Hecke algebra of type A

$\rightsquigarrow$ two variable $\mathbf{q}, \mathbf{a}$ polynomial invariant (HOMFLYPT polynomial).

$\mathbf{q}$=Hecke parameter ; $\mathbf{a}$=trace parameter .

The clas

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I will come back to this with more details for general genus g.
For the time being: This works quite well!

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The clas

Khovanov ∼2005; categorification.

Hochschild homology on complexes of the Hecke category of type A

$\rightsquigarrow$ "three variable $\mathbf{q},\mathbf{t},\mathbf{a}$ homological invariant" (HOMFLYPT homology).

$\mathbf{q}$=Hecke parameter ; $\mathbf{t}$=homological parameter ; $\mathbf{a}$=Hochschild parameter .

Artin ∼1925. This gives an isomorphism of groups $AT(A_{n-1}) \to \mathcal{B}r(0,n)$.

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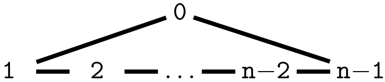
For the time being: This works quite well!

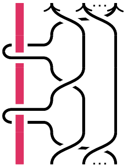
$\cos(\pi/3)$ on a circle.

type $\tilde{\mathrm{A}}_{n-1}$:



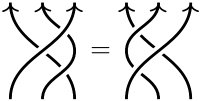
Affine adds genus. Consider the map

$\beta_0 \mapsto$



& $\beta_i \mapsto$

braid rel.:



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tom Dieck ∼1998. (Earlier reference?) This gives an isomorphism of groups $\mathbb{Z} \ltimes \mathrm{AT}(\tilde{\mathrm{A}}_{n-1}) \xrightarrow{\cong} \mathscr{B}\mathrm{r}(1, n)$.

$\cos(\pi/3)$ on a circle.

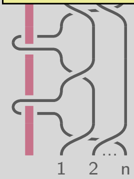
Orellana–Ram ∼2004. (Earlier reference?)

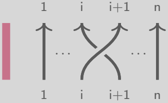
Markov trace on the Hecke algebra of type $\tilde{\mathrm{A}}$

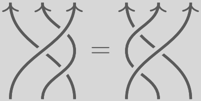
⇝ two variable $\mathbf{q},\mathbf{a}$ polynomial invariant (HOMFLYPT polynomial).

$\mathbf{q}$=Hecke parameter ; $\mathbf{a}$=trace parameter .

Affine a

$\beta_0 \mapsto$ 

& $\beta_i \mapsto$ 

braid rel.: 

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tom Dieck $\sim$1998. (Earlier reference?) This gives an isomorphism of groups $\mathbb{Z} \ltimes \mathrm{AT}(\tilde{\mathrm{A}}_{n-1}) \xrightarrow{\cong} \mathscr{B}r(1,n)$

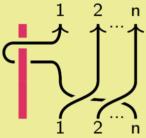
I will come back to this with more details for general genus g.
For the time being: This works quite well!

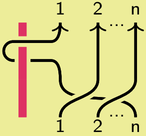
$\cos(\pi/3)$ on a circle.

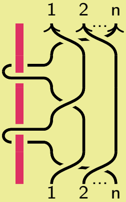
Affine adds ...

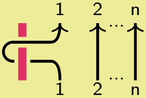
$\beta_0 \mapsto$  $=$ 

Fact. One can recover the (missing) generator of $\mathbb{Z}$ if one works with extended affine type A.

"extended, extra generator" $\mapsto$ 



and


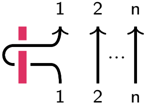
give


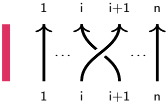
tom Dieck ~ ... m of groups $\mathbb{Z} \ltimes \mathrm{AT}(\tilde{\mathrm{A}}_{n-1}) \xrightarrow{\cong} \mathscr{B}\mathrm{r}(1,n)$.

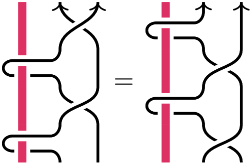
$\cos(\pi/4)$ on a line:

$$\text{type } \mathsf{C}_n: \quad 0 \overset{4}{=\!=} 1 - 2 - \ldots - \mathtt{n}{-}1 - \mathtt{n}$$

The semi-classical case. Consider the map

$\beta_0 \mapsto$ 

& $\beta_i \mapsto$ 

braid rel.: 

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$\cos(\pi/4)$ on a line:

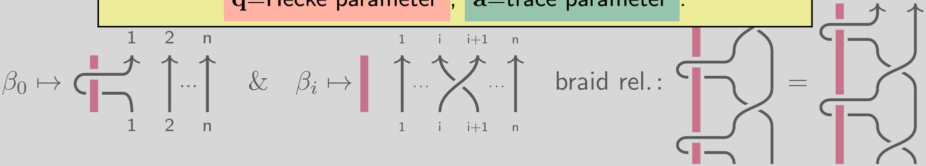
Geck–Lambropoulou ∼1997.

Markov trace on the Hecke algebra of type C

$\rightsquigarrow$ two variable $\mathbf{q},\mathbf{a}$ polynomial invariant (HOMFLYPT polynomial).

$\mathbf{q}$=Hecke parameter ; $\mathbf{a}$=trace parameter .

The sen



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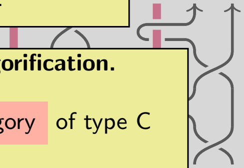
1 2 n

1 i i+1 n



β_0

Rouquier ∼2012, Webster–Williamson ∼2009; categorification.

Hochschild homology on complexes of the Hecke category of type C

⇝ "three variable $\mathbf{q}, \mathbf{t}, \mathbf{a}$ homological invariant" (HOMFLYPT homology).

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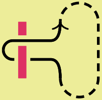
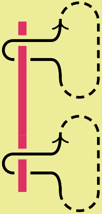
$\cos(\pi/4)$ on a line:

Fact. (Not true in type A.)

There is a whole infinite family of Markov traces,
one for each choice of a value for essential unlinks.

 $\leftrightsquigarrow$ extra parameter and  $\leftrightsquigarrow$ extra parameter *etc.*

However, I only know the categorification of one of these.

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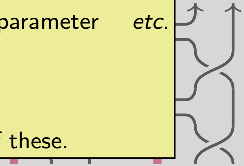
$\beta_0 \mapsto$



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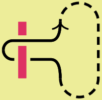
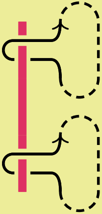
 $\leftrightsquigarrow$ extra parameter and  $\leftrightsquigarrow$ extra parameter *etc.*

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Brieskorn ∼1973. $\ldots C_n) \xrightarrow{\cong} \mathscr{B}\mathrm{r}(1, n)$.

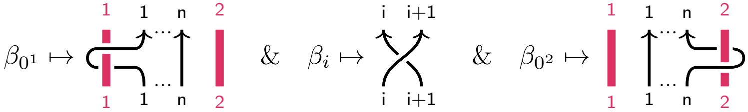
Fact. (Not true in type A.)

There is also a second Hecke parameter, which we do not know how to categorify yet.

$\cos(\pi/4)$ twice on a line:

$$\text{type } \tilde{\mathsf{C}}_n: \quad 0^1 \overset{4}{=\!=} 1 - 2 - \dots - n{-}1 - n \overset{4}{=\!=} 0^2$$

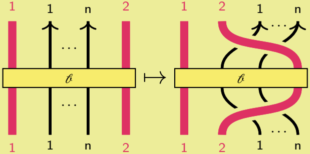
Affine adds genus. Consider the map



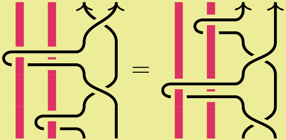
Allcock ∼1999. This gives an isomorphism of groups $\mathrm{AT}(\tilde{\mathsf{C}}_n) \xrightarrow{\cong} \mathscr{B}\mathrm{r}(2,n)$.

$\cos(\pi/4)$ twice

This case is strange – it only arises under conjugation:



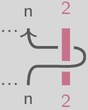
By a miracle, one can avoid the special relation



This relation involves three players and inverses. Bad!

Affine adds ge

$\beta_{0^1} \mapsto$



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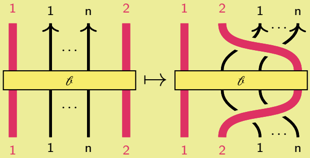

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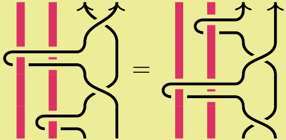

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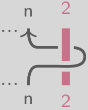

Currently, not much seems to be known, but I think the same story works.

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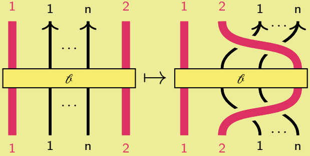
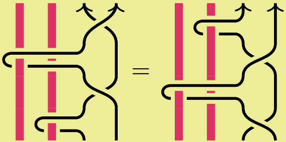


By a miracle, one can avoid the special relation



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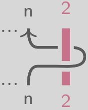
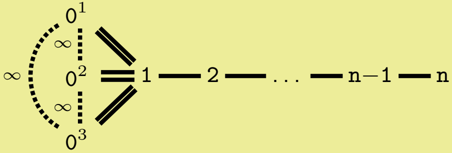


Currently, not much seems to be known, but I think the same story works.

Allcock However, this is where it seems to end, *e.g.* genus $g = 3$ wants to be $n)$.



But the special relation makes it a mere quotient.

So: In the remaining time I tell you what works.

$\cos(\pi/4)$ twice on a line:

Affi

Allo

Currently known (to the best of my knowledge).

Genus	type A	type C
$g=0$	$\mathscr{B}\mathrm{r}(n) \cong \mathrm{AT}(\mathsf{A}_{n-1})$	
$g=1$	$\mathscr{B}\mathrm{r}(1,n) \cong \mathbb{Z} \ltimes \mathrm{AT}(\tilde{\mathsf{A}}_{n-1}) \cong \mathrm{AT}(\hat{\mathsf{A}}_{n-1})$	$\mathscr{B}\mathrm{r}(1,n) \cong \mathrm{AT}(\mathsf{C}_n)$
$g=2$		$\mathscr{B}\mathrm{r}(2,n) \cong \mathrm{AT}(\tilde{\mathsf{C}}_n)$
$g\geq 3$		

And some $\mathbb{Z}/2\mathbb{Z}$-orbifolds ($\mathbb{Z}/\infty\mathbb{Z}$ =puncture):

Genus	type D	type B
$g=0$		
$g=1$	$\mathscr{B}\mathrm{r}(1,n)_{\mathbb{Z}/2\mathbb{Z}} \cong \mathrm{AT}(\mathsf{D}_n)$	$\mathscr{B}\mathrm{r}(1,n)_{\mathbb{Z}/\infty\mathbb{Z}} \cong \mathrm{AT}(\mathsf{B}_n)$
$g=2$	$\mathscr{B}\mathrm{r}(2,n)_{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}} \cong \mathrm{AT}(\tilde{\mathsf{D}}_n)$	$\mathscr{B}\mathrm{r}(2,n)_{\mathbb{Z}/\infty\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}} \cong \mathrm{AT}(\tilde{\mathsf{B}}_n)$
$g\geq 3$		

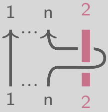
(For orbifolds "genus" is just an analogy.)

$\cos(\pi/4)$ twice on a line:

type ... 0^2

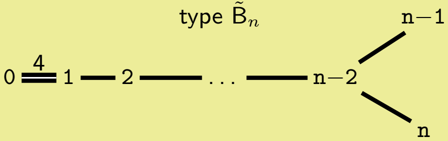
Affine adds genus

$\beta_{0^1} \mapsto$ 

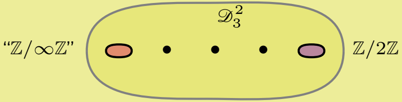


Allcock ∼1999. T ... $\xrightarrow{\cong} \mathscr{B}\mathrm{r}(2,n)$.

Example.



$\rightsquigarrow$



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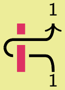
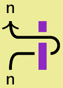


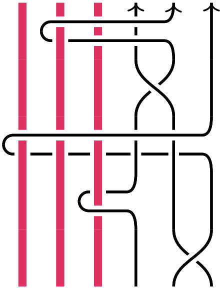
order ∞

order 2



Please stop!

Philosophy 1: Reshetikhin–Turaev with "huge" colors.



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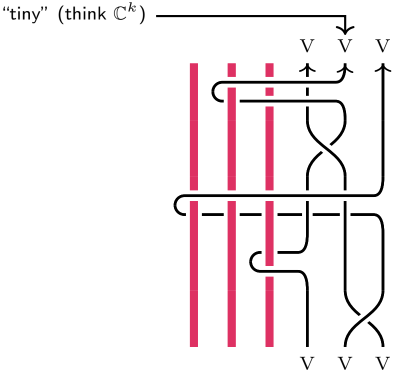

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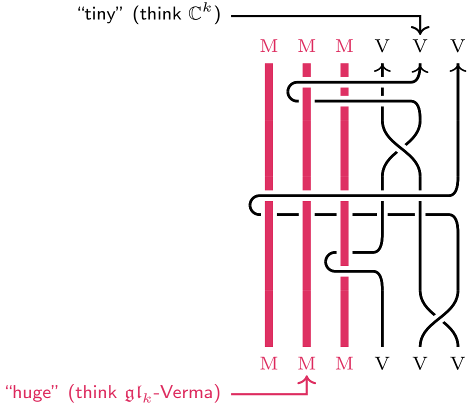

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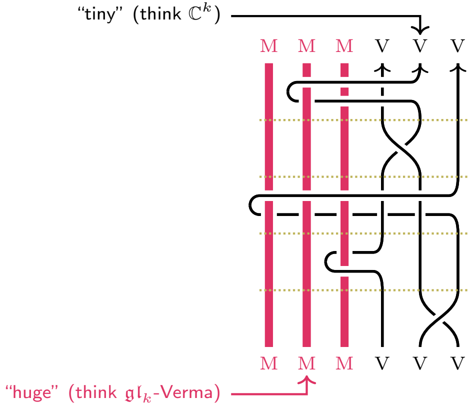

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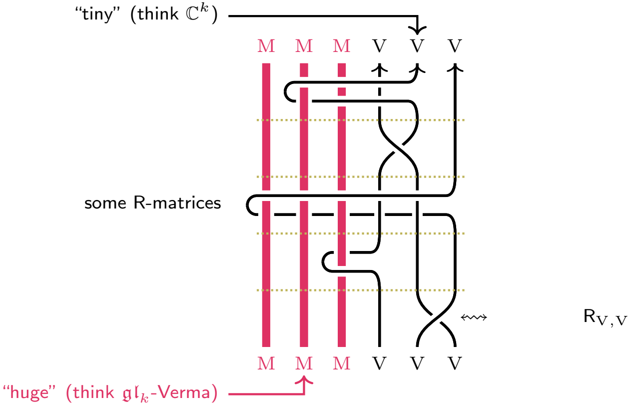

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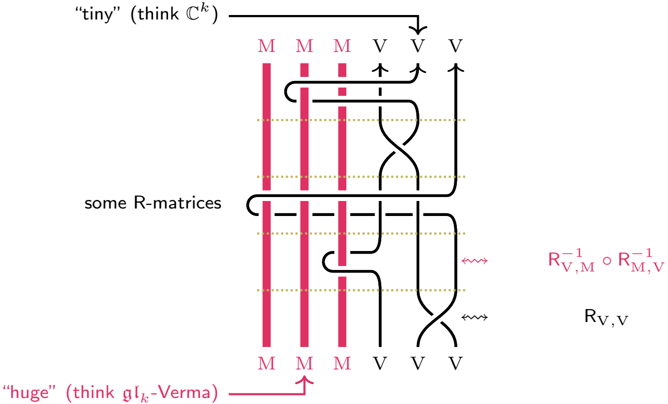

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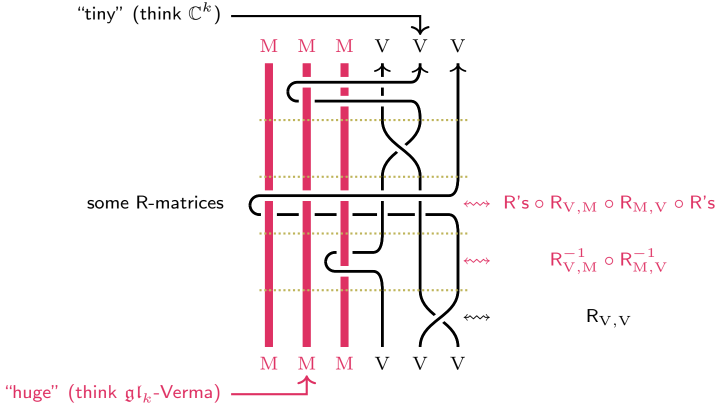

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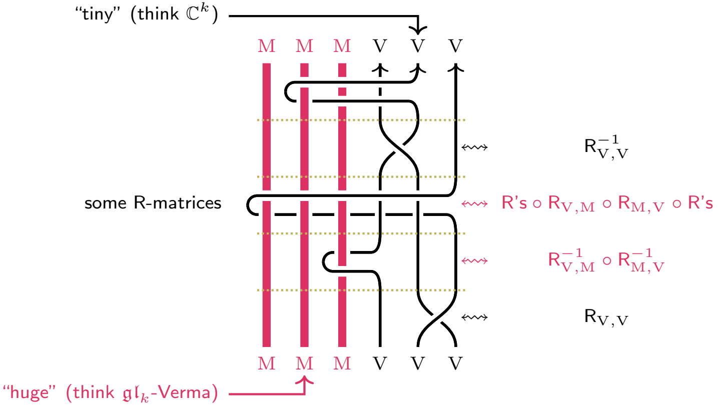

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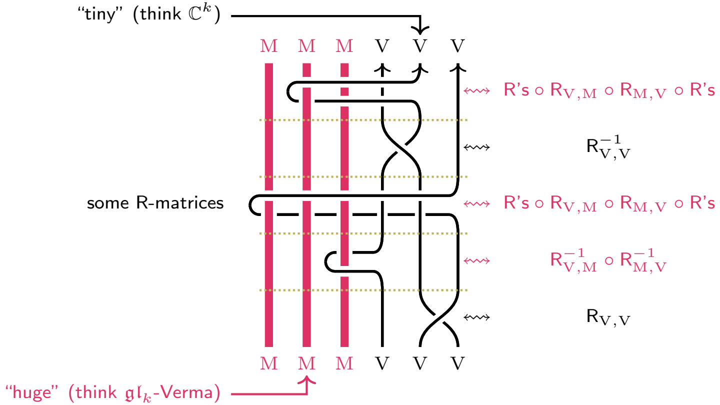
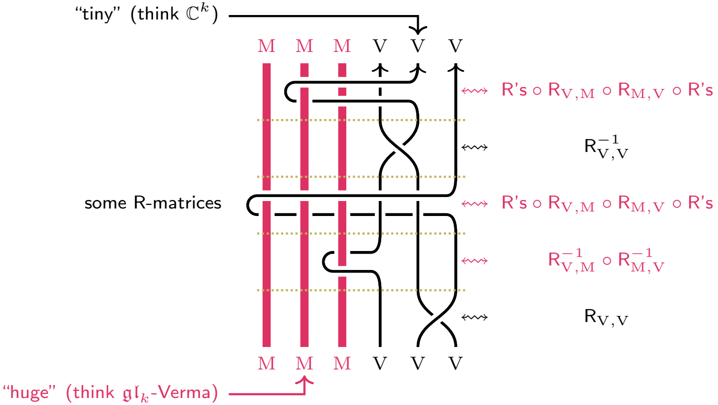


Philosophy 1: Reshe

Note that the type A embedding guarantees that any usual invariant of braids produces an invariant of braids in $\mathscr{H}_g$.



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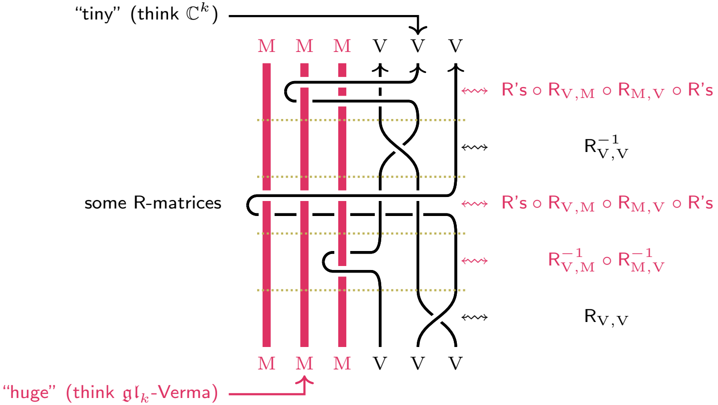

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Works quite well (*e.g.* use Naisse–Vaz's ideas on the categorified level).

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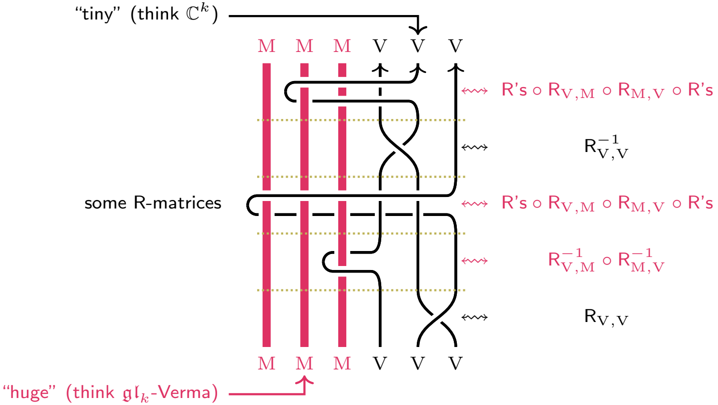


Genus $g = 0, 1$.

Works quite well (*e.g.* use Naisse–Vaz's ideas on the categorified level).

We mimic this for M being "huge, but finite".

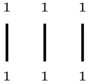
Singular Soergel bimodules $\mathscr{S}_{\mathrm{s}}^{\mathbf{q}}(\mathrm{W})$ for $\mathrm{W} = \mathrm{W}(\mathsf{A}_{N-1})$.

Tuples $\mathtt{I} = (k_1, \dots, k_N) \in \mathbb{N}_{\geq 1}^N$ with $k_1 + \dots + k_N = N \leftrightsquigarrow$ parabolic subgroups

$$\mathrm{W}_{\mathtt{I}} = \mathrm{W}(\mathsf{A}_{k_1-1}) \times \dots \times \mathrm{W}(\mathsf{A}_{k_N-1}) \subset \mathrm{W}.$$

W acts on $\mathrm{R} = \mathrm{R}_N = \mathbb{k}[\mathtt{x}_1, \dots, \mathtt{x}_N]$ via permutation $\rightsquigarrow$ rings of invariants $\mathrm{R}^{\mathtt{I}}$.

Bimodules. Identities, restriction ("merge") and induction ("split"), *e.g.*



$\leftrightsquigarrow \mathrm{R}^{(1,1,1)} = \mathrm{R},$


$\leftrightsquigarrow \mathrm{R}^{(2,1)} = \mathrm{R}^{\sigma_1} = \mathbb{k}[\mathtt{x}_1 + \mathtt{x}_2, \mathtt{x}_1\mathtt{x}_2, \mathtt{x}_3].$

$\leftrightsquigarrow$ shift$\mathrm{R}^{(k+l)} \otimes_{\mathrm{R}^{(k+l)}} \mathrm{R}^{(k,l)},$


$\leftrightsquigarrow \mathrm{R}^{(k,l)} \otimes_{\mathrm{R}^{(k+l)}} \mathrm{R}^{(k+l)}.$

Define $\mathscr{S}_{\mathrm{s}}^{\mathbf{q}}(\mathrm{W})$ as the full 2-subcategory of the rings&bimodules 2-category.

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W acts on $\mathrm{R} = \mathrm{R}_N =$ … gs of invariants $\mathrm{R}^{\mathtt{I}}$.

Everything is $\mathbb{Z}$-graded, called $\mathbf{q}$-grading.
I just omit this for simplicity.

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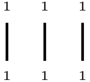

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Singular Soergel bimodules $\mathscr{S}_s^q(W)$ for $W = W(A_{N-1})$.

A monoidal structure is given by

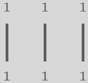
$=$


$\leftarrow$ glue $\rightarrow$


$\rightsquigarrow R \otimes_{R^{\sigma_1}} R \cong R \otimes_{R^{\sigma_1}} R^{\sigma_1} \otimes_{R^{\sigma_1}} R.$

This gives a way to define bimodules associated to any web built out of merge and split.

Bimodules. Identities, restriction ("merge") and induction ("split"), *e.g.*



$\rightsquigarrow R^{(1,1,1)} = R,$


$\rightsquigarrow R^{(2,1)} = R^{\sigma_1} = \mathbb{k}[x_1 + x_2, x_1x_2, x_3].$

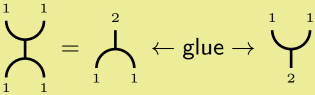
$\rightsquigarrow \text{shift}R^{(k+l)} \otimes_{R^{(k+l)}} R^{(k,l)},$


$\rightsquigarrow R^{(k,l)} \otimes_{R^{(k+l)}} R^{(k+l)}.$

Define $\mathscr{S}_s^q(W)$ as the full 2-subcategory of the rings&bimodules 2-category.

Singular Soergel bimodules $\mathscr{S}_s^q(W)$ for $W = W(A_{N-1})$.

A monoidal structure is given by



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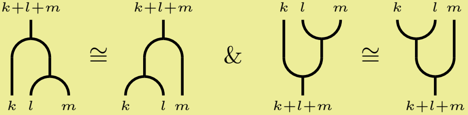
Bimodules. Identi ... split"), *e.g.*

$\rightsquigarrow R^{(}$ ... $_1 + x_2, x_1x_2, x_3]$.

$\rightsquigarrow$ shift ... $\otimes_{R^{(k+l)}} R^{(k+l)}$.

Define $\mathscr{S}_s^q(W)$ as ... es 2-category.

There are several bimodule isomorphisms, *e.g.*



Hence, we can unambiguously write

which one could call thick merge and split.

Singular Soergel bimodules $\mathscr{S}_s^{\mathbf{q}}(\mathrm{W})$ for $\mathrm{W} = \mathrm{W}(\mathsf{A}_{N-1})$.

Soergel ∼1992, Williamson ∼2010.

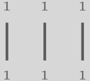
$\mathscr{S}_s^{\mathbf{q}}(\Gamma)$ categorifies the Hecke algebra (or rather, the algebroid).

Tuples $\mathrm{I} = \dots$ subgroups

$$\mathrm{W}_{\mathrm{I}} = \mathrm{W}(\mathsf{A}_{k_1-1}) \times \cdots \times \mathrm{W}(\mathsf{A}_{k_N-1}) \subset \mathrm{W}.$$

W acts on $\mathrm{R} = \mathrm{R}_N = \Bbbk[\mathrm{x}_1, \dots, \mathrm{x}_N]$ via permutation $\rightsquigarrow$ rings of invariants R^{I}.

Bimodules. Identities, restriction (“merge”) and induction (“split”), *e.g.*



$\longleftrightarrow \mathrm{R}^{(1,1,1)} = \mathrm{R}$,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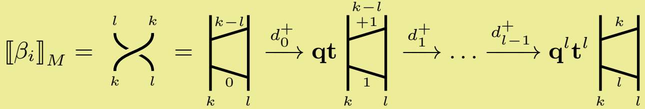
Tuples $\mathrm{I} =$ … subgroups

Rouquier ∼2004, Mackaay–Stošić–Vaz ∼2008, Webster–Williamson ∼2009, etc.

There are certain complex ("**t**-graded") of singular Soergel bimodules, *e.g.*



providing a categorical action of the Artin–Tits group of type A.

1 1 1 2 1

$\rightsquigarrow$ shift$\mathrm{R}^{(k+l)} \otimes_{\mathrm{R}^{(k+l)}} \mathrm{R}^{(k,l)}$,

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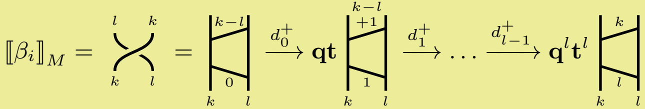
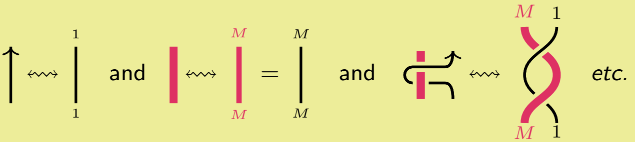


providing a categorical action of the Artin–Tits group of type A.

Hence, we are in business by taking $M \ggg n$:



Define $\mathscr{S}$ … gory.

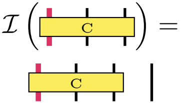
Fact. This gives a faithful representation of $[\![b]\!]_M$ of $b \in \mathscr{B}\mathrm{r}(g, n)$.

Partial Hochschild homology (à la Hogancamp ∼2015). $\mathrm{R}\text{-}f\mathscr{B}\mathrm{im}_N^{\mathbf{atq}}$ category of (▸ bicomplexes of) **q**-graded, free R_N-bimodules. Adjoint pair $(\mathcal{I},\mathcal{T})$:

$$\mathcal{I}\colon \mathrm{R}\text{-}f\mathscr{B}\mathrm{im}_{N-1}^{\mathbf{atq}} \to \mathrm{R}\text{-}f\mathscr{B}\mathrm{im}_N^{\mathbf{atq}}$$

$$\mathrm{B} \mapsto \mathrm{B} \otimes_{\mathrm{R}^{\mathrm{e}}_{N-1}} (\mathrm{R}^{\mathrm{e}}_N/(\mathtt{x}_N \otimes 1 - 1 \otimes \mathtt{x}_N)) \quad \leftrightsquigarrow$$

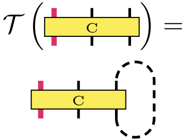
extending scalars



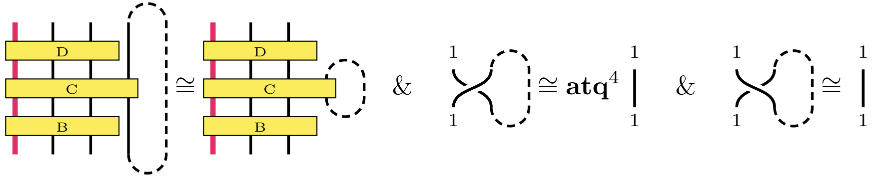
$$\mathcal{T}\colon \mathrm{R}\text{-}f\mathscr{B}\mathrm{im}_N^{\mathbf{atq}} \to \mathrm{R}\text{-}f\mathscr{B}\mathrm{im}_{N-1}^{\mathbf{atq}}$$

$$\mathrm{B} \mapsto (\mathrm{B} \xrightarrow{\mathtt{x}_N.\mathtt{b} - \mathtt{b}.\mathtt{x}_N} \mathbf{aq}^2\mathrm{B}) \quad \leftrightsquigarrow$$

identifying left-right action



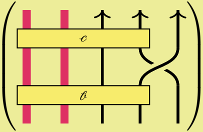
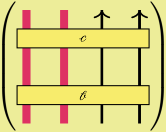
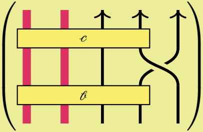
Skein relations. One gets *e.g.*



Partial Hochschild homology (à la Hogancamp ~2015). $\mathrm{R}\text{-}f\mathscr{B}\mathrm{im}_N^{\mathbf{atq}}$ category of (▸ bicomplexes of) q-graded, free R_N-bimodules. Adjoint pair $(\mathcal{I},\mathcal{T})$:

Theorem (after normalization).

We get a triply-graded invariant $\mathrm{HHH}^{\star}_{M}(\mathscr{b}) \in \Bbbk\text{-}\mathcal{V}\mathrm{ect}^{\mathbf{atq}}$ for $\mathscr{b} \in \mathscr{B}\mathrm{r}(g,n)$, which respects Markov stabilization, *i.e.*

$\mathrm{HHH}^{\star}_{M}$  $\cong \mathrm{HHH}^{\star}_{M}$  $\cong \mathrm{HHH}^{\star}_{M}$ 

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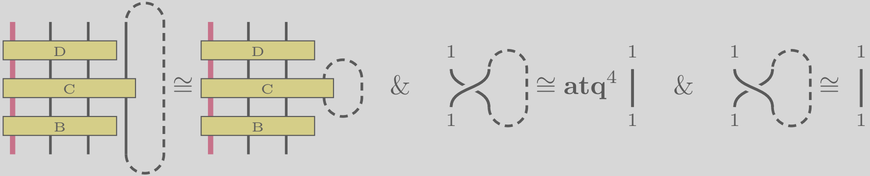

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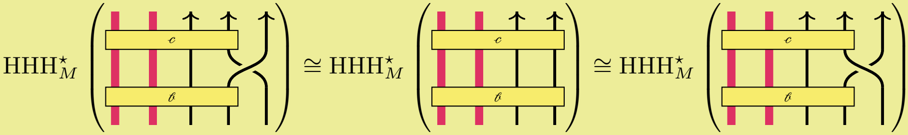

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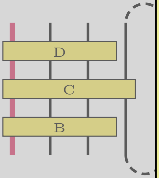
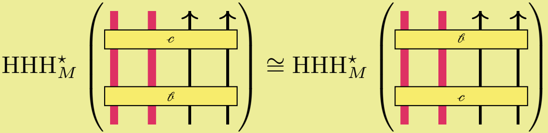
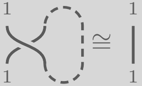


However, we are not quite there:
one gets a too strong Markov conjugation, *i.e.*





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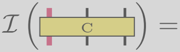
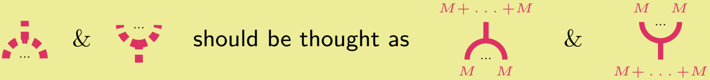
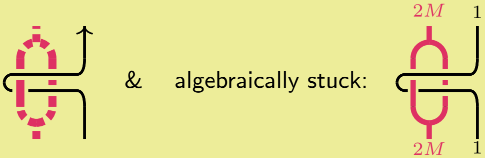


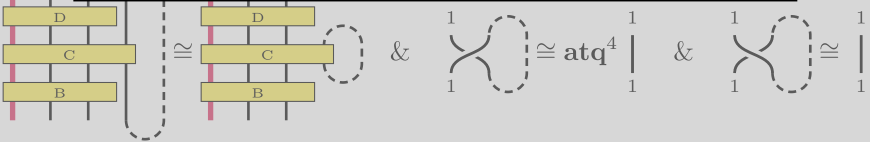
Idea: Flank them!



and things get stuck, *e.g.*



Skein re



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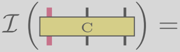

Theorem (after normalization and flanking).

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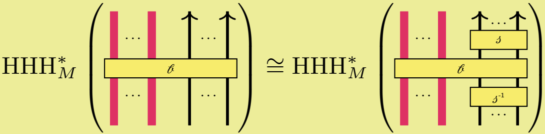
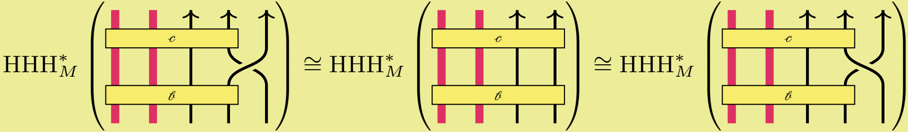
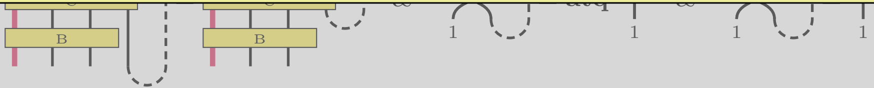
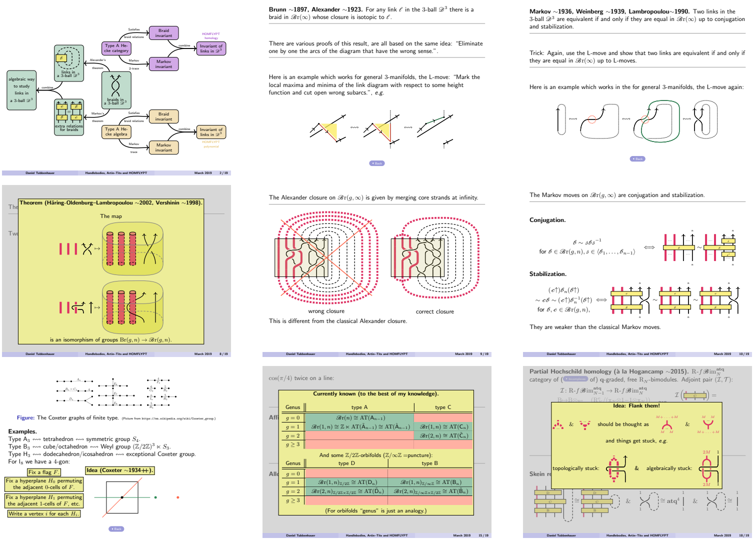
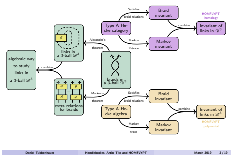


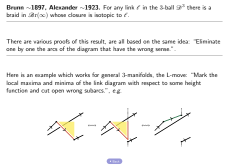


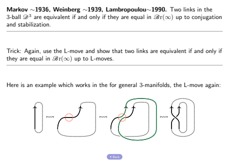


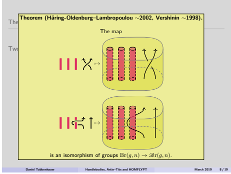


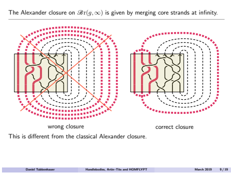
There is still much to do...

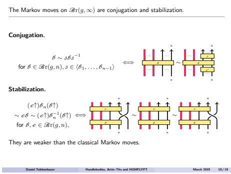


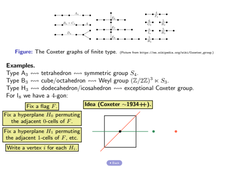


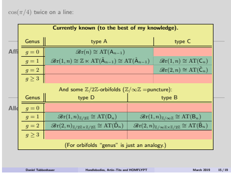


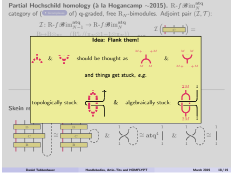












Thanks for your attention!

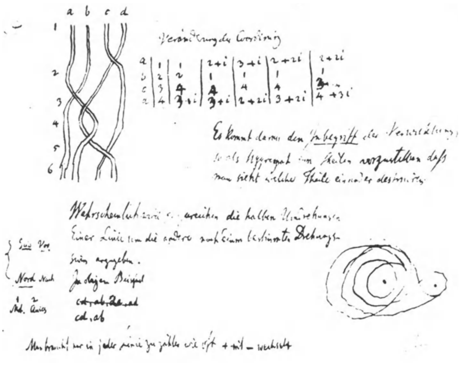
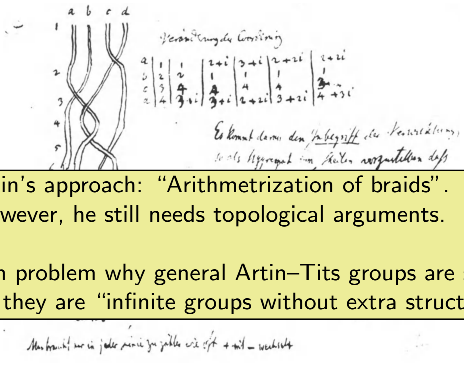


Figure: The first ever "published" braid diagram. (Page 283 from Gauß' handwritten notes, volume seven, $\leq$1830).

Tits $\sim$1961$++$. Gauß' braid group is the type A case of more general groups. (We come back to this later.)

Back



Artin's approach: "Arithmetrization of braids".
However, he still needs topological arguments.

And this is one main problem why general Artin–Tits groups are so complicated:
Basically, they are "infinite groups without extra structure".

Figure: The first ever "published" braid diagram. (Page 283 from Gauß' handwritten notes, volume seven, $\leq$1830).

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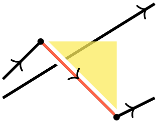
Back

Brunn ∼1897, Alexander ∼1923. For any link ℓ in the 3-ball $\mathscr{D}^3$ there is a braid in $\mathscr{B}\mathrm{r}(\infty)$ whose closure is isotopic to ℓ.

There are various proofs of this result, are all based on the same idea: "Eliminate one by one the arcs of the diagram that have the wrong sense.".

Here is an example which works for general 3-manifolds, the L-move: "Mark the local maxima and minima of the link diagram with respect to some height function and cut open wrong subarcs.", *e.g.*



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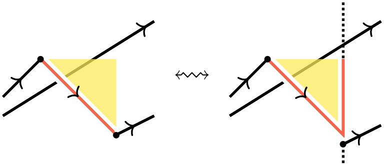

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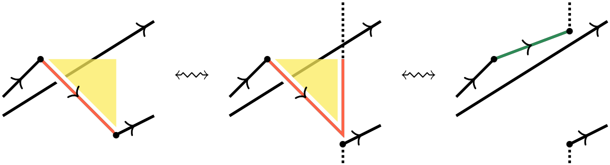

Markov ∼1936, Weinberg ∼1939, Lambropoulou∼1990. Two links in the 3-ball $\mathscr{D}^3$ are equivalent if and only if they are equal in $\mathscr{B}\mathrm{r}(\infty)$ up to conjugation and stabilization.

Trick: Again, use the L-move and show that two links are equivalent if and only if they are equal in $\mathscr{B}\mathrm{r}(\infty)$ up to L-moves.

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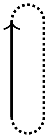

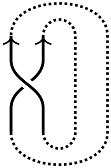

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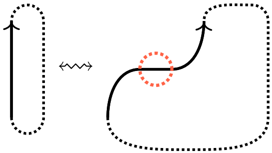

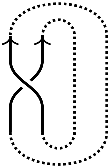

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Markov ∼1936, Weinberg ∼1939, Lambropoulou∼1990. Two links in the 3-ball $\mathscr{D}^3$ are equivalent if and only if they are equal in $\mathscr{B}\mathrm{r}(\infty)$ up to conjugation and stabilization.

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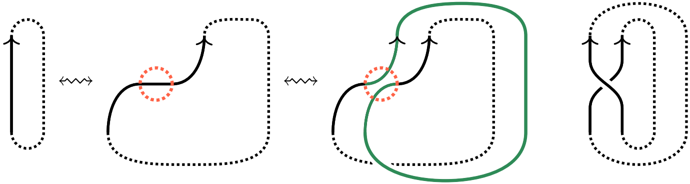

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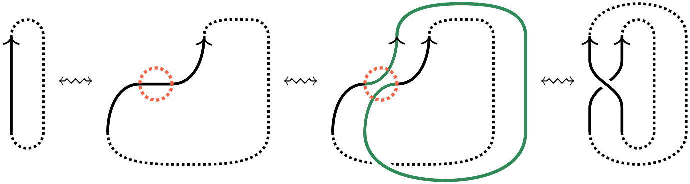

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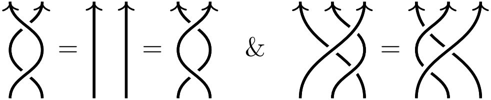
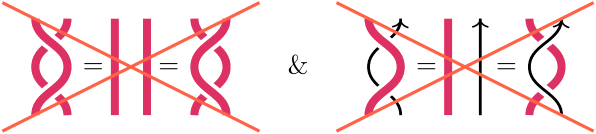
The Reidemeister braid relations:



These hold for usual strands only since core strands do not cross each other, *e.g.*



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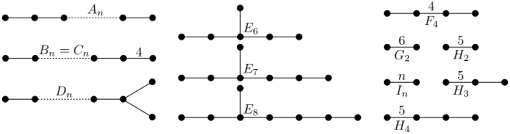
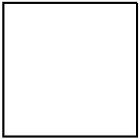


Figure: The Coxeter graphs of finite type. (Picture from https://en.wikipedia.org/wiki/Coxeter_group.)

Examples.

Type A_3 $\leftrightsquigarrow$ tetrahedron $\leftrightsquigarrow$ symmetric group S_4.
Type B_3 $\leftrightsquigarrow$ cube/octahedron $\leftrightsquigarrow$ Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \ltimes S_3$.
Type H_3 $\leftrightsquigarrow$ dodecahedron/icosahedron $\leftrightsquigarrow$ exceptional Coxeter group.
For I_8 we have a 4-gon:

Idea (Coxeter ∼1934++).



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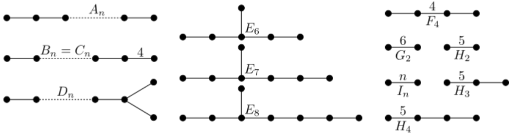

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Type A_3 ⇝ tetrahedron ⇝ symmetric group S_4.
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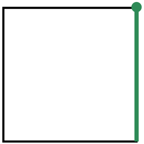
For I_8 we have a 4-gon:

Fact. The symmetries are given by exchanging flags.

Fix a flag F.

Idea (Coxeter ∼1934++).



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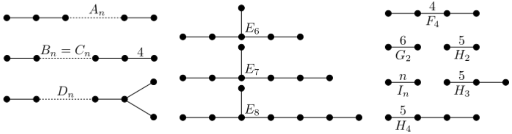

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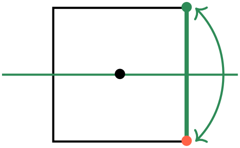
For I_8 we have a 4-gon:

Fix a flag F.

Fix a hyperplane H_0 permuting the adjacent 0-cells of F.

Idea (Coxeter ∼1934++).



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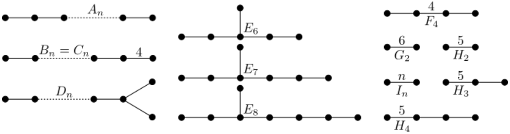

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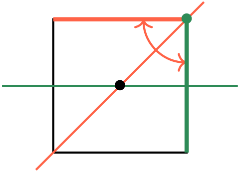

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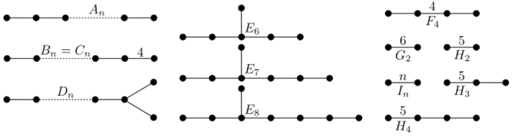

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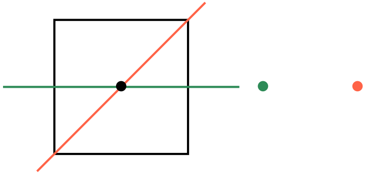
Fix a hyperplane H_1 permuting the adjacent 1-cells of F, etc.

Write a vertex i for each H_i.

Idea (Coxeter ∼1934++).



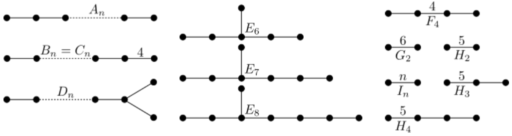


Figure: The Coxeter graphs of finite type. (Picture from https://en.wikipedia.org/wiki/Coxeter_group.)

Examples.

This gives a generator-relation presentation.

Type A_3 ⇝ tetrahedron ⇝ symmetric group S_4.

Type B_3 ⇝ And the braid relation measures the angle between hyperplanes.

Type H_3 ⇝ dodecahedron/icosahedron ⇝ exceptional Coxeter group.

For I_8 we have a 4-gon:

Fix a flag F.

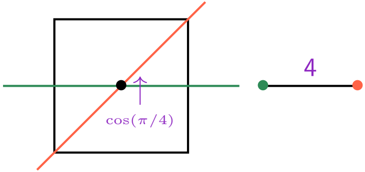
Fix a hyperplane H_0 permuting the adjacent 0-cells of F.

Fix a hyperplane H_1 permuting the adjacent 1-cells of F, etc.

Write a vertex i for each H_i.

Connect i, j by an n-edge for H_i, H_j having angle $\cos(\pi/n)$.

Idea (Coxeter ∼1934++).



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Three gradings:

$\mathbf{q} \leftrightsquigarrow$ internal & $\mathbf{t} \leftrightsquigarrow$ homological & $\mathbf{a} \leftrightsquigarrow$ Hochschild

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Example. To compute Hochschild cohomology take the Koszul resolution

$$\bigotimes_{i=1}^{N} \left(\mathrm{R}^{\mathrm{e}} = \mathrm{R} \otimes \mathrm{R}^{\mathrm{op}} \xrightarrow{\cdot(\mathtt{x}_i \otimes 1 - 1 \otimes \mathtt{x}_i)} \mathbf{aq}^2 \mathrm{R}^{\mathrm{e}} \right),$$

Tensor it with B, gives a complex with differentials $\mathtt{x}_i \otimes 1 - 1 \otimes \mathtt{x}_i$, of which we think as identifying the variables. This gives a chain complex having non-trivial chain groups in $\mathbf{a}$-degree $0, \dots, n$. Here the i^{th} chain group consists of $\binom{n}{i}$ copies of B, with differentials given by the various ways of identifying i variables. The a^{th} cohomology $= a^{\mathrm{th}}$ Hochschild cohomology.

Example. If B is already a $\mathbf{t}$-graded complex, then one can take homology of it and gets "triple H".

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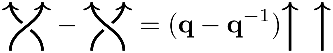
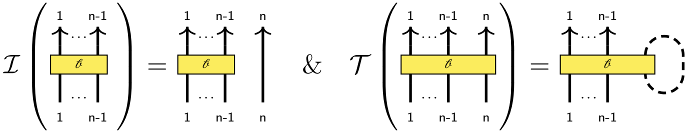
The type A Hecke algebra H_n is the quotient of $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]\mathscr{B}\mathrm{r}(n)$ by:



H_n is of dimension $n!$. (Proof: Over- and undercrossing are linear dependent. Hence, there is a basis given by diagrams in the symmetric group.)

Theorem (Jones ∼1987; Skein theory). There is a unique pair $\mathcal{I}\colon \mathrm{H}_{n-1} \to \mathrm{H}_n$ and $\mathcal{T}\colon \mathrm{H}_n \to \mathrm{H}_{n-1}$ of “adjoint functors”



which satisfy the Markov moves and are determined by

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