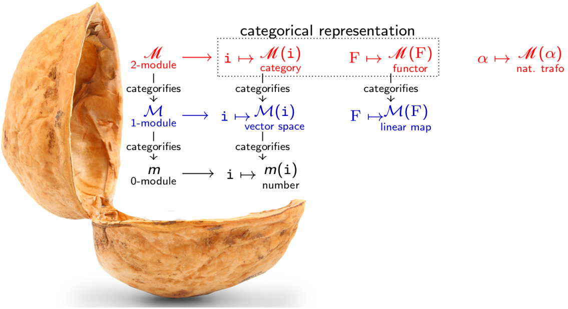
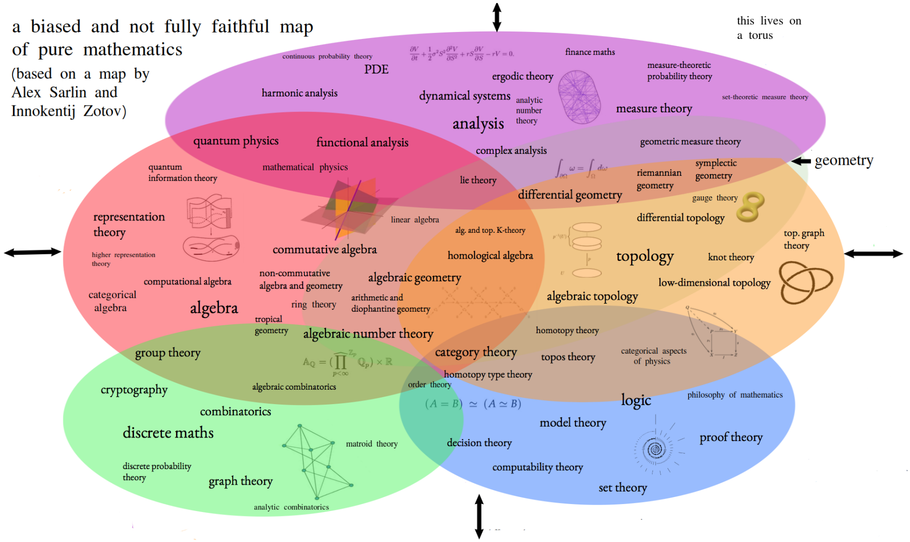
What is...2-representation theory?

Or: Of matrices and functors

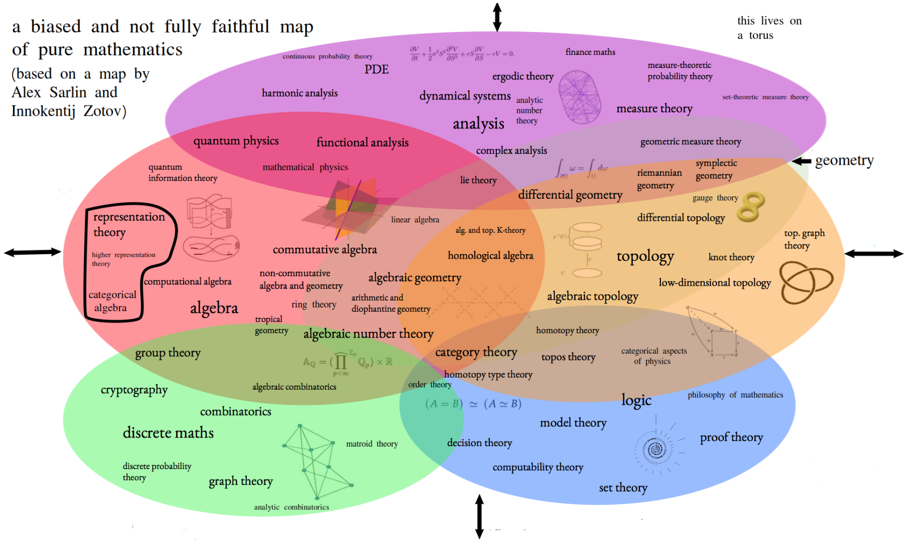
Daniel Tubbenhauer



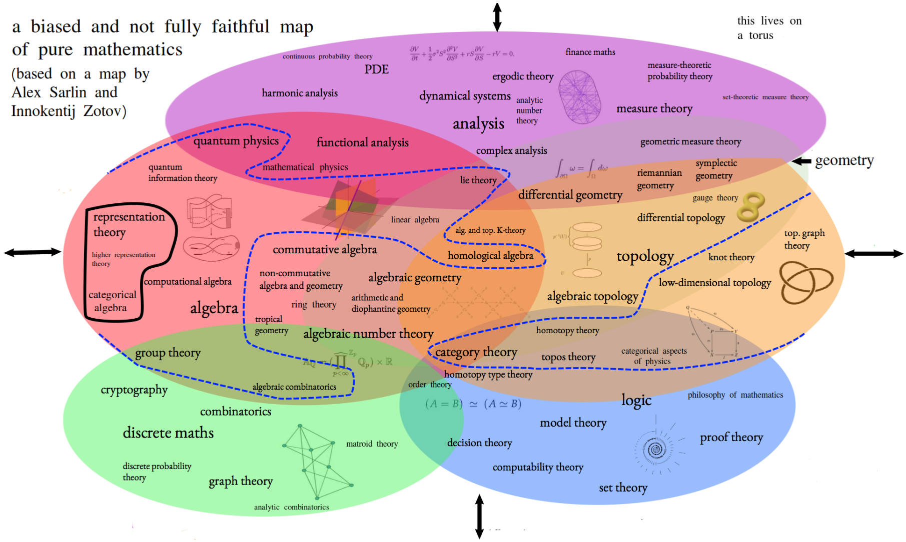
The map of pure mathematics.



The map of pure mathematics—my part of it.



The map of pure mathematics—my part of it and ramifications.

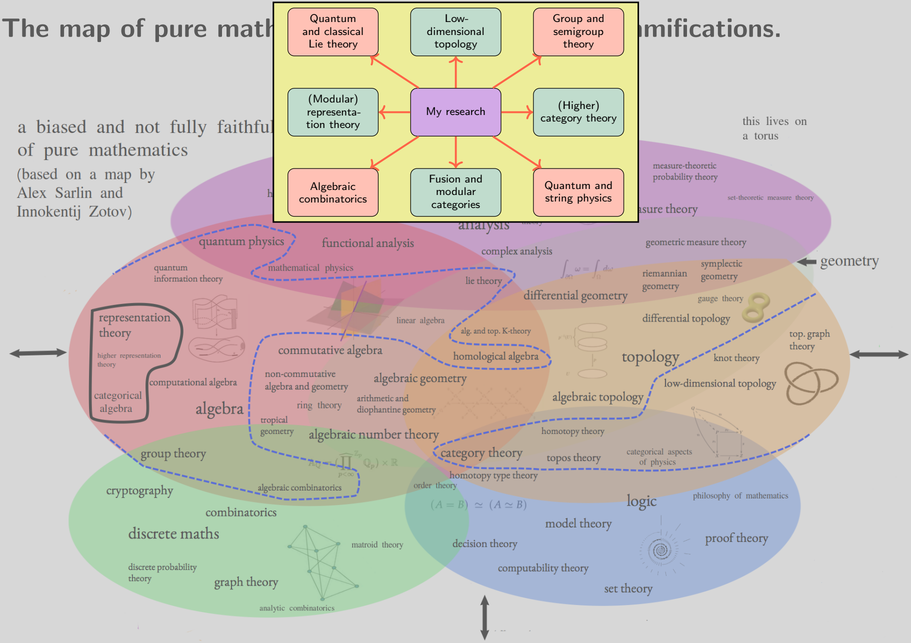


The map of pure math … mifications.

a biased and not fully faithful … of pure mathematics

(based on a map by Alex Sarlin and Innokentij Zotov)

this lives on a torus



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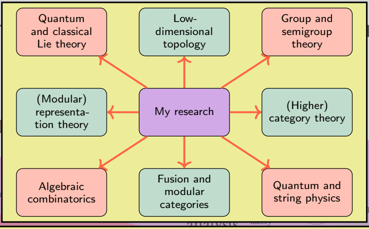

Some honorable mentions.

2-representation theory is a very modern version
of representation theory.

Categorification and positivity
are of importance in cluster theory.

My interest in semigroup theory (cells)
has overlap with ring and ideal theory.

2-representation theory (via modular representation theory)
connects to algebraic number theory and fractal geometry.

Various other connection (topology, category theory, mathematical physics...)
are currently explored.

The map of pure mathematics—my part of it and ramifications.

Today.

A short tour through 2-representation theory (black)

keeping an eye on its ramifications (blue).

a biased and not fu... of pure mathematic...

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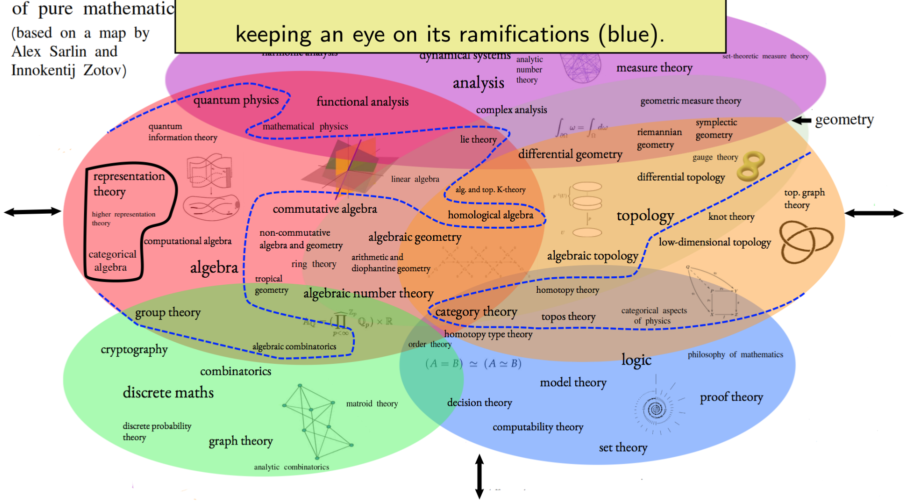
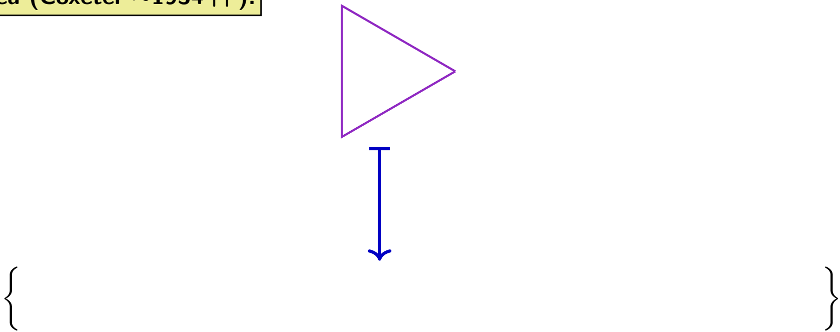


Slogan. Representation theory is group theory in vector spaces.

symmetries of n-gons $\subset \mathcal{A}\mathrm{ut}(\mathbb{R}^2)$

Idea (Coxeter ∼1934++).



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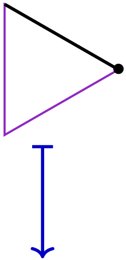

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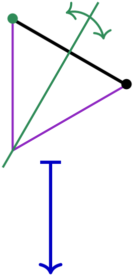
Fix a hyperplane H_0 permuting the adjacent 0-cells of F.



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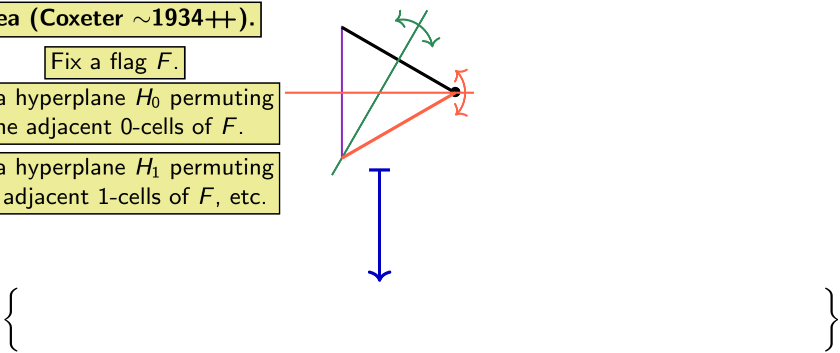

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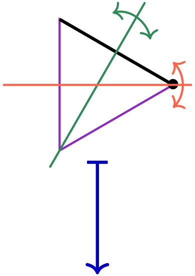
This gives a generator-relation presentation.



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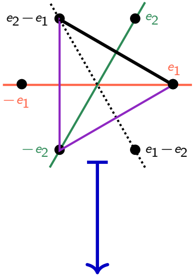

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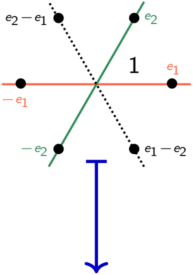

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \right\}$$

$$1$$

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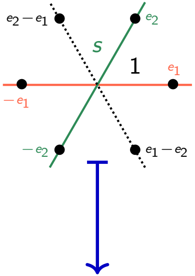

$$\left\{ \underset{1}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{s}{\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}}, \qquad \right\}$$

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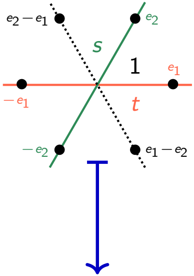

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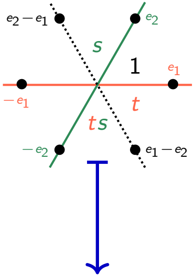

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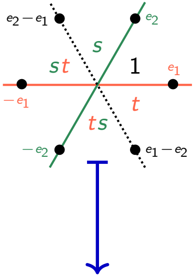
symmetries of *n*-gons $\subset \mathcal{A}\mathrm{ut}(\mathbb{R}^2)$



$$\left\{ \underset{1}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{s}{\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}}, \underset{t}{\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}}, \underset{ts}{\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}}, \underset{st}{\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}}, \qquad \right\}$$

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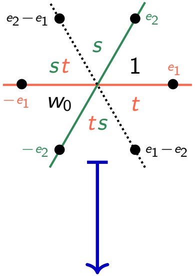

$$\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}-1&0\\1&1\end{pmatrix},\begin{pmatrix}1&1\\0&-1\end{pmatrix},\begin{pmatrix}0&1\\-1&-1\end{pmatrix},\begin{pmatrix}-1&-1\\1&0\end{pmatrix},\begin{pmatrix}0&-1\\-1&0\end{pmatrix}\right\}$$

1 $\quad s$ $\quad t$ $\quad ts$ $\quad st$ $\quad sts=tst$ (w_0)

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The representation theory approach.

Reduce a non-linear problem to questions in linear algebra.

Problem involving a group action $\mathrm{G} \circlearrowright \mathtt{X}$

"linearize"

Problem involving a linear group action $\mathbb{K}[\mathrm{G}] \circlearrowright \mathbb{K}\mathtt{X}$

Decomposition of the problem into simples

new insights?

$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right.$ … $\left. \begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix} \right) \Big\}$

1 … $ts = tst$

w_0

Pioneers of representation theory

Let G be a finite group.

Frobenius ∼1895++, Burnside ∼1900++. Representation theory is the ▸ useful? study of linear group actions

$$\mathcal{M}\colon \mathrm{G} \longrightarrow \mathcal{A}\mathrm{ut}(\mathtt{V}), \qquad \boxed{\text{“}\mathcal{M}(g) = \text{a matrix in } \mathcal{A}\mathrm{ut}(\mathtt{V})\text{”}}$$

with $\mathtt{V}$ being some vector space. (Called modules or representations.)

The “atoms” of such an action are called simple. A module is called semisimple if it is a direct sum of simples.

Maschke ∼1899. All modules are built out of simples (“Jordan–Hölder” filtration).

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We want to have a categorical version of this!

Pioneers of representation theory

Let A be a finite-dimensional algebra.

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$$\mathcal{M}\colon \mathrm{A} \longrightarrow \mathcal{E}\mathrm{nd}(\mathtt{V}),$$

"$\mathcal{M}(a)$ = a matrix in $\mathcal{E}\mathrm{nd}(\mathtt{V})$"

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We want to have a categorical version of this.

I am going to explain what we can do at present.

collection ("category") of modules $\leftrightsquigarrow$ the world

---

modules $\leftrightsquigarrow$ chemical compounds

---

simples $\leftrightsquigarrow$ elements

---

semisimple (*e.g.* groups & $\mathbb{C}$) $\leftrightsquigarrow$ only trivial compounds

---

non-semisimple (*e.g.* semigroups/algebras) $\leftrightsquigarrow$ non-trivial compounds

---

Main goal of representation theory. Find the periodic table of simples.

collection (

modules ⇝

simples ⇝

Example.

$$\mathbb{Z}/2\mathbb{Z} \to \mathcal{E}\mathrm{nd}(\mathbb{C}^2),\ 0 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ \& \ 1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Common eigenvectors: $(1,1)$ and $(1,-1)$ and base change gives

$$0 \mapsto \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right) \ \& \ 1 \mapsto \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right)$$

and the module decomposes.

semisimple (*e.g.* groups & $\mathbb{C}$) ⇝ only trivial compounds

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$$\mathbb{C}[X]/(X^2) \to \mathcal{A}\mathrm{ut}(\mathbb{C}^2),\ 1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ \&\ X \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

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and the module is non-simple, yet does not decompose.

collection (

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Morally: representation theory of algebra is "rarely" semisimple.

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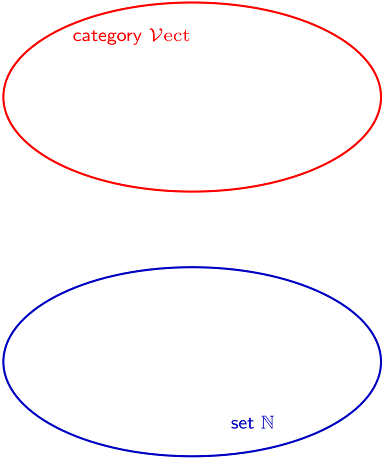
Categorification in a nutshell



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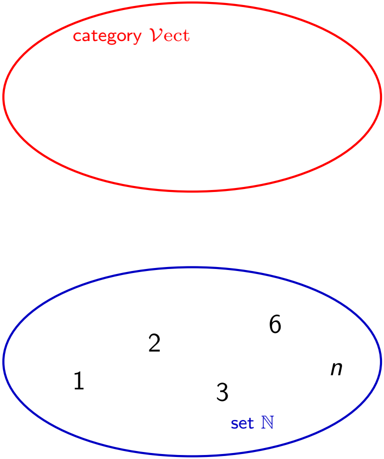

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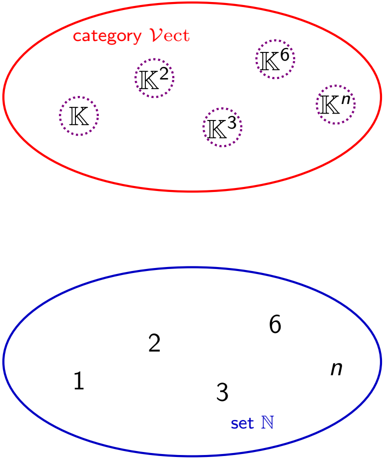

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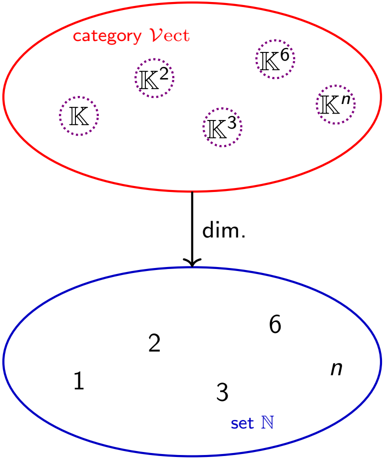

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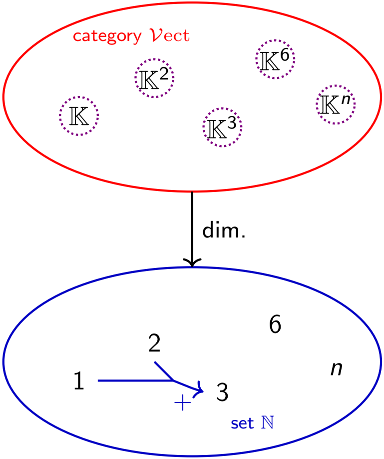

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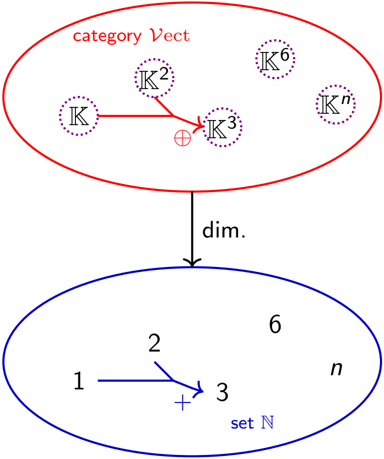

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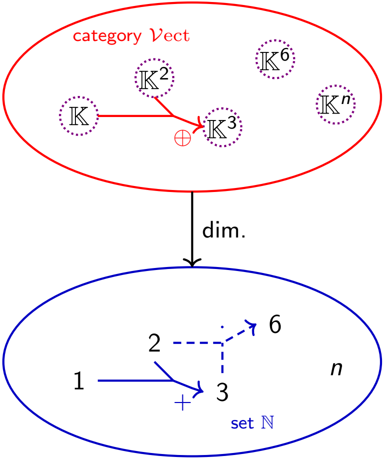

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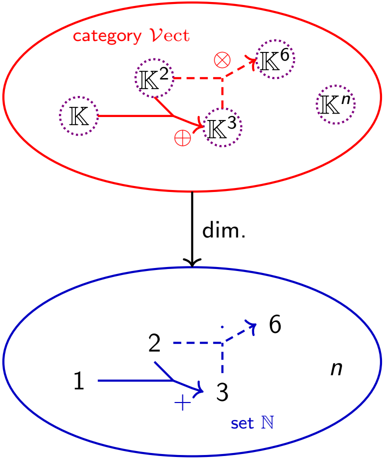

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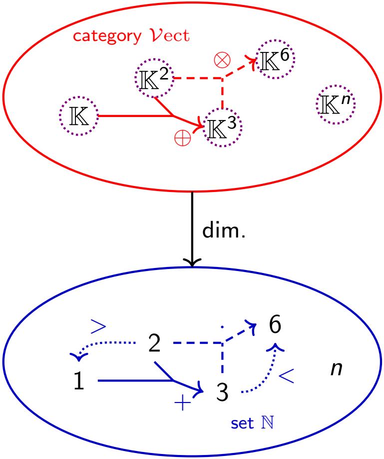

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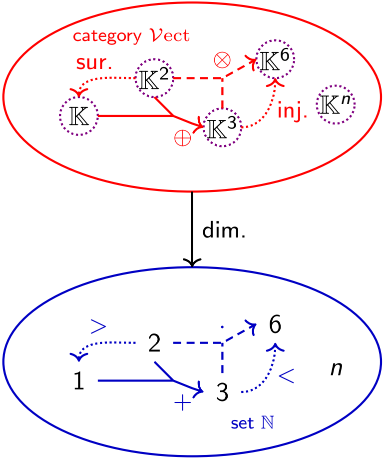

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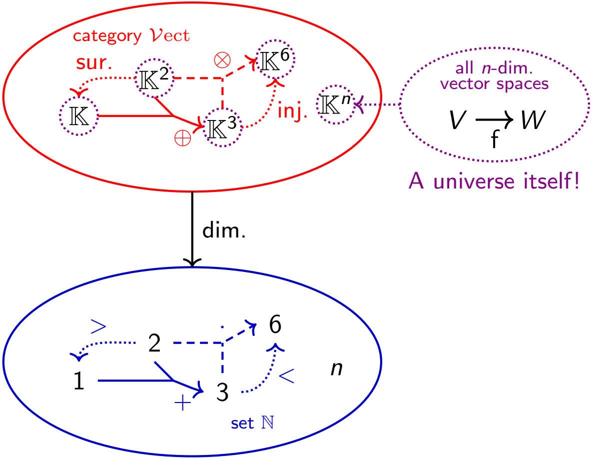

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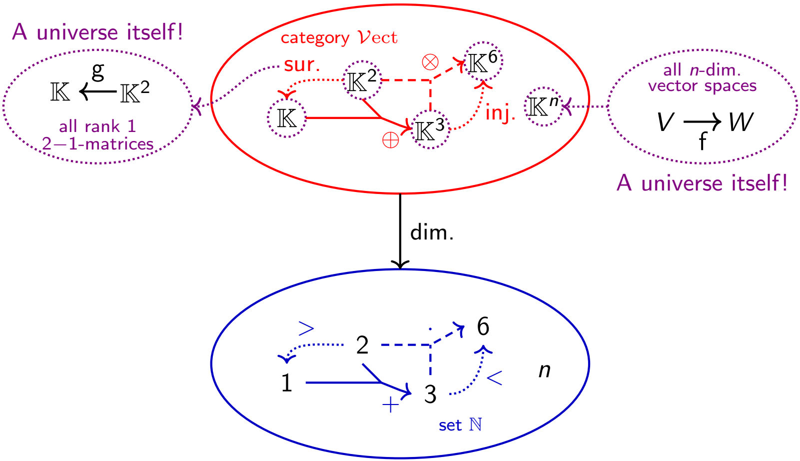
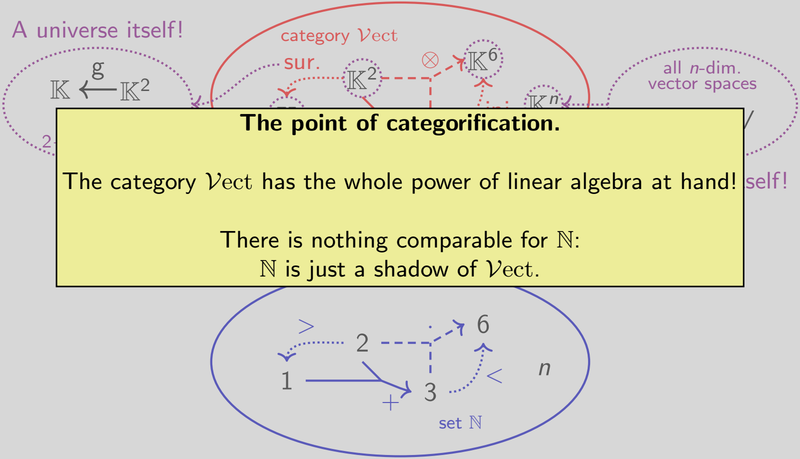


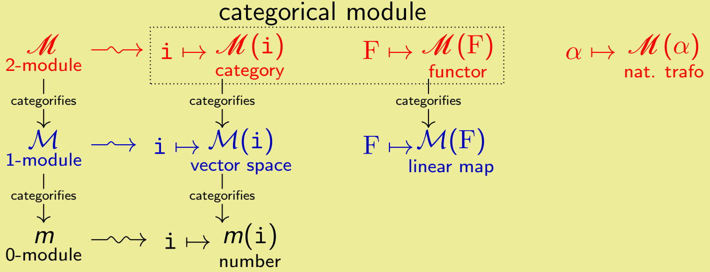


The point of categorification.

The category $\mathcal{V}$ect has the whole power of linear algebra at hand!

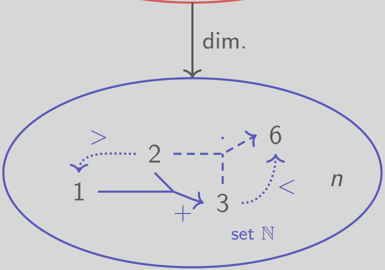
There is nothing comparable for $\mathbb{N}$:
$\mathbb{N}$ is just a shadow of $\mathcal{V}$ect.

Slogan. 2-representation theory is group theory in categories.



A

A universe itself!



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categorical module

$\mathscr{M}$ 2-module $\rightsquigarrow$	$\mathtt{i} \mapsto \mathscr{M}(\mathtt{i})$ category	$\mathrm{F} \mapsto \mathscr{M}(\mathrm{F})$ functor	$\alpha \mapsto \mathscr{M}(\alpha)$ nat. trafo
categorifies	categorifies	categorifies	
$\mathcal{M}$ 1-module $\rightsquigarrow$	$\mathtt{i} \mapsto \mathcal{M}(\mathtt{i})$ vector space	$\mathrm{F} \mapsto \mathcal{M}(\mathrm{F})$ linear map	
categorifies	categorifies		
m 0-module $\rightsquigarrow$	$\mathtt{i} \mapsto m(\mathtt{i})$ number		

A universe itself!

What one can hope for.

Problem involving a group action $\mathrm{G} \circlearrowright \mathtt{X}$ ⇢ "categorify" ⇢ Problem involving a categorical group action ⇢ Decomposition of the problem into 2-simples ⇢ new insights? ⇢ $\mathrm{G} \circlearrowright \mathtt{X}$

What I am working on—three flavors of categorical representation theory.

Clearly, there are many ways to go from here. My main paths at the moment:

(A) Finitary 2-representation theory. (I will discuss this in a second.)
Comment. This categorifies the representation theory of finite-dimensional algebras.
Main goals. Find the periodic tables of 2-simples, advance the abstract theory.
Ramifications. (Modular) representation theory, categorical algebra, (higher) category theory, group and semigroup theory.

(B) 2-representation theory in Lie theory.
Comment. Related to various flavors of geometric representation theory.
Main goals. Study classical categories by studying functors acting on them.
Ramifications. Classical and Lie theoretic algebra, (modular) representation theory, algebraic combinatorics, Kazhdan–Lusztig theory.

(C) 2-representation theory in topology.
Comment. Related to the celebrated link homologies and categorical braid group actions.
Main goals. Find "hidden or higher structures" in 3- or 4-dimensional topology.
Ramifications. Low-dimensional topology, representation theory, quantum Lie theory, quantum and string physics, homological algebra.

Slogan. Finitary 2-representation theory is the study of algebras via categories.

Let $\mathscr{C}$ be a ▸ finitary 2-category.

Etingof–Ostrik, Chuang–Rouquier, many others ∼2000++. Higher representation theory is the ▸ useful? study of actions of 2-categories:

$$\mathscr{M}\colon \mathscr{C} \longrightarrow \mathscr{E}\mathrm{nd}(\mathcal{V}), \quad \boxed{\text{“}\mathscr{M}(\mathrm{F}) = \text{a functor in } \mathscr{E}\mathrm{nd}(\mathcal{V})\text{”}}$$

with $\mathcal{V}$ being some finitary category. (Called 2-modules or 2-representations.)

The "atoms" of such an action are called 2-simple.

Mazorchuk–Miemietz ∼2014. All (suitable) 2-modules are built out of 2-simples ("weak 2-Jordan–Hölder filtration").

Slogan. Finitary 2-representation theory is the study of algebras via categories.

Let $\mathscr{C}$ be a ▸ finitary 2-category.

Etingof–Ostrik, Chuang–Rouquier, many others ∼2000++. Higher representation theory is the ▸ useful? study of actions of 2-categories:

$$\mathscr{M} : \mathscr{C} \longrightarrow \mathscr{E}\mathrm{nd}(\mathcal{V}),$$

with

Main goal of 2-representation theory. Find the periodic table of 2-simples.

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Let $\mathscr{C}$

Example. (Group-like) $\mathscr{C} = \mathcal{V}\mathrm{ec}_G$ or $\mathcal{R}\mathrm{ep}(G)$.
Status. Semisimple, classification of 2-simples well-understood. ▸ Example
Comments. $\mathcal{V}\mathrm{ec}_G$ can be see as the categorical analog of G.

Example. (Group-like) $\mathscr{C} = \mathcal{R}\mathrm{ep}_q^{sesi}(g)_{\text{level } n}$.
Status. Semisimple, finitely many 2-simples,
classification of 2-simples only known for $g = \mathrm{SL}_2$, some guesses for general g.
Comments. These categories and their 2-representations arose from quantum physics.

with $\mathcal{V}$ being some finitary category. (Called 2-modules or 2-representations.)

Example. (Semigroup-like) $\mathscr{C}$ = Hecke category.
Status. Non-semisimple, we (finally—after 10 years) have now a complete classification by reducing the problem to the above examples.
Comments. The Hecke category (a categorification of the Hecke algebra) and its 2-representation play a crucial role in modern mathematics.
Our main result is a categorification of the theory of representations of Hecke algebras.

Slogan. Finitary 2-representation theory is the study of algebras via categories.

Let $\mathscr{C}$ be a

Question ("2-representation theory").

Find the periodic table 2-simples of a fixed finitary 2-category.

In a lot of cases this has a very satisfactory answer.

Etingof–O r

representat

$$\mathscr{M} : \mathscr{C} \longrightarrow \mathscr{E}\mathrm{nd}(\mathcal{V}),$$

with $\mathcal{V}$ being so entations.)

This is the categorification of

'Classify all simples a fixed finite-dimensional algebra',

but much harder, e.g. it really is like working over $\mathbb{Z}$ and non-semisimple.

The "atoms" of

Mazorchuk–M out of 2-simples ("weak 2-Jordan–Holder filtration").

Slogan. Finitary 2-representation theory is the study of algebras via categories.

Let $\mathscr{C}$ be a finitary 2-category.

Applications and ramifications—examples.

The theory generalizes various classical construction as *e.g.* representation theory of Hopf algebras.

The theory is intrinsically connected to Lie theory and modular representation theory, where we proved and disproved several “conjectures”.

The theory has applications in low-dimensional topology and mathematical physics.

The theory has relations to computational algebra and a lot of data is collected numerically.

Example

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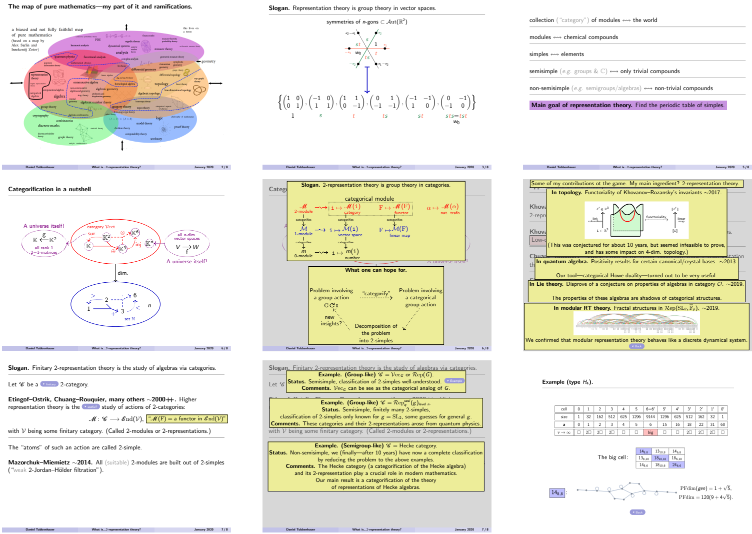
The “a

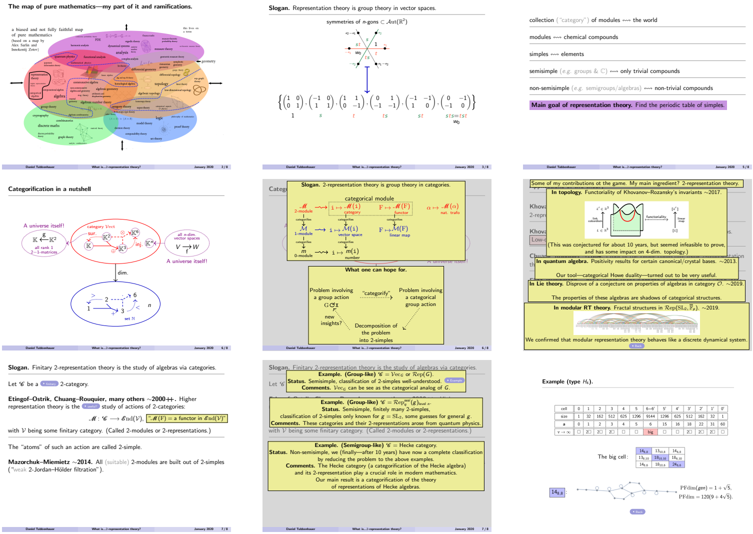
Take away message.

We systematically study abstract 2-analogs of finite-dimensional algebras, hoping to “unite” the vast theory of 2-representations.

Mazorchuk–Miemietz ∼2014. All (suitable) 2-modules are built out of 2-simples (“weak 2-Jordan–Hölder filtration”).



There is still much to do...



Thanks for your attention!

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

Figure: Quotes from "Theory of Groups of Finite Order" by Burnside. Top: first edition (1897); bottom: second edition (1911).

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Nowadays representation theory is pervasive across mathematics, and beyond.

of linear transformations.

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But this wasn't clear at all when Frobenius started it.

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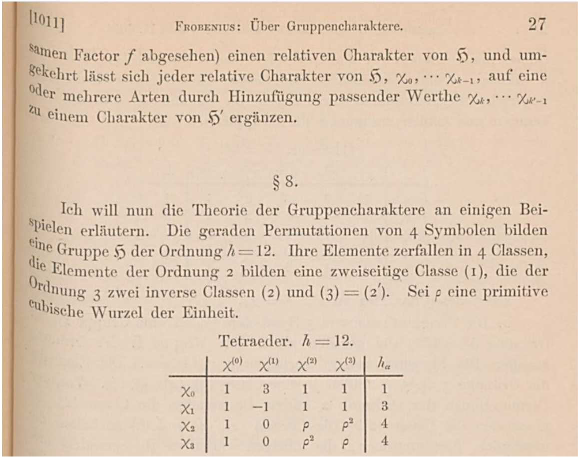

samen Factor f abgesehen) einen relativen Charakter von $\mathfrak{H}$, und umgekehrt lässt sich jeder relative Charakter von $\mathfrak{H}$, $\chi_0, \cdots \chi_{k-1}$, auf eine oder mehrere Arten durch Hinzufügung passender Werthe $\chi_k, \cdots \chi_{k'-1}$ zu einem Charakter von $\mathfrak{H}'$ ergänzen.

§ 8.

Ich will nun die Theorie der Gruppencharaktere an einigen Beispielen erläutern. Die geraden Permutationen von 4 Symbolen bilden eine Gruppe $\mathfrak{H}$ der Ordnung $h=12$. Ihre Elemente zerfallen in 4 Classen, die Elemente der Ordnung 2 bilden eine zweiseitige Classe (1), die der Ordnung 3 zwei inverse Classen (2) und (3) = (2′). Sei ρ eine primitive cubische Wurzel der Einheit.

Tetraeder. $h=12$.

	$\chi^{(0)}$	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	h_α
χ_0	1	3	1	1	1
χ_1	1	−1	1	1	3
χ_2	1	0	ρ	ρ^2	4
χ_3	1	0	ρ^2	ρ	4

Figure: "Über Gruppencharaktere (i.e. characters of groups)" by Frobenius (1896). Bottom: first published character table.

Note the root of unity ρ!

Back

An additive, $\mathbb{K}$-linear, idempotent complete, Krull–Schmidt category $\mathcal{C}$ is called finitary if it has only finitely many isomorphism classes of indecomposable objects and the morphism sets are finite-dimensional. A 2-category $\mathscr{C}$ with finitely many objects is finitary if its hom-categories are finitary, $\circ_h$-composition is additive and linear, and identity 1-morphisms are indecomposable.

A simple transitive 2-module (2-simple) of $\mathscr{C}$ is an additive, $\mathbb{K}$-linear 2-functor

$$\mathscr{M}\colon \mathscr{C} \to \mathscr{A}^{\mathrm{f}} (= \text{2-cat of finitary cats}),$$

such that there are no non-zero proper $\mathscr{C}$-stable ideals.

There is also the notion of 2-equivalence.

Example. (Semi)groups and their representations, quantum groups and their categorifications and the Hecke category, tensor, fusion and modular categories (generalizing Hopf algebras), 2-Kac–Moody categories...

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2-Simples $\leftrightsquigarrow$ simples (*e.g.* weak 2-Jordan–Hölder filtration),

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There is

Example.

$\mathrm{B}\text{-}\mathrm{p}\mathcal{M}\mathrm{od}$ (with B finite-dimensional) is a prototypical object of $\mathscr{A}^{\mathrm{f}}$.

A 2-module usually is given by endofunctors on $\mathrm{B}\text{-}\mathrm{p}\mathcal{M}\mathrm{od}$.

Example … their categorif … egories (generalizing Hopf algebras), 2-Kac–Moody categories...

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Applications of the theory

Khovanov & others ∼1999++. Knot homologies are instances of 2-representation theory. Low-dim. topology & Math. Physics

Khovanov–Seidel & others ∼2000++. Faithful 2-modules of braid groups. Low-dim. topology & Symplectic geometry

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Many more...

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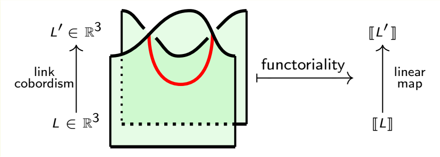

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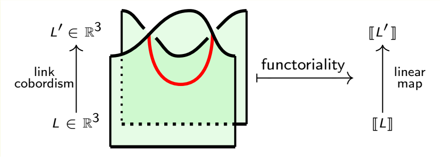

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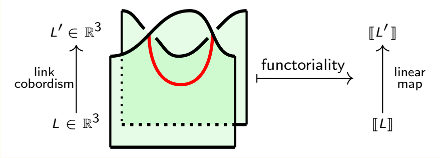

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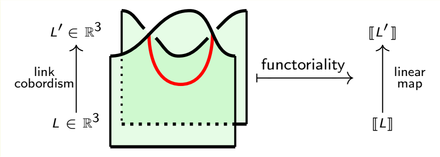

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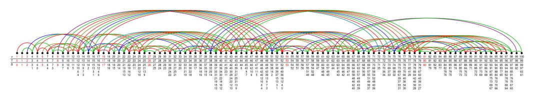
In modular RT theory. Fractal structures in $\mathcal{R}\mathrm{ep}(\mathrm{SL}_2, \overline{\mathbb{F}}_p)$. ∼2019.



We confirmed that modular representation theory behaves like a discrete dynamical system.

Back

Example ($\mathscr{R}\mathrm{ep}(G)$).

- Let $\mathscr{C} = \mathscr{R}\mathrm{ep}(G)$ (G a finite group).
- $\mathscr{C}$ is fusion (fiat and semisimple). For any $\mathtt{M}, \mathtt{N} \in \mathscr{C}$, we have $\mathtt{M} \otimes \mathtt{N} \in \mathscr{C}$:

$$g(m \otimes n) = gm \otimes gn$$

  for all $g \in G, m \in \mathtt{M}, n \in \mathtt{N}$. There is a trivial representation 1.
- The regular 2-representation $\mathscr{M} : \mathscr{C} \to \mathscr{E}\mathrm{nd}(\mathscr{C})$:

$$\begin{array}{ccc} \mathtt{M} & \longrightarrow & \mathtt{M} \otimes \_ \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes \_} \\ \mathtt{N} & \longrightarrow & \mathtt{N} \otimes \_ \end{array} .$$

- The decategorification is a $\mathbb{N}$-representation, the regular representation.
- The associated (co)algebra object is $\mathtt{A}_{\mathscr{M}} = 1 \in \mathscr{C}$.

Example ($\mathscr{R}\mathrm{ep}(G)$).

- Let $K \subset G$ be a subgroup.
- $\mathcal{R}\mathrm{ep}(K)$ is a 2-representation of $\mathscr{R}\mathrm{ep}(G)$, with action

$$\mathcal{R}es_K^G \otimes \_ \colon \mathscr{R}\mathrm{ep}(G) \to \mathscr{E}\mathrm{nd}(\mathcal{R}\mathrm{ep}(K))$$

  which is indeed a 2-action because $\mathcal{R}es_K^G$ is a $\otimes$-functor.
- The decategorifications are $\mathbb{N}$-representations.
- The associated (co)algebra object is $\mathtt{A}_{\mathscr{M}} = \mathcal{I}nd_K^G(1_K) \in \mathscr{C}$.

Example ($\mathscr{R}\mathrm{ep}(G)$).

- Let $\psi \in H^2(K, \mathbb{C}^*)$. Let $\mathcal{V}(K, \psi)$ be the category of projective K-modules with Schur multiplier ψ, *i.e.*vector spaces $\mathtt{V}$ with $\rho\colon K \to \mathcal{E}\mathrm{nd}(\mathtt{V})$ such that

$$\rho(g)\rho(h) = \psi(g, h)\rho(gh), \text{ for all } g, h \in K.$$

- Note that $\mathcal{V}(K, 1) = \mathcal{R}\mathrm{ep}(K)$ and

$$\otimes\colon \mathcal{V}(K, \phi) \boxtimes \mathcal{V}(K, \psi) \to \mathcal{V}(K, \phi\psi).$$

- $\mathcal{V}(K, \psi)$ is also a 2-representation of $\mathscr{C} = \mathscr{R}\mathrm{ep}(G)$:

$$\mathscr{R}\mathrm{ep}(G) \boxtimes \mathcal{V}(K, \psi) \xrightarrow{\mathcal{R}es^G_K \boxtimes \mathrm{Id}} \mathcal{R}\mathrm{ep}(K) \boxtimes \mathcal{V}(K, \psi) \xrightarrow{\otimes} \mathcal{V}(K, \psi).$$

- The decategorifications are $\mathbb{N}$-representations.
- The associated (co)algebra object is $\mathtt{A}_{\mathscr{M}} = \mathcal{I}nd^G_K(1_K) \in \mathscr{C}$, but with ψ-twisted multiplication.

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Theorem (folklore?).

Completeness. All 2-simples of $\mathscr{R}\mathrm{ep}(G)$ are of the form $\mathcal{V}(K,\psi)$.

Non-redundancy. We have $\mathcal{V}(K,\psi) \cong \mathcal{V}(K',\psi')$

$\Leftrightarrow$

the subgroups are conjugate or $\psi' = \psi^g$, where $\psi^g(k,l) = \psi(gkg^{-1}, glg^{-1})$.

◂ Back

$$\mathscr{R}\mathrm{ep}(G) \boxtimes \mathcal{V}(K,\psi) \xrightarrow{\mathrm{Res}_K \boxtimes \mathrm{id}} \mathcal{R}\mathrm{ep}(K) \boxtimes \mathcal{V}(K,\psi) \xrightarrow{\otimes} \mathcal{V}(K,\psi).$$

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Example (type H_4).

cell	0	1	2	3	4	5	6=6′	5′	4′	3′	2′	1′	0′
size	1	32	162	512	625	1296	9144	1296	625	512	162	32	1
a	0	1	2	3	4	5	6	15	16	18	22	31	60
$\mathrm{v} \to \infty$	$\square$	$2\square$	$2\square$	$2\square$	$\square$	$\square$	big	$\square$	$\square$	$2\square$	$2\square$	$2\square$	$\square$

The big cell :

$14_{8,8}$	$13_{10,8}$	$14_{6,8}$
$13_{8,10}$	$18_{10,10}$	$18_{6,10}$
$14_{8,6}$	$18_{10,6}$	$24_{6,6}$

$14_{8,8}$:

$\mathrm{PFdim}(gen) = 1 + \sqrt{5},$

$\mathrm{PFdim} = 120(9 + 4\sqrt{5}).$

Back