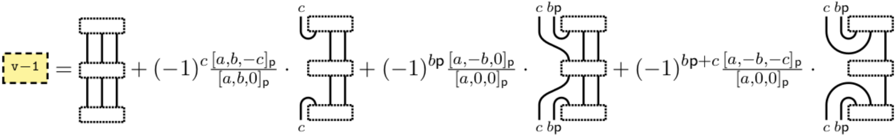
On categories of tilting modules

Or: Mind your poles

Daniel Tubbenhauer



Joint with Paul Wedrich

August 2020

Folklore, Lucas ∼1878. Let $q \in \mathbb{K}^*$, $q\mathrm{char}(\mathbb{K}) = p$, $a = mp + a_0$ and $b = np + b_0$ (a_0, b_0 zeroth digit of the p-adic expansion). Then

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \binom{m}{n} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}_q .$$

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Corollary. We understand finite-dimensional modules for $\mathrm{SL}_2 = \mathrm{SL}_2(\mathbb{K} = \overline{\mathbb{K}})$

- generically;
- for the quantum group over $\mathbb{C}$ at $q^{2\ell} = 1$;
- the quantum group over $\mathbb{K}$, $\mathrm{char}(\mathbb{K}) = p$ and $q^{2\ell} = 1$ (mixed case);
- in prime characteristic $\mathrm{char}(\mathbb{K}) = p$.

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Example/Remark.

$\mathbb{K} = \overline{\mathbb{F}}_p$, $q = 1$ (known as characteristic p),
and $a = [a_r, ..., a_0]_p$, $b = [b_r, ..., b_0]_p$ (the p-adic expansions), then

$$\binom{a}{b} = \begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{bmatrix} a_r \\ b_r \end{bmatrix}_q \cdots \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}_q = \binom{a_r}{b_r} \cdots \binom{a_0}{b_0}.$$

Folklore, Lucas ∼1878. Let $q \in \mathbb{K}^*$, $q\mathrm{char}(\mathbb{K}) = p$, $a = mp + a_0$ and

Examples for $a = 1331 = 11^3$ and $b = a - 1$.

If $\mathbb{K} = \mathbb{C}$, $q = 1$, then $q\mathrm{char}(\mathbb{K}) = 0$, $a = [1331]_0$ and $b = [1330]_0$
$\Rightarrow \begin{bmatrix} 1331 \\ 1330 \end{bmatrix}_q = \binom{1331}{1330} = 1331$ does not vanish.

If $\mathbb{K} = \mathbb{C}$, $q = \exp(2\pi i/11)$, then $q\mathrm{char}(\mathbb{K}) = 11$, $a = [121, 0]_{11}$ and $b = [120, 10]_{11}$
$\Rightarrow \begin{bmatrix} 1331 \\ 1330 \end{bmatrix}_q = 121 \cdot \begin{bmatrix} 0 \\ 10 \end{bmatrix}_q$ vanishes of order one.

If $\mathbb{K} = \overline{\mathbb{F}}_{11}$, $q = 3$, then $q\mathrm{char}(\mathbb{K}) = 5$, $a = [266, 1]_5$, $b = [266, 0]_5$ and $\frac{a-1}{5} = \frac{b}{5} = [2, 2, 2]_{11}$
$\Rightarrow [1331]_q = 1 \cdot 1 \cdot 1 \cdot [1]_q$ does not vanish.

If $\mathbb{K} = \overline{\mathbb{F}}_{11}$, $q = 1$, then $q\mathrm{char}(\mathbb{K}) = 11$, $a = [1, 0, 0, 0]_{11}$ and $b = [0, 10, 10, 10]_{11}$
$\Rightarrow [1331]_q = 1 \cdot 0 \cdot 0 \cdot [0]_q$ vanishes of order three.

- the quantum group over $\mathbb{K}$, $\mathrm{char}(\mathbb{K}) = p$ and $q^{2\ell} = 1$ (mixed case);
- in prime characteristic $\mathrm{char}(\mathbb{K}) = p$.

Example/Remark.

$\mathbb{K} = \overline{\mathbb{F}}_p$, $q = 1$ (known as characteristic p),
and $a = [a_r, \ldots, a_0]_p$, $b = [b_r, \ldots, b_0]_p$ (the p-adic expansions), then

$$\binom{a}{b} = \begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{bmatrix} a_r \\ b_r \end{bmatrix}_q \cdots \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}_q = \binom{a_r}{b_r} \cdots \binom{a_0}{b_0}.$$

Folklore, Lucas ∼1878. Let $q \in \mathbb{K}^*$, $q\mathrm{char}(\mathbb{K}) = p$, $a = mp + a_0$ and $b = np + b_0$ (a_0, b_0 zeroth digit of the p-adic expansion). Then

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Corollary. We ... $_2(\mathbb{K} = \overline{\mathbb{K}})$

- genericall...
- for the qu...
- the quant... case);
- in prime characteristic $\mathrm{char}(\mathbb{K}) = p$.

I will stick with characteristic p, but

the quantum group is a "zeroth digit only" version of it;

the mixed cases is a mixture of the two.

Weyl ∼1923. The SL_2 Weyl modules $\Delta(v-1)$.

$\Delta(1-1)$				X^0Y^0			
$\Delta(2-1)$			X^1Y^0		X^0Y^1		
$\Delta(3-1)$			X^2Y^0	X^1Y^1	X^0Y^2		
$\Delta(4-1)$		X^3Y^0	X^2Y^1		X^1Y^2	X^0Y^3	
$\Delta(5-1)$		X^4Y^0	X^3Y^1	X^2Y^2	X^1Y^3	X^0Y^4	
$\Delta(6-1)$	X^5Y^0	X^4Y^1	X^3Y^2		X^2Y^3	X^1Y^4	X^0Y^5
$\Delta(7-1)$	X^6Y^0	X^5Y^1	X^4Y^2	X^3Y^3	X^2Y^4	X^1Y^5	X^0Y^6

$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto$ matrix who's columns are expansions of $(aX+cY)^{v-i}(bX+dY)^{i-1}$.

▸ The simples

Weyl

Example $\Delta(7-1) = \mathbb{K}X^6Y^0 \oplus \cdots \oplus \mathbb{K}X^0Y^6$.

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \text{ acts as } \begin{pmatrix} a^6 & a^5b & a^4b^2 & \dots\dots\dots & d^6 \\ 6a^5c & 5a^4bc+a^5d & 4a^3b^2c+2a^4bd & \dots\dots\dots & 6bd^5 \\ 15a^4c^2 & 10a^3bc^2+5a^4cd & 6a^2b^2c^2+8a^3bcd+a^4d^2 & \dots\dots\dots & 15b^2d^4 \\ 20a^3c^3 & 10a^2bc^3+10a^3c^2d & 12a^2bc^2d+4a^3cd^2 & \dots\dots\dots & 20b^3d^3 \\ 15a^2c^4 & 5abc^4+10a^2c^3d & b^2c^4+8abc^3d+6a^2c^2d^2 & \dots\dots\dots & 15b^4d^2 \\ 6ac^5 & 5ac^4d+bc^5 & 2bc^4d+4ac^3d^2 & \dots\dots\dots & 6b^5d \\ c^6 & c^5d & c^4d^2 & \dots\dots\dots & b^6 \end{pmatrix}$$

The columns are expansions of $(aX+cY)^{7-i}(bX+dY)^{i-1}$. Binomials!

$\Delta(4-1)$ $\quad X^3Y^0 \quad X^2Y^1 \quad X^1Y^2 \quad X^0Y^3$

$\Delta(5-1)$ $\quad X^4Y^0 \quad X^3Y^1 \quad X^2Y^2 \quad X^1Y^3 \quad X^0Y^4$

$\Delta(6-1)$ $\quad X^5Y^0 \quad X^4Y^1 \quad X^3Y^2 \quad X^2Y^3 \quad X^1Y^4 \quad X^0Y^5$

$\Delta(7-1)$ $\quad X^6Y^0 \quad X^5Y^1 \quad X^4Y^2 \quad X^3Y^3 \quad X^2Y^4 \quad X^1Y^5 \quad X^0Y^6$

$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto$ matrix who's columns are expansions of $(aX+cY)^{v-i}(bX+dY)^{i-1}$.

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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ acts as } \begin{pmatrix}
a^6 & a^5b & a^4b^2 & \dots\dots\dots & d^6 \\
6a^5c & 5a^4bc+a^5d & 4a^3b^2c+2a^4bd & \dots\dots\dots & 6bd^5 \\
15a^4c^2 & 10a^3bc^2+5a^4cd & 6a^2b^2c^2+8a^3bcd+a^4d^2 & \dots\dots\dots & 15b^2d^4 \\
20a^3c^3 & 10a^2bc^3+10a^3c^2d & 12a^2bc^2d+4a^3cd^2 & \dots\dots\dots & 20b^3d^3 \\
15a^2c^4 & 5abc^4+10a^2c^3d & b^2c^4+8abc^3d+6a^2c^2d^2 & \dots\dots\dots & 15b^4d^2 \\
6ac^5 & 5ac^4d+bc^5 & 2bc^4d+4ac^3d^2 & \dots\dots\dots & 6b^5d \\
c^6 & c^5d & c^4d^2 & \dots\dots\dots & b^6
\end{pmatrix}$$

The columns are expansions of $(aX + cY)^{7-i}(bX + dY)^{i-1}$. Binomials!

Example $\Delta(7{-}1)$, characteristic 0.

$$\text{“}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\text{” acts as } \begin{pmatrix}
1 & \dots\dots\dots\dots\dots & 1 \\
6 & \dots\dots\dots\dots\dots & 6 \\
15 & \dots\dots\dots\dots\dots & 15 \\
20 & \dots\dots\dots\dots\dots & 20 \\
15 & \dots\dots\dots\dots\dots & 15 \\
6 & \dots\dots\dots\dots\dots & 6 \\
1 & \dots\dots\dots\dots\dots & 1
\end{pmatrix}$$

No zeros $\Rightarrow \Delta(7{-}1)$ simple.

Example $\Delta(7{-}1)$, characteristic 5.

$$\text{“}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\text{” acts as } \begin{pmatrix}
1 & \dots\dots\dots\dots\dots & 1 \\
1 & \dots\dots\dots\dots\dots & 1 \\
0 & \dots\dots\dots\dots\dots & 0 \\
0 & \dots\dots\dots\dots\dots & 0 \\
0 & \dots\dots\dots\dots\dots & 0 \\
1 & \dots\dots\dots\dots\dots & 1 \\
1 & \dots\dots\dots\dots\dots & 1
\end{pmatrix}$$

We found a submodule.

$\Delta(4{-}1)$

$\Delta(5{-}1)$ X^0Y^4

$\Delta(6{-}1)$ X^0Y^5

$\Delta(7{-}1)$ X^1Y^5 X^0Y^6

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto$ matrix who's ... $Y)^{v-i}(bX + dY)^{i-1}$.

The simples

Weyl ∼1923. The SL_2 Weyl modules $\Delta(v-1)$.

$\Delta(1-1)$

$\Delta(2-1)$

$\Delta(3-1)$ Y^2

$\Delta(4-1)$ X^0Y^3

$\Delta(5-1)$ X^4Y^0 X^3Y^1 X^2Y^2 X^1Y^3 X^0Y^4

$\Delta($

$\Delta($ Y^6

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto$ matri $X + dY)^{i-1}$.

When is $\Delta(v-1)$ simple?

$\Delta(v-1)$ is simple

$\Leftrightarrow$

$\binom{v-1}{w-1} \neq 0$ for all $w \leq v$

$\Leftrightarrow$ (Lucas's theorem)

$v = [a_r, 0, \dots, 0]_p$.

General.
Weyl $\Delta(\lambda)$ and dual Weyl $\nabla(\lambda)$
are easy a.k.a. standard;
are parameterized by dominant integral weights;
are highest weight modules;
are defined over $\mathbb{Z}$;
have the classical Weyl characters;
form a basis of the Grothendieck group unitriangular w.r.t. simples;
satisfy (a version of) Schur's lemma $\dim_{\mathbb{K}} \mathrm{Ext}^i(\Delta(\lambda), \nabla(\mu)) = \delta_{i,0}\delta_{\lambda,\mu}$;
are simple generically;
have a root-binomial-criterion to determine whether they are simple (Jantzen's thesis ∼1973).

The simples

Ringel, Donkin ∼1991. The indecomposable SL_2 tilting modules $\mathtt{T}(v{-}1)$ are the indecomposable summands of $\Delta(1)^{\otimes i}$.

General.
These facts hold in general, and the first bullet point is the general definition.

Tilting modules $\mathtt{T}(v{-}1)$

- are those modules with a $\Delta(w{-}1)$- and a $\nabla(w{-}1)$-filtration;
- are parameterized by dominant integral weights;
- are highest weight modules;
- satisfy reciprocity $\big(\mathtt{T}(v{-}1) : \Delta(w{-}1)\big) = \big(\mathtt{T}(v{-}1) : \nabla(w{-}1)\big) = [\Delta(w'{-}1) : \mathtt{L}(v'{-}1)] = [\nabla(w'{-}1) : \mathtt{L}(v'{-}1)]$;
- form a basis of the Grothendieck group unitriangular w.r.t. simples;
- satisfy (a version of) Schur's lemma $\dim_{\mathbb{K}} \mathrm{Hom}\big(\mathtt{T}(v{-}1), \mathtt{T}(w{-}1)\big) = \sum_{x<\min(v,w)} \big(\mathtt{T}(v{-}1) : \Delta(x{-}1)\big)\big(\mathtt{T}(w{-}1) : \nabla(x{-}1)\big)$ ▸ Why the name?;
- are simple generically;
- have a root-binomial-criterion to determine whether they are simple.

Let $\mathcal{T}\mathrm{ilt}$ be the category of tilting modules.

Goal. Describe $\mathcal{T}\mathrm{ilt}$ by generators and relations.

Ringel, Donkin $\sim$1991. The indecomposable SL_2 tilting modules $\mathtt{T}(v{-}1)$ are the indecomposable summands of $\Delta(1)^{\otimes i}$.

Tilting modu

- are thos
- are para
- are high
- satisfy ... $[\Delta(w'{-}1) :$ $\mathrm{L}(v'{-}1$
- form a ... es;
- satisfy ... $) =$ $\sum_{x<\min(v,w}$ ... ame? ;
- are simple
- have a roo ... simple.

Let $\mathcal{T}\mathrm{ilt}$ be the

C ... s.

How many Weyl factors does $\mathtt{T}(v{-}1)$ have?

$\#$ Weyl factors of $\mathtt{T}(v{-}1)$ is 2^k where

$k = \max\{\nu_p(\binom{v-1}{w-1}), w \leq v\}$. (Order of vanishing of $\binom{v-1}{w-1}$.)

determined by (Lucas's theorem)

non-zero non-leading digits of $v = [a_r, a_{r-1}, \dots, a_0]_p$.

Example $\mathtt{T}(220540{-}1)$ for $p = 11$?

$v = 220540 = [1, 4, 0, 7, 7, 1]_{11}$;

Maximal vanishing for $w = 75594 = [0, 5, 1, 8, 8, 2]_{11}$;

$\binom{v-1}{w-1} = (HUGE) = [\dots, \neq 0, 0, 0, 0, 0]_{11}$.

$\Rightarrow \mathtt{T}(220540{-}1)$ has 2^4 Weyl factors.

Ringel, Donkin ∼1991. The indecomposable SL_2 tilting modules $\mathtt{T}(v{-}1)$ are the indecomposable summands of $\Delta(1)^{\otimes i}$.

Tilting modules $\mathtt{T}(v{-}1)$

- are those modules with a $\Delta(w{-}1)$- and a $\nabla(w{-}1)$-filtration;
- a
- a
- s $\ldots -1)$:
- satisfy (a version of) Schur's lemma $\dim_{\mathbb{K}} \mathrm{Hom}\big(\mathtt{T}(v{-}1), \mathtt{T}(w{-}1)\big) = \sum_{x<\min(v,w}$ … name? ;
- are simple g…
- have a root … simple.

Which Weyl factors does $\mathtt{T}(v{-}1)$ have a.k.a. the negative digits game?

Weyl factors of $\mathtt{T}(v{-}1)$ are

$$\Delta([a_r, \pm a_{r-1}, \ldots, \pm a_0]_p{-}1) \text{ where } v = [a_r, \ldots, a_0]_p.$$

Example $\mathtt{T}(220540{-}1)$ for $p = 11$?

$$v = 220540 = [1,4,0,7,7,1]_{11};$$

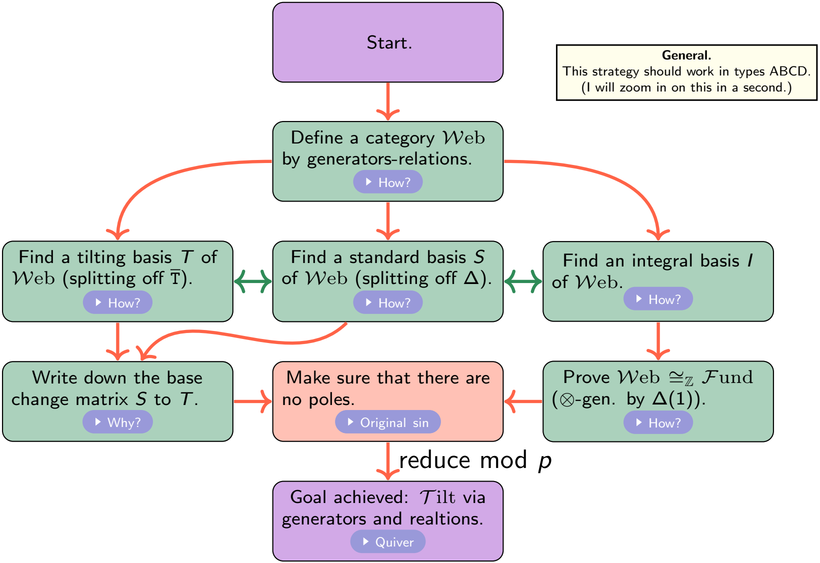
has Weyl factors $[1, \pm 4, 0, \pm 7, \pm 7, \pm 1]_{11}$;

e.g. $\Delta(218690 = [1,4,0,-7,-7,-1]_{11}{-}1)$ appears.

Let $\mathcal{T}\mathrm{ilt}$ be the …

G… s.

Strategical interlude.



Strategical interlude.

Start.

What remains to be done?

No more sins!

What is the diagrammatic incarnation of the Frobenius $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$?

The mixed case will be easier but might be a pain to write down.

Up next: the first steps towards higher ranks,
i.e. let us try $U_q(\mathfrak{sl}_3)$ for q a primitive complex 2ℓth root of unity.

Write down the base change matrix S to T.
Why?

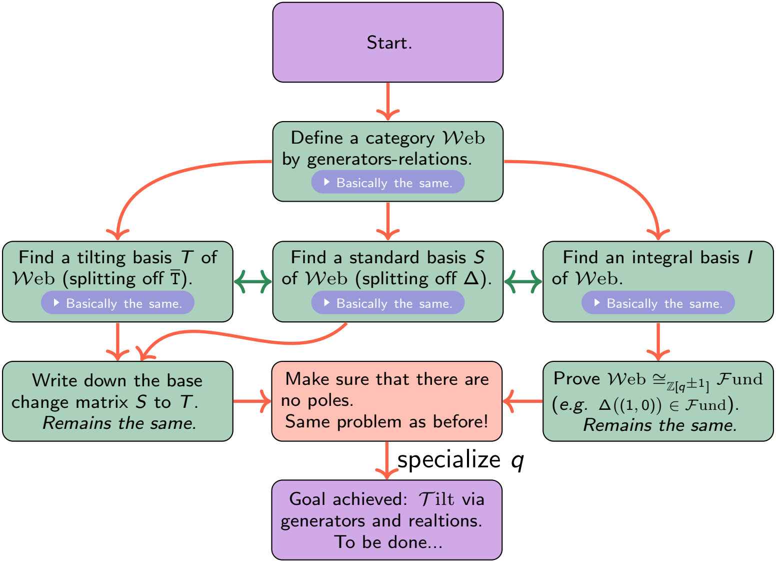
Make sure that there are no poles.
Original sin

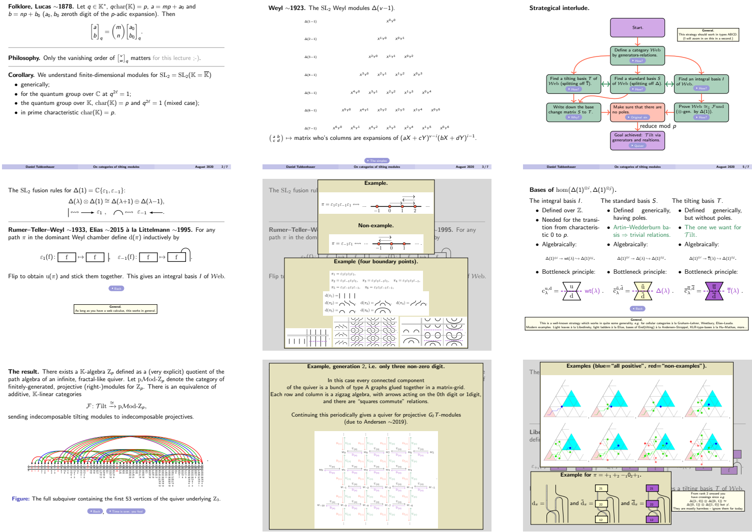
Prove $\mathcal{W}\mathrm{eb} \cong_{\mathbb{Z}} \mathcal{F}\mathrm{und}$ ($\otimes$-gen. by $\Delta(1)$).
How?

reduce mod p

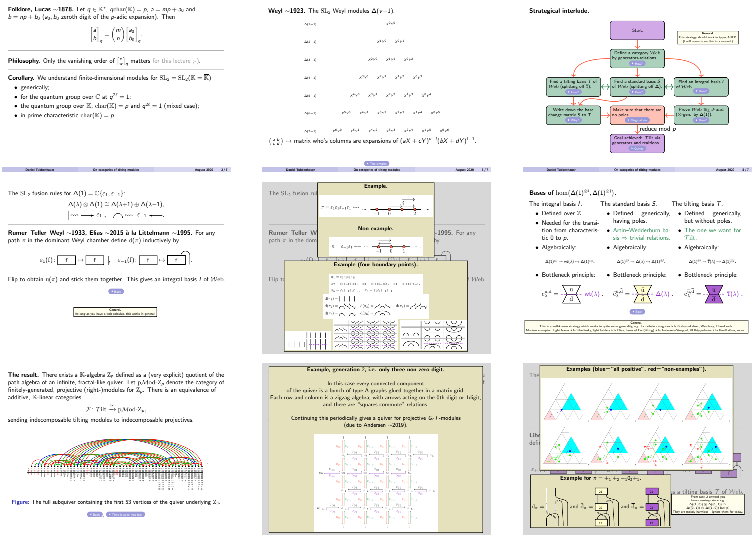
Goal achieved: $\mathcal{T}\mathrm{ilt}$ via generators and realtions.
Quiver

Strategical interlude.





There is still much to do...



Thanks for your attention!

Weyl ∼1923. The SL_2 simples $\mathtt{L}(v{-}1)$ in $\nabla(v{-}1)$ for $p = 5$.

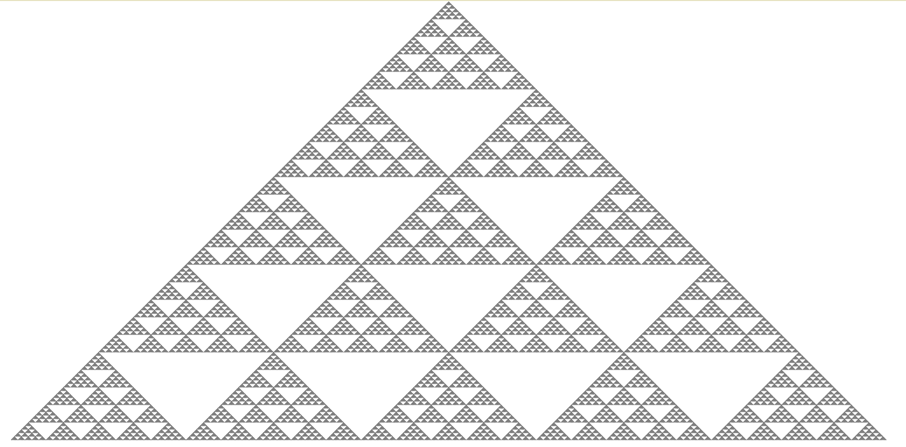
$\nabla(1{-}1)$	X^0Y^0							$L(1{-}1)$
$\nabla(2{-}1)$	X^1Y^0	X^0Y^1						$L(2{-}1)$
$\nabla(3{-}1)$	X^2Y^0	X^1Y^1	X^0Y^2					$L(3{-}1)$
$\nabla(4{-}1)$	X^3Y^0	X^2Y^1	X^1Y^2	X^0Y^3				$L(4{-}1)$
$\nabla(5{-}1)$	X^4Y^0	X^3Y^1	X^2Y^2	X^1Y^3	X^0Y^4			$L(5{-}1)$
$\nabla(6{-}1)$	X^5Y^0	X^4Y^1	X^3Y^2	X^2Y^3	X^1Y^4	X^0Y^5		$L(6{-}1)$
$\nabla(7{-}1)$	X^6Y^0	X^5Y^1	X^4Y^2	X^3Y^3	X^2Y^4	X^1Y^5	X^0Y^6	$L(7{-}1)$

$\nabla(7{-}1)$ has $\mathtt{L}(7{-}1)$ and $\mathtt{L}(3{-}1)$. Note $7 = [1, 2]_5$ and $3 = [3, -2]_5$.

Back

Weyl ∼1923. The SL_2 simples $\mathrm{L}(v-1)$ in $\nabla(v-1)$ for $p = 5$.

$\nabla(1-1)$ X^0Y^0 $L(1-1)$



Pascals triangle modulo $p = 5$ picks out the simples,
e.g. an unbroken east-west line is a Weyl module which is simple.

Picture from `https://commons.wikimedia.org/wiki/File:Pascal_triangle_modulo_5.png`

Weyl ∼1923. The SL_2 simples $\mathtt{L}(v-1)$ in $\nabla(v-1)$ for $p=5$.

$\nabla(1-1)$				X^0Y^0				$L(1-1)$
$\nabla(2-1)$			X^1Y^0		X^0Y^1			$L(2-1)$
$\nabla(3-1)$			X^2Y^0	X^1Y^1	X^0Y^2			$L(3-1)$

I should have told you that the zeros in the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \dots & \dots & \dots & \dots & \dots & 1 \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 1 & \dots & \dots & \dots & \dots & \dots & 1 \\ 1 & \dots & \dots & \dots & \dots & \dots & 1 \end{pmatrix}$$

are of order 1. Keeping track of these orders lets you pinpoint all simples.

I will come back to this for the tilting modules.

$\nabla(6-1)$	X^5Y^0	X^4Y^1	X^3Y^2	X^2Y^3	X^1Y^4	X^0Y^5		$L(6-1)$
$\nabla(7-1)$	X^6Y^0	X^5Y^1	X^4Y^2	X^3Y^3	X^2Y^4	X^1Y^5	X^0Y^6	$L(7-1)$

$\nabla(7-1)$ has $\mathtt{L}(7-1)$ and $\mathtt{L}(3-1)$. Note $7=[1,2]_5$ and $3=[3,-2]_5$.

Back

“Schur’s tilting lemma a.k.a. Weyl clustering”.

In the Grothendieck group: $[\mathtt{T}(\lambda)] = [\Delta(\lambda)] + \sum_{\mu<\lambda} \big(\mathtt{T}(\lambda) : \Delta(\mu)\big)[\Delta(\mu)]$.

Let $\overline{\mathtt{T}}(\lambda) = \Delta(\lambda) \oplus \bigoplus_{\mu<\lambda} \big(\mathtt{T}(\lambda) : \Delta(\mu)\big)\Delta(\mu)$, seen generically.

Philosophy. Never ever go to characteristic p – its too complicated. Work with $\overline{\mathtt{T}}(\lambda)$ instead, “the characteristic 0 cousin of $\mathtt{T}(\lambda)$”.

Then

$$\mathrm{dim}_{\mathbb{K}} \operatorname{End}\big(\mathtt{T}(\lambda)\big) = \mathrm{dim}_{gen} \operatorname{End}\big(\overline{\mathtt{T}}(\lambda)\big) = 1 + \textstyle\sum_{\mu<\lambda} \big(\mathtt{T}(\lambda) : \Delta(\mu)\big)^2,$$

by Schur’s lemma. (Similarly for hom-spaces, of course.)

Back

“Schur’s tilting lemma a.k.a. Weyl clustering”.

In the Grothendie … $\lambda) : \Delta(\mu)\big)[\Delta(\mu)]$.

Let $\overline{\mathtt{T}}(\lambda) = \Delta(\lambda) \oplus \bigoplus$ … lly.

Philosophy. Never e … plicated. Work with $\overline{\mathtt{T}}(\lambda)$ instead, “the ch …

Weyl clustering algorithm.

$\Delta(1)^k$ has the following tilting summands.

Take the highest appearing weight $v-1$;
set $\overline{\mathtt{T}}(v{-}1) = \bigoplus_{w \in \mathsf{NDG}} \Delta(w{-}1)$;
repeat.

Then

$$\dim_{\mathbb{K}} \mathrm{End}\big(\mathtt{T}(\lambda)\big) = \dim_{gen} \mathrm{End}\big(\overline{\mathtt{T}}(\lambda)\big) = 1 + \textstyle\sum_{\mu<\lambda} \big(\mathtt{T}(\lambda) : \Delta(\mu)\big)^2,$$

by Schur’s lemma. (Similarly for hom-spaces, of course.)

Back

“Schur’s tilting lemma a.k.a. Weyl clustering”.

In the Grothendie $\lambda) : \Delta(\mu)\big)[\Delta(\mu)]$.

Let $\overline{\mathtt{T}}(\lambda) = \Delta(\lambda) \oplus \bigoplus$ lly.

Philosophy. Never e plicated. Work with $\overline{\mathtt{T}}(\lambda)$ instead, “the ch

Weyl clustering algorithm.

$\Delta(1)^k$ has the following tilting summands.

Take the highest appearing weight $v-1$; set $\overline{\mathtt{T}}(v{-}1) = \bigoplus_{w \in \mathsf{NDG}} \Delta(w{-}1)$; repeat.

Then

$$\dim_{\mathbb{K}} \mathrm{End}\big(\mathtt{T} \quad (\lambda) : \Delta(\mu)\big)^2,$$

by Schur’s lemma.

$\mathtt{T}(v{-}1)$ vs. $\overline{\mathtt{T}}(v{-}1)$.

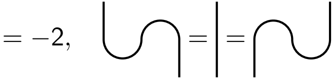
The idempotents in $\mathrm{End}(\overline{\mathtt{T}}(v{-}1))$ inducing the splitting into summands have poles, and $\mathtt{T}(v{-}1)$ does not split into Weyl factors.

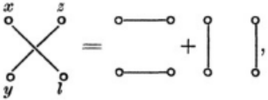
Back

Rumer–Teller–Weyl ∼1932, Temperley–Lieb ∼1971, Kauffman ∼1987.

The category $\mathcal{W}\mathrm{eb}$ is the monoidal $\mathbb{Z}$-linear category monoidally generated by

object generators : $\bullet$, morphism generators :  : $\mathbb{1} \to \bullet^{\otimes 2}$,  : $\bullet^{\otimes 2} \to \mathbb{1}$,

relations :  $= -2$, .



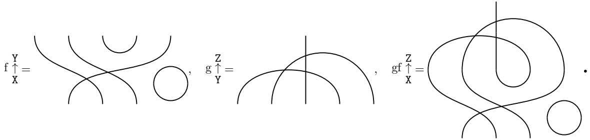


Figure: Conventions and examples. The crossing is from "G. Rumer, E. Teller, H. Weyl. Eine für die Valenztheorie geeignete Basis der binären Vektorinvarianten. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse (1932), Volume: 1932, pages 499–504.".

Back

General.
For type A we have webs
à la Kuperberg ∼1997, Cautis–Kamnitzer–Morrison ∼2012.
For types BCD there are some partial results,
e.g. Brauer ∼1937, Kuperberg ∼1997,
Sartori ∼2017, Rose–Tatham ∼2020.
Outside of these types I do not even
expect our approach to work anyway.

The SL_2 fusion rules for $\Delta(1) = \mathbb{C}\{\varepsilon_1, \varepsilon_{-1}\}$:

$$\Delta(\lambda) \otimes \Delta(1) \cong \Delta(\lambda{+}1) \oplus \Delta(\lambda{-}1),$$

$$\Big| \leftrightsquigarrow \longrightarrow \varepsilon_1\ , \quad \frown \leftrightsquigarrow \varepsilon_{-1} \longleftarrow .$$

Rumer–Teller–Weyl ∼1933, Elias ∼2015 à la Littelmann ∼1995. For any path π in the dominant Weyl chamber define $\mathrm{d}(\pi)$ inductively by

$$\varepsilon_1(\mathrm{f})\colon \boxed{\ \mathrm{f}\ } \mapsto \boxed{\ \mathrm{f}\ }\ \Big|, \qquad \varepsilon_{-1}(\mathrm{f})\colon \boxed{\ \mathrm{f}\ } \mapsto \boxed{\ \mathrm{f}\ }\ \Big| .$$

Flip to obtain $\mathrm{u}(\pi)$ and stick them together. This gives an integral basis I of $\mathcal{W}\mathrm{eb}$.

Back

General.
As long as you have a web calculus, this works in general.

The SL_2 fusion rul

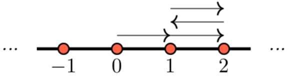
Rumer–Teller–W $\sim$**1995.** For any path π in the dom by

$\varepsilon_1(\mathrm{f})$: f $\mapsto$ f $\varepsilon_{-1}(\mathrm{f})$: f $\mapsto$ f

Flip to f $\mathcal{W}$eb.

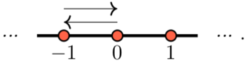
Example.

$\pi = \varepsilon_1\varepsilon_1\varepsilon_{-1}\varepsilon_1 \rightsquigarrow$



Non-example.

$\pi = \varepsilon_{-1}\varepsilon_1 \rightsquigarrow$

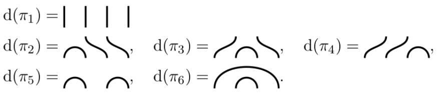


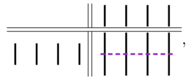
Example (four boundary points).

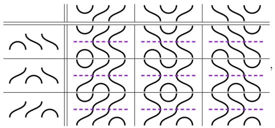
$\pi_1 = \varepsilon_1\varepsilon_1\varepsilon_1\varepsilon_1,$

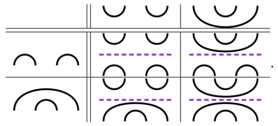
$\pi_2 = \varepsilon_1\varepsilon_{-1}\varepsilon_1\varepsilon_1, \quad \pi_3 = \varepsilon_1\varepsilon_1\varepsilon_{-1}\varepsilon_1, \quad \pi_4 = \varepsilon_1\varepsilon_1\varepsilon_1\varepsilon_{-1},$

$\pi_5 = \varepsilon_1\varepsilon_{-1}\varepsilon_1\varepsilon_{-1}, \quad \pi_6 = \varepsilon_1\varepsilon_1\varepsilon_{-1}\varepsilon_{-1},$





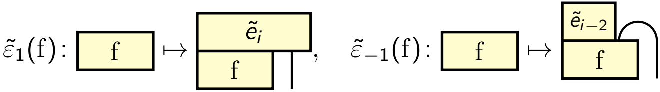




The SL_2 fusion rules for $\Delta(1) = \mathbb{C}\{\varepsilon_1, \varepsilon_{-1}\}$:

$$\Delta(\lambda) \otimes \Delta(1) \cong \Delta(\lambda+1) \oplus \Delta(\lambda-1),$$

Jones ∼1985, Wenzl ∼1989, Cooper–Hogancamp ∼2012. For any path π in the dominant Weyl chamber define $\tilde{\mathrm{d}}(\pi)$ inductively by



Flip to obtain $\tilde{\mathrm{u}}(\pi)$ and stick them together. This gives a standard basis S of $\mathcal{W}\mathrm{eb}$.

◂ Back

General.
As long as you have a web calculus, this works in general,
e.g. Elias has explained how to define the highest weight projectors "$\tilde{\mathrm{e}}$".

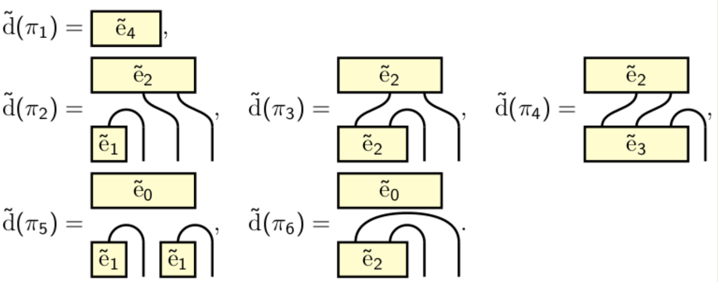
The SL_2

Jones $\sim$

the dom

ath π in

Example.



Flip to obtain $\tilde{\mathrm{u}}(\pi)$ and stick them together. This gives a standard basis S of $\mathcal{W}\mathrm{eb}$.

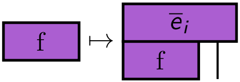
Back

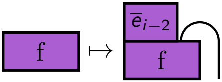
The SL_2 fusion rules for $\Delta(1) = \mathbb{C}\{\varepsilon_1, \varepsilon_{-1}\}$:

$$\Delta(\lambda) \otimes \Delta(1) \cong \Delta(\lambda{+}1) \oplus \Delta(\lambda{-}1),$$

$$| \rightsquigarrow \longrightarrow \varepsilon_1 \,, \quad \frown \rightsquigarrow \varepsilon_{-1} \longleftarrow .$$

Burrull–Libedinsky–Sentinelli ∼2019. For any path π in the dominant Weyl chamber define $\overline{\mathrm{d}}(\pi)$ inductively by

$\overline{\varepsilon}_1(\mathrm{f})$: 

, $\overline{\varepsilon}_{-1}(\mathrm{f})$: 

Flip to obtain $\overline{\mathrm{u}}(\pi)$ and stick them together. This gives a tilting basis T of $\mathcal{W}\mathrm{eb}$.

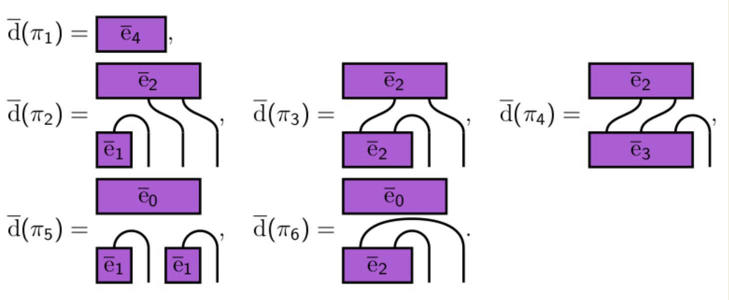
Back

General.
As long as you know the tilting characters, this works in general
e.g. one can define the highest weight tilting projectors "$\overline{\mathrm{e}}$".

The SL_2 fusion rules for $\Delta(1) = \mathbb{C}\{\varepsilon_1, \varepsilon_{-1}\}$:

$$\Delta(\lambda) \otimes \Delta(1) \cong \Delta(\lambda+1) \oplus \Delta(\lambda-1).$$

Example.



Burrull ... Weyl chambe...

Flip to ... of $\mathcal{W}$eb.

Back

In order to prove $\mathcal{W}\mathrm{eb} \cong \mathcal{F}\mathrm{und}$ we need

- a functor $\Gamma\colon \mathcal{W}\mathrm{eb} \to \mathcal{F}\mathrm{und}$ defined integrally;
- an integral basis I of $\mathcal{W}\mathrm{eb}$;
- that $\Delta(1)$ is tilting regardless of $\mathbb{K}$ (by a very general argument, which I learned from Andersen–Stroppel ∼2015, this implies that hom-spaces in $\mathcal{F}\mathrm{und}$ are flat);
- to prove fully faithfulness Γ generically.

Back

General.
The first, second and last bullet points are known in type A and should work more generally.
The third bullet point works *verbatim* for tensor products of any minuscule modules.
Example. Exterior powers of $\Delta(\omega_1)$ in type A
$\Rightarrow$ the Cautis–Kamnitzer–Morrison exterior web calculus works *verbatim* in characteristic p (as observed by Elias ∼2015).
Non-example. Symmetric powers of $\Delta(\omega_1)$ in type A
$\Rightarrow$ the Rose (Vaz–Wedrich) symmetric web calculus in characteristic p is still to be found.

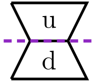
Bases of $\hom(\Delta(1)^{\otimes i}, \Delta(1)^{\otimes j})$.

The integral basis I.

- Defined over $\mathbb{Z}$.
- Needed for the transition from characteristic 0 to p.
- Algebraically:

$$\Delta(1)^{\otimes i} \twoheadrightarrow \mathrm{wt}(\lambda) \hookrightarrow \Delta(1)^{\otimes j}.$$

- Bottleneck principle:

$c_\lambda^{u,d} =$ 

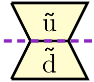
$\mathrm{wt}(\lambda)$.

The standard basis S.

- Defined generically, having poles.
- Artin–Wedderburn basis $\Rightarrow$ trivial relations.
- Algebraically:

$$\Delta(1)^{\otimes i} \twoheadrightarrow \Delta(\lambda) \hookrightarrow \Delta(1)^{\otimes j}.$$

- Bottleneck principle:

$\tilde{c}_\lambda^{\tilde{u},\tilde{d}} =$ 

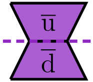
$\Delta(\lambda)$.

The tilting basis T.

- Defined generically, but without poles.
- The one we want for $\mathcal{T}\mathrm{ilt}$.
- Algebraically:

$$\Delta(1)^{\otimes i} \twoheadrightarrow \mathrm{T}(\lambda) \hookrightarrow \Delta(1)^{\otimes j}.$$

- Bottleneck principle:

$\overline{c}_\lambda^{\overline{u},\overline{d}} =$ 

$\mathrm{T}(\lambda)$.

Back

General.
This is a well-known strategy which works in quite some generality, *e.g.* for cellular categories à la Graham–Lehrer, Westbury, Elias–Lauda.
Modern examples. Light leaves à la Libedinsky, light ladders à la Elias, bases of End(tilting) à la Andersen–Stroppel, KLR-type-bases à la Hu–Mathas, more...

Bases of $\hom\big(\Delta(1)^{\otimes i}, \Delta(1)^{\otimes j}\big)$.

Base change for $\overline{\mathrm{T}}([1,1]_{11}) = \Delta([1,1]_{11}) \oplus \Delta([1,-1]_{11})$.

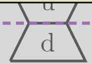
$S = \{\tilde{c}_{[1,1]_{11}}, \tilde{c}_{[1,-1]_{11}}\}$, $\tilde{c}_{[1,1]_{11}}$ and $\tilde{c}_{[1,-1]_{11}}$ are orthogonal idempotents .

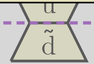
$T = \{\overline{c}_{[1,1]_{11}}, \overline{c}_{[1,-1]_{11}}\}$, and relations to be found.

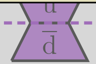
Base change matrix $T \to S$ is $\left(\begin{smallmatrix} 1 & 0 \\ 1 & \kappa^{-1/2} \end{smallmatrix}\right)$, where $\kappa = [1,-1]_{11}/[1,0]_{11} = 10/11$, gives

$$\begin{gathered}
\overline{c}^2_{[1,1]_{11}} = (\tilde{c}_{[1,1]_{11}} + \tilde{c}_{[1,-1]_{11}})^2 = \tilde{c}_{[1,1]_{11}} + \tilde{c}_{[1,-1]_{11}} = \overline{c}_{[1,1]_{11}}, \\
\overline{c}_{[1,1]_{11}}\overline{c}_{[1,-1]_{11}} = \overline{c}_{[1,-1]_{11}}\overline{c}_{[1,1]_{11}}, \\
\overline{c}^2_{[1,-1]_{11}} = 11/10 \cdot \tilde{c}_{[1,-1]_{11}} = 0 \mod 11 .
\end{gathered}$$

Thus, the endomorphism space is $\mathbb{K}[X]/(X^2)$.

$c_\lambda^{u,d} =$ 

$\mathrm{wt}(\lambda)$. $\tilde{c}_\lambda^{\tilde{u},d} =$ 

$\Delta(\lambda)$. $\overline{c}_\lambda^{\overline{u},d} =$ 

$\overline{\mathrm{T}}(\lambda)$.

Back

Original sin. In order to get $\overline{\mathrm{T}}(\lambda)$ I need to know the tilting characters.

So I cannot use the presentation of $\mathcal{T}\mathrm{ilt}$ to say anything new about the objects, a.k.a. tilting modules.

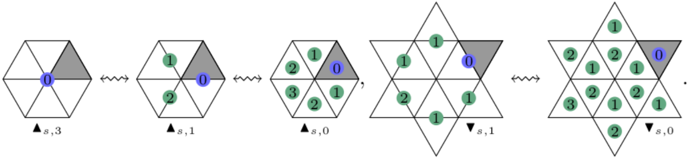


Figure: The quantum tilting characters for SL_3, due to Soergel and Stroppel $\sim$1997.

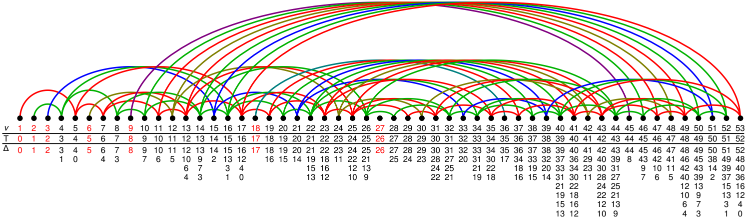
Not much more is known in general, but there are some notable exceptions *e.g.* Jensen $\sim$2000, Parker $\sim$2008, Lusztig–Williamson $\sim$2017.

Back

The result. There exists a $\mathbb{K}$-algebra Z_p defined as a (very explicit) quotient of the path algebra of an infinite, fractal-like quiver. Let $\mathrm{p}\mathcal{M}\mathrm{od}\text{-}\mathrm{Z}_p$ denote the category of finitely-generated, projective (right-)modules for Z_p. There is an equivalence of additive, $\mathbb{K}$-linear categories

$$\mathcal{F}\colon \mathcal{T}\mathrm{ilt} \xrightarrow{\cong} \mathrm{p}\mathcal{M}\mathrm{od}\text{-}\mathrm{Z}_p,$$

sending indecomposable tilting modules to indecomposable projectives.



.

Figure: The full subquiver containing the first 53 vertices of the quiver underlying Z_3.

Back, Time is over, you fool

The result. There exists a $\mathbb{K}$-algebra Z_p defined as a (very explicit) quotient of the path algebra of an infinite, fractal-like quiver. Let $\mathrm{p}\mathcal{M}\mathrm{od}\text{-}\mathrm{Z}_p$ denote the category of finitely-generated, projective (right-)modules for Z_p. There is an equivalence of additiv

Example, generation 0, i.e. only one non-zero digit.

In this case the quiver has no edges.

Continuing this periodically gives a quiver for $\mathcal{T}\mathrm{ilt}$ in characteristic zero.

(This is the semisimple case: the quiver has to be boring.)

sending

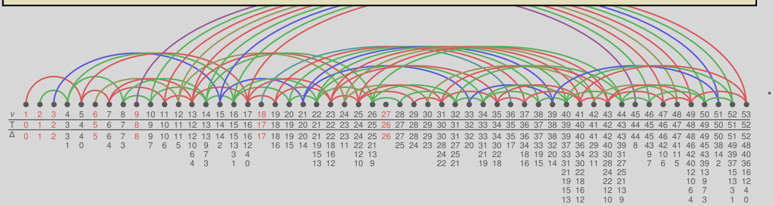


Figure: The full subquiver containing the first 53 vertices of the quiver underlying Z_3.

Back, Time is over, you fool

Example, generation 1, i.e. only two non-zero digit.

In this case the quiver is a bunch of type A graphs. The algebra is a zigzag algebra, with arrows acting on the 0th digit.

Continuing this periodically gives a quiver for $\mathcal{T}\mathrm{ilt}$ for the quantum group at a complex root of unity (due to Andersen ∼2014).

$$(v_0-1) \underset{\mathrm{D}_{\{0\}}}{\overset{\mathrm{U}_{\{0\}}}{\rightleftarrows}} (v_1-1) \underset{\mathrm{D}_{\{0\}}}{\overset{\mathrm{U}_{\{0\}}}{\rightleftarrows}} (v_2-1) \underset{\mathrm{D}_{\{0\}}}{\overset{\mathrm{U}_{\{0\}}}{\rightleftarrows}} (v_3-1) \underset{\mathrm{D}_{\{0\}}}{\overset{\mathrm{U}_{\{0\}}}{\rightleftarrows}} \dots ,$$

$$\mathrm{D}_{\{0\}}\mathrm{D}_{\{0\}}\mathsf{e}_{v-1}=0, \mathrm{U}_{\{0\}}\mathrm{U}_{\{0\}}\mathsf{e}_{v-1}=0,\ \mathrm{D}_{\{0\}}\mathrm{U}_{\{0\}}\mathsf{e}_{v-1}=\mathrm{U}_{\{0\}}\mathrm{D}_{\{0\}}\mathsf{e}_{v-1} \text{ for } v\neq 1,\ \mathrm{D}_{\{0\}}\mathrm{U}_{\{0\}}\mathsf{e}_{0}=0.$$

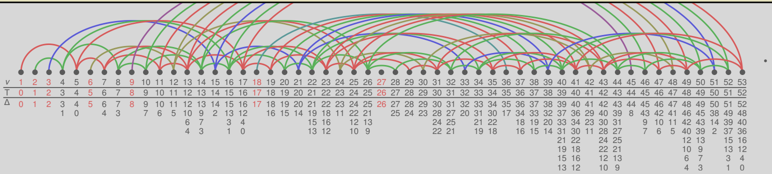
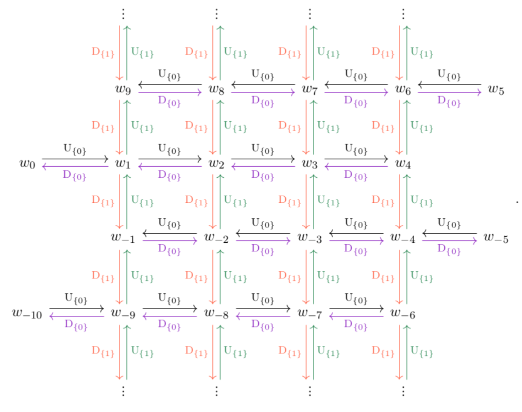


Figure: The full subquiver containing the first 53 vertices of the quiver underlying Z_3.

Example, generation 2, i.e. only three non-zero digit.

In this case every connected component
of the quiver is a bunch of type A graphs glued together in a matrix-grid.
Each row and column is a zigzag algebra, with arrows acting on the 0th digit or 1digit,
and there are "squares commute" relations.

Continuing this periodically gives a quiver for projective G_2T-modules
(due to Andersen ∼2019).



.

The result. There exists a $\mathbb{K}$-algebra Z_p defined as a (very explicit) quotient of the path algebra of an infinite, fractal-like quiver. Let $\mathrm{p}\mathcal{M}\mathrm{od}\text{-}\mathrm{Z}_p$ denote the category of finitely-generated, projective (right-)modules for Z_p. There is an equivalence of additive, $\mathbb{K}$-linear categories

$$\mathcal{F}\colon \mathcal{T}\mathrm{ilt} \xrightarrow{\cong} \mathrm{p}\mathcal{M}\mathrm{od}\text{-}\mathrm{Z}_p,$$

sendin

In general, Z_p is basically a bunch of zigzag algebras

(there are scalars and a lower-order-error term, but never mind)

glued together in a fractal-way, according to the digits of $v = [a_r, \ldots, a_0]_p$.

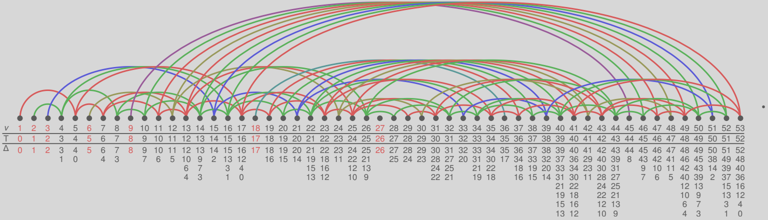
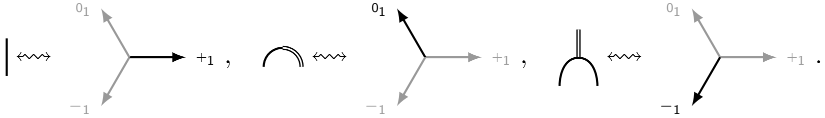


Figure: The full subquiver containing the first 53 vertices of the quiver underlying Z_3.

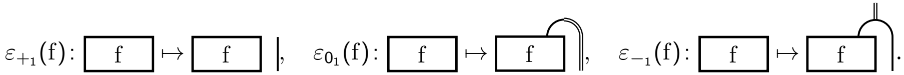
Back, Time is over, you fool

The SL_3 fusion rules for $\Delta\big((1,0)\big) = \mathbb{C}\{\varepsilon_1, \varepsilon_0, \varepsilon_{-1}\}$:

$$\Delta(\lambda) \otimes \Delta\big((1,0)\big) \cong \Delta(\lambda{+}(1,0)) \oplus \Delta(\lambda{+}(-1,1)) \oplus \Delta(\lambda{+}(0,-1)),$$



Elias ∼2015 à la Littelmann ∼1995. For any path π in the dominant Weyl chamber define $\mathrm{d}(\pi)$ inductively by



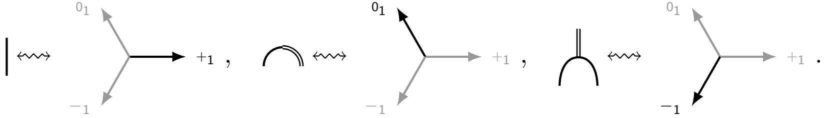
Flip to obtain $\mathrm{u}(\pi)$ and stick them together. This gives an integral basis *I* of $\mathcal{W}\mathrm{eb}$.

Back

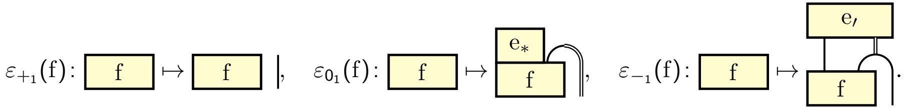
There is of course the dual picture for the second fundamental module – it is omitted to make this slide less cumbersome.

The SL_3 fusion rules for $\Delta\big((1,0)\big) = \mathbb{C}\{\varepsilon_1, \varepsilon_0, \varepsilon_{-1}\}$:

$$\Delta(\lambda) \otimes \Delta\big((1,0)\big) \cong \Delta(\lambda+(1,0)) \oplus \Delta(\lambda+(-1,1)) \oplus \Delta(\lambda+(0,-1)),$$



Kuperberg ∼1995, Kim ∼2006, Elias ∼2015. For any path π in the dominant Weyl chamber define $\tilde{\mathrm{d}}(\pi)$ inductively by

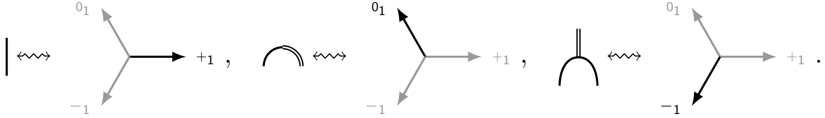


Flip to obtain $\tilde{\mathrm{u}}(\pi)$ and stick them together. This gives a standard basis S of $\mathcal{W}\mathrm{eb}$.

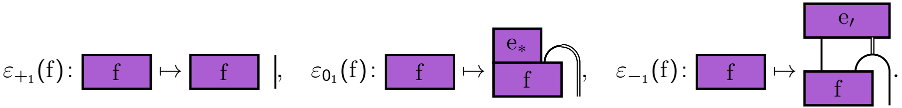
Back

The SL_3 fusion rules for $\Delta\big((1,0)\big) = \mathbb{C}\{\varepsilon_1, \varepsilon_0, \varepsilon_{-1}\}$:

$$\Delta(\lambda) \otimes \Delta\big((1,0)\big) \cong \Delta(\lambda+(1,0)) \oplus \Delta(\lambda+(-1,1)) \oplus \Delta(\lambda+(0,-1)),$$



Libedinsky–Patimo ∼2020. For any path π in the dominant Weyl chamber define $\overline{\mathrm{d}}(\pi)$ inductively by



Flip to obtain $\overline{\mathrm{u}}(\pi)$ and stick them together. This gives a tilting basis T of $\mathcal{W}\mathrm{eb}$.

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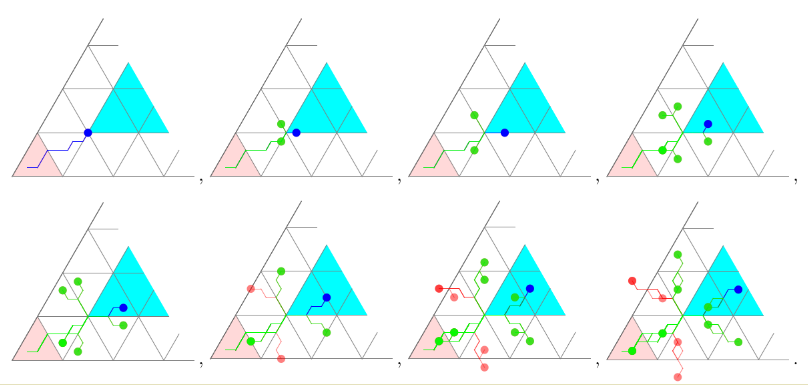
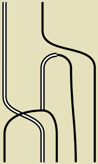
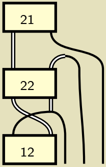
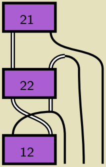
Examples (blue=“all positive”, red=“non-examples”).



Example for $\pi = +_1 +_2 -_1 0_2 +_1$.

$\mathrm{d}_\pi =$  and $\tilde{\mathrm{d}}_\pi =$ 

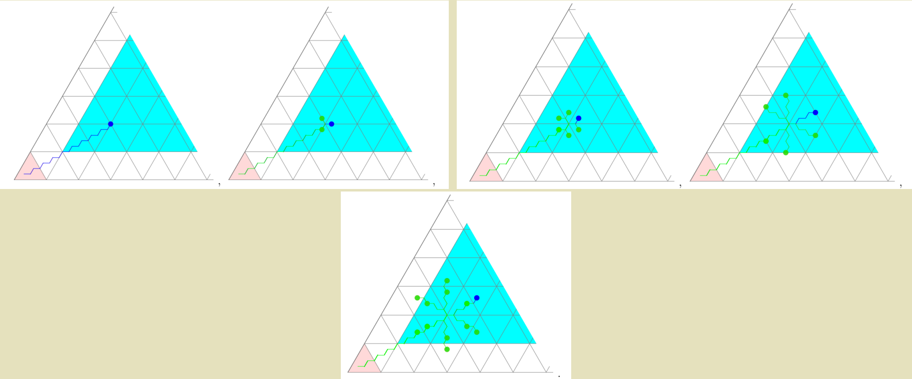
and $\overline{\mathrm{d}}_\pi =$ 

es a tilting basis T of $\mathcal{W}\mathrm{eb}$.

From rank 2 onward you have crossings since *e.g.* $\Delta((1,0)) \otimes \Delta((0,1)) \cong \Delta((0,1)) \otimes \Delta((1,0))$ but $\not\equiv$.
They are mostly harmless – ignore them for today.

The tilting characters, and thus the tilting projectors, are given by path folding.

Examples (blue=“leading summand”, green=“other summands”).



Flip to obtain $\overline{\mathrm{u}}(\pi)$ and stick them together. This gives a tilting basis T of $\mathcal{W}\mathrm{eb}$.

Back