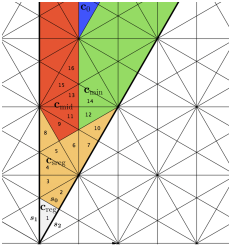
Green's theory of cells in categorification

Or: Mind your cells!

Daniel Tubbenhauer

Shamelessly stolen from https://arxiv.org/abs/1707.07740:



Joint with Marco Mackaay, Volodymyr Mazorchuk, Vanessa Miemietz and Xiaoting Zhang

December 2020

Clifford, Munn, Ponizovskiĭ, Green ∼1942++. Finite semigroups or monoids.

Example. $\mathbb{N}$, $\mathrm{Aut}(\{1,\dots,n\}) = S_n \subset T_n = \mathrm{End}(\{1,\dots,n\})$, groups, groupoids, categories, any · closed subsets of matrices, "everything" ▸ click, *etc.*

The cell orders and equivalences:

$$x \leq_L y \Leftrightarrow \exists z\colon y = zx, \quad x \sim_L y \Leftrightarrow (x \leq_L y) \wedge (y \leq_L x),$$
$$x \leq_R y \Leftrightarrow \exists z'\colon y = xz', \quad x \sim_R y \Leftrightarrow (x \leq_R y) \wedge (y \leq_R x),$$
$$x \leq_{LR} y \Leftrightarrow \exists z, z'\colon y = zxz', \quad x \sim_{LR} y \Leftrightarrow (x \leq_{LR} y) \wedge (y \leq_{LR} x).$$

Left, right and two-sided cells (a.k.a. $\mathcal{L}$-, $\mathcal{R}$- and $\mathcal{J}$-cells): Equivalence classes.

Example (group-like). The unit 1 is always in the lowest cell – *e.g.* $1 \leq_L y$ because we can take $z = y$. Invertible elements g are always in the lowest cell – *e.g.* $g \leq_L y$ because we can take $z = yg^{-1}$.

$\mathcal{L}$-cells ⇜⇝ left modules / left ideals.
$\mathcal{R}$-cells ⇜⇝ right modules / right ideals.
$\mathcal{J}$-cells "$\mathcal{L} \otimes_{\mathbb{K}} \mathcal{R}$" ⇜⇝ bimodules / ideals.
$\mathcal{H}$-cells "$\mathcal{R} \otimes_S \mathcal{L}$" ⇜⇝ subalgebras.

Clifford, Munn, Ponizovskiĭ, Green ∼1942++. Finite semigroups or monoids.

Example (the transformation monoid T_3). Cells – $\mathcal{L}$ (columns), $\mathcal{R}$ (rows), $\mathcal{J}$ (big rectangles), $\mathcal{H}$ (small rectangles).

$\mathcal{J}_{\text{biggest}}$	(111)	(222)	(333)	$\mathcal{H} \cong S_1$
$\mathcal{J}_{\text{middle}}$	(122), (221)	(133), (331)	(233), (322)	
	(121), (212)	(313), (131)	(323), (232)	$\mathcal{H} \cong S_2$
	(221), (112)	(113), (311)	(223), (332)	
$\mathcal{J}_{\text{lowest}}$	(123), (213), (132) (231), (312), (321)			$\mathcal{H} \cong S_3$

Cute facts.

- Each $\mathcal{H}$ contains precisely one idempotent e or no idempotent. Each e is contained in some $\mathcal{H}(e)$. (Idempotent separation.)
- Each $\mathcal{H}(e)$ is a maximal subgroup. (Group-like.)
- Each simple has a unique maximal $\mathcal{J}(e)$ whose $\mathcal{H}(e)$ does not kill it. (Apex.)

Clifford, Mun… s or monoids.

$\mathcal{H}$-reduction. (Mind your cells!)—stated for monoids

There is a one-to-one correspondence

$$\left\{\begin{matrix}\text{simples with}\\ \text{apex } \mathcal{J}(e)\end{matrix}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{matrix}\text{simples of (any)}\\ \mathcal{H}(e)\subset\mathcal{J}(e)\end{matrix}\right\}.$$

In other words,

$$S\text{-smod}_{\mathcal{J}(e)} \simeq \mathcal{H}(e)\text{-smod}.$$

smod means the category of simples.

Example (the … $\mathcal{R}$ (rows), $\mathcal{J}$ (big rectangles …

$\mathcal{J}_{\text{biggest}}$ … $\mathcal{H} \cong S_1$

$\mathcal{J}_{\text{middle}}$ … (221), (11 … 23), (332) … $\mathcal{H} \cong S_2$

$\mathcal{J}_{\text{lowest}}$: (123), (213), (132), (231), (312), (321) … $\mathcal{H} \cong S_3$

Cute facts.

- Each $\mathcal{H}$ contains precisely one idempotent e or no idempotent. Each e is contained in some $\mathcal{H}(e)$. (Idempotent separation.)
- Each $\mathcal{H}(e)$ is a maximal subgroup. (Group-like.)
- Each simple has a unique maximal $\mathcal{J}(e)$ whose $\mathcal{H}(e)$ does not kill it. (Apex.)

Clifford, Mun… s or monoids.

Example (the … $\mathcal{R}$ (rows), $\mathcal{J}$ (big rectangles …

$\mathcal{J}_{\text{biggest}}$ … $\mathcal{H} \cong S_1$

$\mathcal{J}_{\text{middle}}$ … $\mathcal{H} \cong S_2$

(221), (112) (113), (311) (223), (332)

$\mathcal{J}_{\text{l}}$… $\cong S_3$

$\mathcal{H}$-reduction. (Mind your cells!)—stated for monoids

There is a one-to-one correspondence

$$\left\{\begin{matrix}\text{simples with}\\ \text{apex } \mathcal{J}(e)\end{matrix}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{matrix}\text{simples of (any)}\\ \mathcal{H}(e) \subset \mathcal{J}(e)\end{matrix}\right\}.$$

In other words,

$$S\text{-smod}_{\mathcal{J}(e)} \simeq \mathcal{H}(e)\text{-smod}.$$

Example. (T_3.) More

$\mathcal{H}(e) = S_3, S_2, S_1$ gives $3 + 2 + 1 = 6$ associated simples (over $\mathbb{C}$).

Cute facts.

This is a general philosophy in representation theory.

Buzz words. Idempotent truncations, Kazhdan–Lusztig cells, quasi-hereditary algebras, cellular algebras, *etc.*

- Each $\mathcal{H}$ … Each e is contain…
- Each $\mathcal{H}($…
- Each sim… kill it. (Apex.)

Note. Whenever one has a (reasonable) antiinvolution *, the $\mathcal{H}$-cells to consider are the diagonals $\mathcal{H} = \mathcal{L} \cap \mathcal{L}^\star$. I will almost ignore non-contributing $\mathcal{H}$-cells from now on.

Kazhdan–Lusztig (KL) and others ∼1979++. Green's theory in linear.

Choose a basis. For a finite-dimensional algebra S fix a basis B_{S}. For $x, y, z \in B_{\mathrm{S}}$ write $y \Subset zx$ if y appears in zx with non-zero coefficient.

The cell orders and equivalences:

$$x \leq_L y \Leftrightarrow \exists z\colon y \Subset zx, \quad x \sim_L y \Leftrightarrow (x \leq_L y) \wedge (y \leq_L x),$$
$$x \leq_R y \Leftrightarrow \exists z'\colon y \Subset xz', \quad x \sim_R y \Leftrightarrow (x \leq_R y) \wedge (y \leq_R x),$$
$$x \leq_{LR} y \Leftrightarrow \exists z, z'\colon y \Subset zxz', \quad x \sim_{LR} y \Leftrightarrow (x \leq_{LR} y) \wedge (y \leq_{LR} x).$$

$\mathcal{L}$-, $\mathcal{R}$- and $\mathcal{J}$-cells: Equivalence classes. $\mathrm{S}_{\mathcal{H}} = \mathbb{K}\{B_{\mathcal{H}}\}$/bigger friends.

Example (group-like). For $\mathrm{S} = \mathbb{Z}[G]$ and the choice of the group element basis $B_{\mathrm{S}} = G$, cell theory is boring.

$\mathcal{L}$-cells ⇜⇝ left modules / left ideals.
$\mathcal{R}$-cells ⇜⇝ right modules / right ideals.
$\mathcal{J}$-cells "$\mathcal{L} \otimes_{\mathbb{K}} \mathcal{R}$" ⇜⇝ bimodules / ideals.
$\mathcal{H}$-cells "$\mathcal{R} \otimes_{\mathrm{S}} \mathcal{L}$" ⇜⇝ subalgebras.

Kazhdan–Lusztig (KL) and others ∼1979++. Green's theory in linear.

Example ($\mathrm{H}(1 \overset{4}{\text{———}} 2)$**,** B_{S}=**KL basis,** $[2],[4]\neq 0$ **and** $2\neq 0$**).**

$\mathcal{J}_{w_0}$	b_{1212}	$\mathrm{S}_{\mathcal{H}}\cong\mathbb{K}$
$\mathcal{J}_{\text{middle}}$	b_1, b_{121} ∣ b_{21} / b_{12} ∣ b_2, b_{212}	$\mathrm{S}_{\mathcal{H}}\cong\mathbb{K}[\mathbb{Z}/2\mathbb{Z}]$
$\mathcal{J}_{\emptyset}$	$b_{\emptyset}$	$\mathrm{S}_{\mathcal{H}}\cong\mathbb{K}$

We count the wrong number of simples, namely $1+2+1=4<5$.

Kazhdan–Lusztig (KL) and others $\sim$1979++. Green's theory in linear.

Example ($\mathrm{H}(1 \overset{4}{\text{——}} 2)$, B_{S}=KL basis with $b'_{121} = b_{121} + b_1$ and $b'_{212} = b_{212} - b_2$), $[2] \neq 0$ and $2 \neq 0$.

$\mathcal{J}_{(\emptyset,(2))}$	b_{1212}	$\mathrm{S}_{\mathcal{H}} \cong \mathbb{K}$
$\mathcal{J}_{(\emptyset,(1,1))}$	b'_{212}	$\mathrm{S}_{\mathcal{H}} \cong \mathbb{K}$
$\mathcal{J}_{((1),(1))}$	$\begin{array}{c\|c} b'_{121} & b_{21} \\ \hline b_{12} & b_2 \end{array}$	$\mathrm{S}_{\mathcal{H}} \cong \mathbb{K}$
$\mathcal{J}_{((1,1),\emptyset)}$	b_1	$\mathrm{S}_{\mathcal{H}} \cong \mathbb{K}$
$\mathcal{J}_{((2),\emptyset)}$	$b_\emptyset$	$\mathrm{S}_{\mathcal{H}} \cong \mathbb{K}$

We count the correct number of simples, namely $1 + 1 + 1 + 1 + 1 = 5$.

Kazhdan–Lusztig (KL) and others ∼1979++. Green's theory in linear.

Example ($\mathrm{H}(1 \overset{5}{\text{———}} 2)$, B_{S}=KL basis, $[2],[5]\neq 0$ and $2,5\neq 0$).

$\mathcal{J}_{w_0}$	b_{12121}	$\mathrm{S}_{\mathcal{H}} \cong \mathbb{K}$
$\mathcal{J}_{\text{middle}}$	b_1, b_{121} | b_{21}, b_{2121} b_{12}, b_{1212} | b_2, b_{212}	$\mathrm{S}_{\mathcal{H}} \cong \mathbb{K}[\mathbb{Z}/2\mathbb{Z}]$
$\mathcal{J}_{\emptyset}$	$b_{\emptyset}$	$\mathrm{S}_{\mathcal{H}} \cong \mathbb{K}$

We count the correct number of simples, namely $1+2+1=4$.

$\mathcal{H}$-reduction in linear.

Problem 1. Everything depends on the choice of basis.

Problem 2. If $\mathcal{H}$-cells are of varying size within a $\mathcal{J}$-cell, you might count a too low number of simples.

Aside: The case where all $\mathcal{H}$-cells are of size one is called cellular.

$\mathcal{H}$-reduction in linear.

Problem 1. Everything depends on the choice of basis.

Problem 2. If $\mathcal{H}$-cells are of varying size within a $\mathcal{J}$-cell, you might count a too low number of simples.

Aside: The case where all $\mathcal{H}$-cells are of size one is called cellular.

Spoiler.

On the categorified level all problems vanish and (a version of) the $\mathcal{H}$-reduction can be recovered.

Mazorchuk–Miemietz and others ∼2010++. Green's theory in categories.

There is a good basis. For a finitary monoidal category $\mathscr{S}$, and $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ indecomposable write $\mathrm{Y} \subset_{\oplus} \mathrm{ZX}$ if Y is a direct summand of ZX.

The cell orders and equivalences:

$$\mathrm{X} \leq_{L} \mathrm{Y} \Leftrightarrow \exists \mathrm{Z}\colon \mathrm{Y} \subset_{\oplus} \mathrm{ZX}, \quad \mathrm{X} \sim_{L} \mathrm{Y} \Leftrightarrow (\mathrm{X} \leq_{L} \mathrm{Y}) \wedge (\mathrm{Y} \leq_{L} \mathrm{X}),$$
$$\mathrm{X} \leq_{R} \mathrm{Y} \Leftrightarrow \exists \mathrm{Z}'\colon \mathrm{Y} \subset_{\oplus} \mathrm{XZ}', \quad \mathrm{X} \sim_{R} \mathrm{Y} \Leftrightarrow (\mathrm{X} \leq_{R} \mathrm{Y}) \wedge (\mathrm{Y} \leq_{R} \mathrm{X}),$$
$$\mathrm{X} \leq_{LR} \mathrm{Y} \Leftrightarrow \exists \mathrm{Z}, \mathrm{Z}'\colon \mathrm{Y} \subset_{\oplus} \mathrm{ZXZ}', \quad \mathrm{X} \sim_{LR} \mathrm{Y} \Leftrightarrow (\mathrm{X} \leq_{LR} \mathrm{Y}) \wedge (\mathrm{Y} \leq_{LR} \mathrm{X}).$$

$\mathcal{L}$-, $\mathcal{R}$- and $\mathcal{J}$-cells: Equivalence classes. $\mathscr{S}_{\mathcal{H}} = \mathrm{add}(\mathcal{H}, \mathbb{1})/$"bigger friends".

Example (group-like). For $\mathscr{S} = \mathscr{V}\mathrm{ect}_{G}$ cell theory is boring. (In general cell theory is boring for fusion categories.)

$\mathcal{L}$-cells ⇝ left modules / left ideals.
$\mathcal{R}$-cells ⇝ right modules / right ideals.
$\mathcal{J}$-cells "$\mathcal{L} \otimes_{\mathbb{K}} \mathcal{R}$" ⇝ bimodules / ideals.
$\mathcal{H}$-cells "$\mathcal{R} \otimes_{\mathrm{S}} \mathcal{L}$" ⇝ subalgebras.

Examples.

- Cells in $\mathscr{S}$ give $\otimes$-ideals.
- If $\mathscr{S}$ is semisimple, then $\mathrm{X}\mathrm{X}^{\star}$ and $\mathrm{X}^{\star}\mathrm{X}$ both contain the identity, so cell theory is trivial.
- For Soergel bimodules cells are Kazhdan–Lusztig cells.
- For 2-Kac–Moody algebras you can push everything to cyclotomic KLR algebras, and $\mathcal{H}$-cells are of size one.

Mazorchuk–Miemietz and others ∼2010++. Green's theory in categories.

Example ($\mathrm{H}(1 \overset{4}{\text{———}} 2)$, but now Soergel bimodules over $\mathbb{C}$ with their indecomposables).

$\mathcal{J}_{w_0}$	B_{1212}		$\mathscr{S}_{\mathcal{H}}$ "$\simeq$" $\mathscr{V}\mathrm{ect}$
$\mathcal{J}_{\mathrm{middle}}$	$\mathrm{B}_{1}, \mathrm{B}_{121}$	B_{21}	$\mathscr{S}_{\mathcal{H}}$ "$\simeq$" $\mathscr{V}\mathrm{ect}_{\mathbb{Z}/2\mathbb{Z}}$
	B_{12}	$\mathrm{B}_{2}, \mathrm{B}_{212}$	
$\mathcal{J}_{\emptyset}$	$\mathrm{B}_{\emptyset}$		$\mathscr{S}_{\mathcal{H}} \simeq \mathscr{V}\mathrm{ect}$

We count the correct number of 2-simples, namely $1+2+1=4$.

To make the "$\simeq$" above precise is a whole paper...but it works.
For example, $\mathrm{B}_{1212}\mathrm{B}_{1212} \cong p\mathrm{B}_{1212}$ for $p=[2][4] \in \mathbb{N}[v,v^{-1}]$ being a shift.
So B_{1212} is a pseudo-idempotent, but you can't easily rescale on the categorical level.

Mazorchuk–M ... **categories.**

Examples.

- Cells in $\mathscr{S}$ ...
- If $\mathscr{S}$ is se ... tity, so cell theory is t...
- For Soerg...
- For 2-Kac-Moody algebras you can push everything to cyclotomic KLR algebras, and $\mathcal{H}$-cells are of size one.

$\mathcal{H}$-reduction $\sim$2018.

There is a one-to-one correspondence

$$\left\{\begin{matrix}\text{2-simples with}\\ \text{apex } \mathcal{J}\end{matrix}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{matrix}\text{2-simples with apex } \mathcal{H}\\ \text{of (any) } \mathcal{S}_{\mathcal{H}}\end{matrix}\right\}.$$

Strong $\mathcal{H}$-reduction $\sim$2020.

$$\mathscr{S}\text{-stmod}_{\mathcal{J}} \simeq \mathscr{S}_{\mathcal{H}}\text{-stmod}_{\mathcal{H}}.$$

stmod means the category of 2-simples.

Mazorchuk–M... categories.

Examples.

- Cells in $\mathscr{S}$...
- If $\mathscr{S}$ is se... tity, so cell theory is t...
- For Soerg...
- For 2-Kac–Moody algebras you can push everything to cyclotomic KLR algebra...

$\mathcal{H}$-reduction ∼2018.

There is a one-to-one correspondence

$$\left\{\begin{matrix}\text{2-simples with}\\ \text{apex } \mathcal{J}\end{matrix}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{matrix}\text{2-simples with apex } \mathcal{H}\\ \text{of (any) } \mathcal{S}_{\mathcal{H}}\end{matrix}\right\}.$$

Strong $\mathcal{H}$-reduction ∼2020.

$$\mathscr{S}\text{-stmod}_{\mathcal{J}} \simeq \mathscr{S}_{\mathcal{H}}\text{-stmod}_{\mathcal{H}}.$$

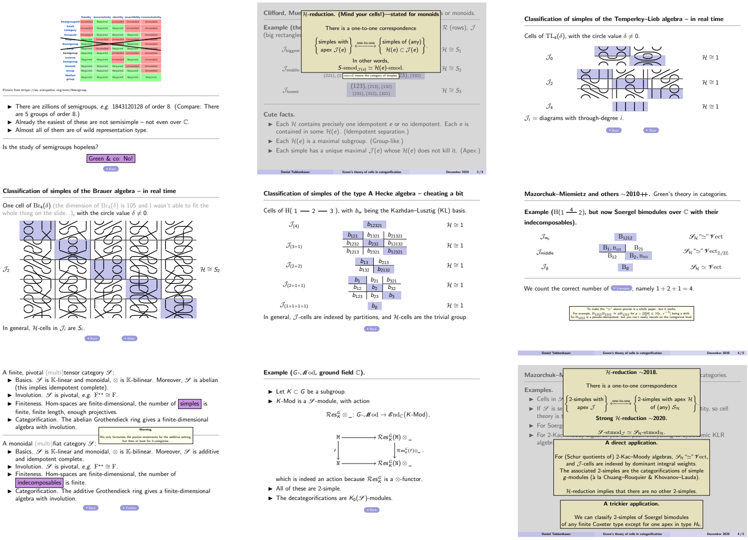
A direct application.

For (Schur quotients of) 2-Kac–Moody algebras, $\mathscr{S}_{\mathcal{H}}$ "$\simeq$" $\mathscr{V}\mathrm{ect}$, and $\mathcal{J}$-cells are indexed by dominant integral weights. The associated 2-simples are the categorifications of simple g-modules (à la Chuang–Rouquier & Khovanov–Lauda).

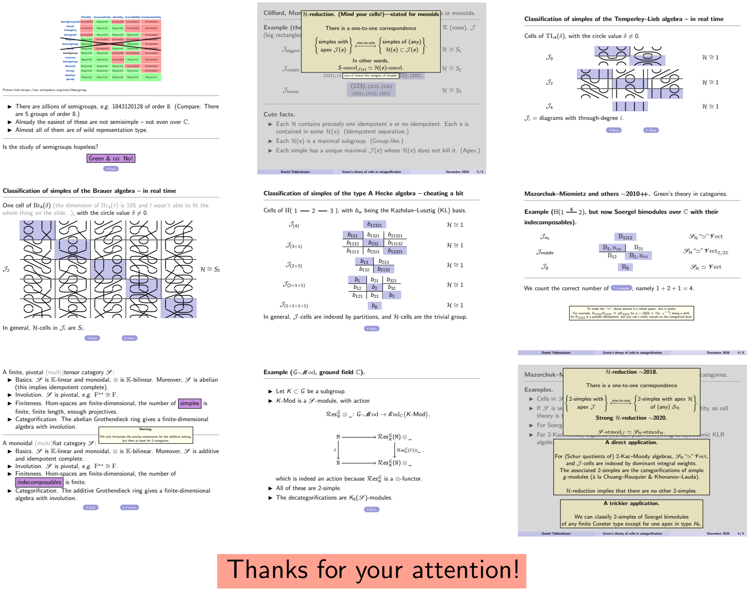
$\mathcal{H}$-reduction implies that there are no other 2-simples.

A trickier application.

We can classify 2-simples of Soergel bimodules of any finite Coxeter type except for one apex in type H_4.



There is still much to do...


Is the study of semigroups hopeless?
Green & co: No!
Classification of simples of the Temperley–Lieb algebra – in real time
Classification of simples of the Brauer algebra – in real time
Classification of simples of the type A Hecke algebra – cheating a bit
Mazorchuk–Miemietz and others ~2010++. Green's theory in categories.
Example (G-Mod, ground field ℂ).
Thanks for your attention!

	Totality	Associativity	Identity	Invertibility	Commutativity
Semigroupoid	Unneeded	Required	Unneeded	Unneeded	Unneeded
Small Category	Unneeded	Required	Required	Unneeded	Unneeded
Groupoid	Unneeded	Required	Required	Required	Unneeded
Magma	Required	Unneeded	Unneeded	Unneeded	Unneeded
Quasigroup	Required	Unneeded	Unneeded	Required	Unneeded
Loop	Required	Unneeded	Required	Required	Unneeded
Semigroup	Required	Required	Unneeded	Unneeded	Unneeded
Inverse Semigroup	Required	Required	Unneeded	Required	Unneeded
Monoid	Required	Required	Required	Unneeded	Unneeded
Group	Required	Required	Required	Required	Unneeded
Abelian group	Required	Required	Required	Required	Required

Picture from `https://en.wikipedia.org/wiki/Semigroup`.

- There are zillions of semigroups, *e.g.* 1843120128 of order 8. (Compare: There are 5 groups of order 8.)
- Already the easiest of these are not semisimple – not even over $\mathbb{C}$.
- Almost all of them are of wild representation type.

Is the study of semigroups hopeless?

Green & co: No!

Back

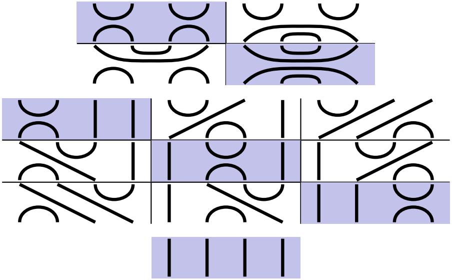
Classification of simples of the Temperley–Lieb algebra – in real time

Cells of $\mathrm{TL}_4(\delta)$, with the circle value $\delta \neq 0$.

$\mathcal{J}_0$ $\mathcal{H} \cong 1$

$\mathcal{J}_2$ $\mathcal{H} \cong 1$

$\mathcal{J}_4$ $\mathcal{H} \cong 1$



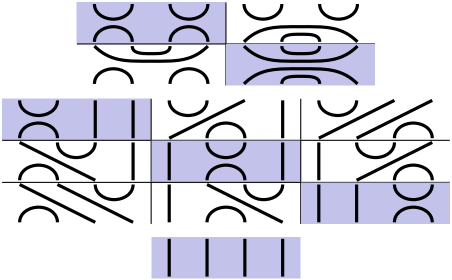
$\mathcal{J}_i$ = diagrams with through-degree i.

Back More

Classification of sim – in real time

There is an antiinvolution (flip pictures),
so $\mathcal{J}$-cells are squares
and $\mathcal{H}$-cells are diagonal.

Cells of $\mathrm{TL}_4(\delta)$, with the circle value $\delta \neq 0$.



$\mathcal{J}_0$ $\mathcal{H} \cong 1$

$\mathcal{J}_2$ $\mathcal{H} \cong 1$

$\mathcal{J}_4$ $\mathcal{H} \cong 1$

$\mathcal{J}_i$ = diagrams with through-degree i.

Back More

Classification of si… – in real time

Note that $\mathcal{H}$-cells are group-like, *e.g.*

so up to rescaling by $1/\delta$,

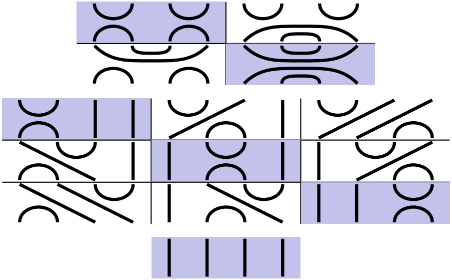
 is the unit in the trivial group.

Cells of $\mathrm{TL}_4(\delta)$, with…

$\mathcal{J}_0$ $\qquad$ $\mathcal{H} \cong 1$

$\mathcal{J}_2$ $\qquad$ $\mathcal{H} \cong 1$

$\mathcal{J}_4$ $\qquad$ $\mathcal{H} \cong 1$

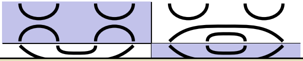


$\mathcal{J}_i$ = diagrams with through-degree i.

Back $\qquad$ More

Classification of simples of the Temperley–Lieb algebra – in real time

Cells of $\mathrm{TL}_4(\delta)$, with the circle value $\delta \neq 0$.

$\mathcal{J}_0$  $\mathcal{H} \cong 1$

Corollary (of Green's theorem).

$$\left\{\begin{array}{c}\text{simples of}\\ \mathrm{TL}_n(\delta)\end{array}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{array}{c}\text{possible}\\ \text{through-degrees}\end{array}\right\}.$$

$$\mathrm{TL}_n(\delta)\text{-}\mathit{smod}_{\mathcal{J}_i} \simeq 1\text{-}\mathit{smod}.$$

$\mathcal{J}_2$ $\mathcal{H} \cong 1$

$\mathcal{J}_4$  $\mathcal{H} \cong 1$

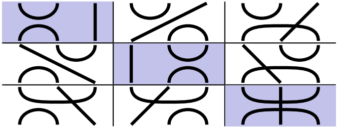
$\mathcal{J}_i$ = diagrams with through-degree i.

Back More

Classification of simples of the Brauer algebra – in real time

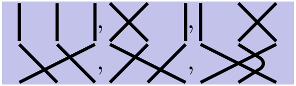
Cells of $\mathrm{Br}_3(\delta)$, with the circle value $\delta \neq 0$.

$\mathcal{J}_1$



$\mathcal{H} \cong S_1$

$\mathcal{J}_3$



$\mathcal{H} \cong S_3$

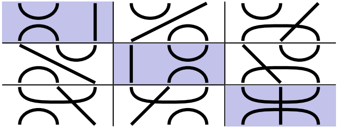
$\mathcal{J}_i$ = diagrams with through-degree i.

Back More

Classification of simples of the Brauer algebra – in real time

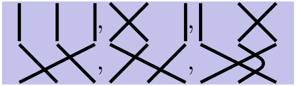
Cells of $\mathrm{Br}_3(\delta)$, with the circle value $\delta \neq 0$.

$\mathcal{J}_1$



$\mathcal{H} \cong S_1$

$\mathcal{J}_3$



$\mathcal{H} \cong S_3$

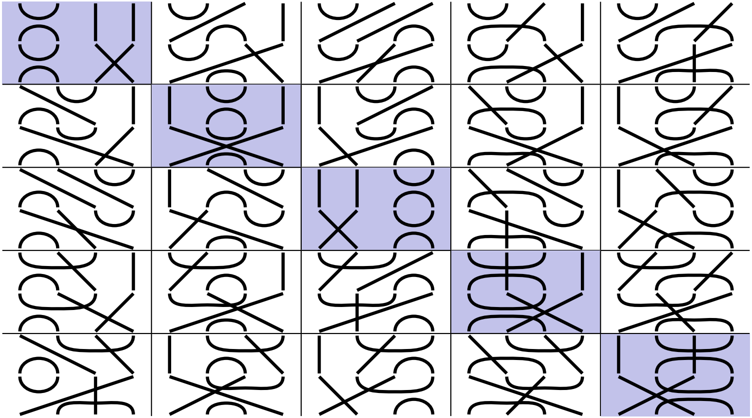
$\mathcal{J}_i$ = diagrams with through-degree i.

There is an antiinvolution (flip pictures),
so $\mathcal{J}$-cells are squares
and $\mathcal{H}$-cells are diagonal.
Moreover, $\mathcal{H}$-cells are group-like.

Classification of simples of the Brauer algebra – in real time

One cell of $\mathrm{Br}_4(\delta)$ (the dimension of $\mathrm{Br}_4(\delta)$ is 105 and I wasn't able to fit the whole thing on the slide...), with the circle value $\delta \neq 0$.

$\mathcal{J}_2$



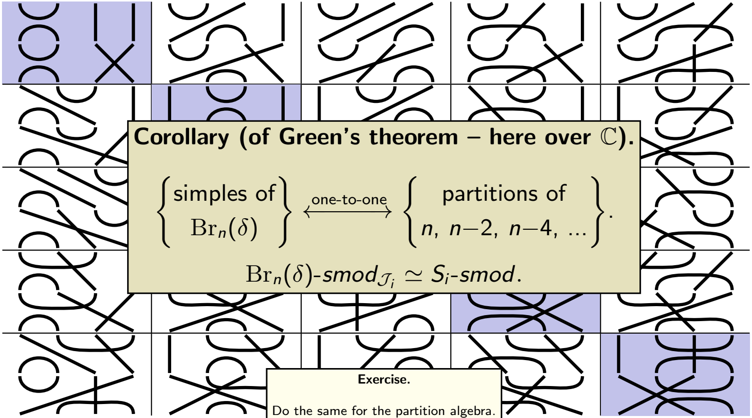
$\mathcal{H} \cong S_2$

In general, $\mathcal{H}$-cells in $\mathcal{J}_i$ are S_i.

Classification of simples of the Brauer algebra – in real time

One cell of $\mathrm{Br}_4(\delta)$ (the dimension of $\mathrm{Br}_4(\delta)$ is 105 and I wasn't able to fit the whole thing on the slide...), with the circle value $\delta \neq 0$.

$\mathcal{J}_2$



$\mathcal{H} \cong S_2$

Corollary (of Green's theorem – here over $\mathbb{C}$).

$$\left\{ \begin{array}{c} \text{simples of} \\ \mathrm{Br}_n(\delta) \end{array} \right\} \xleftrightarrow{\text{one-to-one}} \left\{ \begin{array}{c} \text{partitions of} \\ n,\ n-2,\ n-4,\ \dots \end{array} \right\}.$$

$$\mathrm{Br}_n(\delta)\text{-}\mathit{smod}_{\mathcal{J}_i} \simeq S_i\text{-}\mathit{smod}.$$

Exercise.

Do the same for the partition algebra.

In general, $\mathcal{H}$-cells in $\mathcal{J}_i$ are S_i.

Back More

Classification of simples of the type A Hecke algebra – cheating a bit

Cells of $\mathrm{H}(1 — 2 — 3)$, with b_w being the Kazhdan–Lusztig (KL) basis.

$\mathcal{J}_{(4)}$

b_{12321}

$\mathcal{H} \cong 1$

$\mathcal{J}_{(3+1)}$

b_{121}	b_{1321}	b_{21321}
b_{1232}	b_{232}	b_{12132}
b_{1213}	b_{2321}	b_{12321}

$\mathcal{H} \cong 1$

$\mathcal{J}_{(2+2)}$

b_{13}	b_{213}
b_{132}	b_{2132}

$\mathcal{H} \cong 1$

$\mathcal{J}_{(2+1+1)}$

b_{1}	b_{21}	b_{321}
b_{12}	b_{2}	b_{32}
b_{123}	b_{23}	b_{3}

$\mathcal{H} \cong 1$

$\mathcal{J}_{(1+1+1+1)}$

$b_{\emptyset}$

$\mathcal{H} \cong 1$

In general, $\mathcal{J}$-cells are indexed by partitions, and $\mathcal{H}$-cells are the trivial group.

Back

Classification of simples of the type A Hecke algebra – cheating a bit

Cells of $\mathrm{H}(1 — 2 — 3)$, with b_w being the Kazhdan–Lusztig (KL) basis.

$\mathcal{J}_{(4)}$ $\quad$ b_{12321} $\quad$ $\mathcal{H} \cong 1$

b_{121}	b_{1321}	b_{21321}

There is an antiinvolution (bar involution),
so $\mathcal{J}$-cells are squares
and $\mathcal{H}$-cells are diagonal.
Moreover, $\mathcal{H}$-cells are group-like, *e.g.* $b_{12321}b_{12321} = [3]!b_{12321}$ + bigger friends.

b_{132}	b_{2132}

$\mathcal{J}_{(2+1+1)}$ $\quad$ $\mathcal{H} \cong 1$

b_1	b_{21}	b_{321}
b_{12}	b_2	b_{32}
b_{123}	b_{23}	b_3

$\mathcal{J}_{(1+1+1+1)}$ $\quad$ $b_\emptyset$ $\quad$ $\mathcal{H} \cong 1$

In general, $\mathcal{J}$-cells are indexed by partitions, and $\mathcal{H}$-cells are the trivial group.

Back

Classification of simples of the type A Hecke algebra – cheating a bit

Cells of $\mathrm{H}(\,1 — 2 — 3\,)$, with b_w being the Kazhdan–Lusztig (KL) basis.

$\mathcal{J}_{(4)}$ $\qquad$ b_{12321} $\qquad$ $\mathcal{H} \cong 1$

b_{121}	b_{1321}	b_{21321}

$\mathcal{J}_{(3+1)}$ $\qquad$ $\mathcal{H} \cong 1$

Corollary (of Green's theorem).

$$\left\{\begin{matrix}\text{simples of}\\ \mathrm{H}(S_n)\end{matrix}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{matrix}\text{partitions of}\\ n\end{matrix}\right\}.$$

$$\mathrm{H}(S_n)\text{-}smod_{\mathcal{J}_i} \simeq 1\text{-}smod.$$

$\mathcal{J}_{(2+2)}$ $\qquad$ $\mathcal{H} \cong 1$

$\mathcal{J}_{(2+1+1)}$ $\qquad$ $\mathcal{H} \cong 1$

b_{12}	b_{2}	b_{32}
b_{123}	b_{23}	b_{3}

$\mathcal{J}_{(1+1+1+1)}$ $\qquad$ $\mathcal{H} \cong 1$

Warning.

Outside of type A you need to take a different basis, the KL basis doesn't work.

In general, $\mathcal{J}$-cells are indexed by partitions, and $\mathcal{H}$-cells are the trivial group.

Back

A finite, pivotal (multi)tensor category $\mathscr{S}$:

- Basics. $\mathscr{S}$ is $\mathbb{K}$-linear and monoidal, $\otimes$ is $\mathbb{K}$-bilinear. Moreover, $\mathscr{S}$ is abelian (this implies idempotent complete).
- Involution. $\mathscr{S}$ is pivotal, *e.g.* $\mathrm{F}^{\star\star} \cong \mathrm{F}$.
- Finiteness. Hom-spaces are finite-dimensional, the number of simples is finite, finite length, enough projectives.
- Categorification. The abelian Grothendieck ring gives a finite-dimensional algebra with involution.

Warning.

We only formulate the precise statements for the additive setting, but then at least for 2-categories.

A monoidal (multi)fiat category $\mathscr{S}$:

- Basics. $\mathscr{S}$ is $\mathbb{K}$-linear and monoidal, $\otimes$ is $\mathbb{K}$-bilinear. Moreover, $\mathscr{S}$ is additive and idempotent complete.
- Involution. $\mathscr{S}$ is pivotal, *e.g.* $\mathrm{F}^{\star\star} \cong \mathrm{F}$.
- Finiteness. Hom-spaces are finite-dimensional, the number of indecomposables is finite.
- Categorification. The additive Grothendieck ring gives a finite-dimensional algebra with involution.

Back Further

A finite, pivotal (multi)tensor category $\mathscr{S}$:

- Basics. $\mathscr{S}$ is $\mathbb{K}$-linear and monoidal, $\otimes$ is $\mathbb{K}$-bilinear. Moreover, $\mathscr{S}$ is abelian (this implies idempotent complete).
- Involution. $\mathscr{S}$ is pivotal, *e.g.* $\mathrm{F}^{\star\star} \cong \mathrm{F}$.
- Finiteness. Hom-spaces are finite-dimensional, the number of simples is finite, finite length, enough projectives.

The crucial difference...

...is what we like to consider as "elements" of our theory:

Abelian prefers simples,
additive prefers indecomposables.

This is a huge difference – for example in the fiat case there is no Schur's 2-lemma.

A

ve

- Finiteness. Hom-spaces are finite-dimensional, the number of indecomposables is finite.
- Categorification. The additive Grothendieck ring gives a finite-dimensional algebra with involution.

Back Further

A finite, pivotal (multi)tensor category $\mathscr{S}$:

- Basics. $\mathscr{S}$ is $\mathbb{K}$-linear and monoidal, $\otimes$ is $\mathbb{K}$-bilinear. Moreover, $\mathscr{S}$ is abelian (this implies idempotent complete).
- In...
- Fi... is fi...
- Ca... onal algebra with involution.

Abelian examples.

H-$\mathscr{M}\mathrm{od}$ for H a finite-dimensional Hopf algebra. (Think: $\mathbb{K}G$, G finite.)
Finite Serre quotients of G-$\mathscr{M}\mathrm{od}$ for G being a reductive group.

A... ve

- Involution. ...
- Finiteness. Hom-spaces are finite-dimensional, the number of indecomposables is finite.
- Ca... onal al...

Abelian and additive examples.

H-$\mathscr{M}\mathrm{od}$ for H a finite-dimensional, semisimple Hopf algebra. (Think: $\mathbb{C}G$, G finite.)
$\mathscr{V}\mathrm{ect}_G$ for G graded $\mathbb{K}$-vector spaces, *e.g.* $\mathscr{V}\mathrm{ect} = \mathscr{V}\mathrm{ect}_1$.

Additive examples.

H-$\mathscr{P}\mathrm{roj}$ for H a finite-dimensional Hopf algebra. (Think: $\mathbb{K}G$, G finite.)
Finite quotients of G-$\mathscr{T}\mathrm{ilt}$ for G being a reductive group.

A finite, pivotal (multi)tensor category $\mathscr{S}$:

- Basics. $\mathscr{S}$ is $\mathbb{K}$-linear and monoidal, $\otimes$ is $\mathbb{K}$-bilinear. Moreover, $\mathscr{S}$ is abelian (this implies idempotent complete).
- Involution. $\mathscr{S}$ is pivotal, *e.g.* $\mathrm{F}^{\star\star} \cong \mathrm{F}$.
- Finiteness. Hom-spaces are finite-dimensional, the number of simples is finite, finite length, enough projectives.
- Categorification. The abelian Grothendieck ring gives a finite-dimensional al

---

Why I like the additive case.

All the example I know from my youth are not abelian, but only additive:

Diagram categories, 2-Kac–Moody algebras
and their Schur quotients, Soergel bimodules,
tilting module categories *etc.*

And these only fit into the fiat and not the tensor framework.

A mon

- B additive ar
- In
- Fi indecomposables is finite.
- Categorification. The additive Grothendieck ring gives a finite-dimensional algebra with involution.

Back Further

Example $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (Klein four group).

If $\mathbb{K}$ is not of characteristic 2, $\mathbb{K}G$ is semisimple and additive=abelian. So let us have a look at characteristic 2, where we have $\mathbb{K}G \cong \mathbb{K}[X, Y]/(X^2, Y^2)$

First, abelian:

- X and Y have to act as zero on each simple, so $\mathbb{K}G$ has just $\mathbb{K}$ as a simple.
- $\mathbb{K}G\text{-}\mathscr{M}\mathrm{od}$ has just one element.

Then additive:

- Only X^2 and Y^2 have to act as zero on each indecomposable, and one can cook-up infinitely many, *e.g.*

$$\bullet \xleftarrow{X} \bullet \xrightarrow{Y} \bullet \xleftarrow{X} \bullet \xrightarrow{Y} \bullet \xleftarrow{X} \dots \xrightarrow{Y} \bullet \xleftarrow{X} \bullet$$

- $\mathbb{K}G\text{-}\mathscr{M}\mathrm{od}$ has infinitely many elements.

Back

Example $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (Klein four group).

If $\mathbb{K}$ is not of characteristic 2, $\mathbb{K}G$ is semisimple and additive=abelian. So let us have a look at characteristic 2, where we have $\mathbb{K}G \cong \mathbb{K}[X, Y]/(X^2, Y^2)$

First, abelian:

- ... ple.
-

The

- ... an cook-up infinitely many, *e.g.*

$$\bullet \xleftarrow{X} \bullet \xrightarrow{Y} \bullet \xleftarrow{X} \bullet \xrightarrow{Y} \bullet \xleftarrow{X} \dots \xrightarrow{Y} \bullet \xleftarrow{X} \bullet$$

- $\mathbb{K}G\text{-}\mathcal{M}\mathrm{od}$ has infinitely many elements.

Theorem (Higman ∼1953).

For $\mathrm{char}(\mathbb{K}) = p$, $\mathbb{K}G\text{-}\mathcal{M}\mathrm{od}$ is...

...always a finite, pivotal tensor category.

... monoidal fiat if and only if ($p \nmid |G|$ or the p-Sylow subgroup of G is cyclic).

Back

Abelian. A $\mathscr{S}$-module M:

- Basics. M is $\mathbb{K}$-linear and abelian. The action is a monoidal functor $\mathrm{M}\colon \mathscr{S} \to \mathscr{E}\mathrm{nd}_{\mathbb{K},lex}(\mathrm{M})$ ($\mathbb{K}$-linear, left exactness).
- Finiteness. Hom-spaces are finite-dimensional, the number of simples is finite, finite length, enough projectives.
- Categorification. The abelian Grothendieck group gives a finite-dimensional $G_0(\mathscr{S})$-module.

Additive. A $\mathscr{S}$-module M:

- Basics. M is $\mathbb{K}$-linear, additive and idempotent complete. The action is a monoidal functor $\mathrm{M}\colon \mathscr{S} \to \mathscr{E}\mathrm{nd}_{\mathbb{K}}(\mathrm{M})$ ($\mathbb{K}$-linear).
- Finiteness. Hom-spaces are finite-dimensional, the number of indecomposables is finite.
- Categorification. The additive Grothendieck group gives a finite-dimensional $K_0(\mathscr{S})$-module.

Back Further

Abelian. A $\mathscr{S}$-module M:

- Basics. M is $\mathbb{K}$-linear and abelian. The action is a monoidal functor M: $\mathscr{S} \to \mathscr{E}\mathrm{nd}_{\mathbb{K},lex}(\mathrm{M})$ ($\mathbb{K}$-linear, left exactness).
- Finiteness. … simples is finite, finite …
- Categorifica… …te-dimensional $G_0(\mathscr{S})$-mo…

The easiest of such modules are called simple transitive (2-simple for short) and they satisfy a Jordan–Hölder theorem.

By definition, these are those $\mathscr{S}$-modules without $\mathscr{S}$-stable ideals on the morphism level.

This categorifies the definition of a simple having no S-stable subspaces.

Additive. A $\mathscr{S}$-…

- Basics. M is … The action is a monoidal functor …
- Finiteness. Hom-… …ber of indecomposables …
- Categorification. … s a finite-dimensional $K_0(\mathscr{S})$-module.

Example.

For 2-Kac–Moody algebras the minimal categorifications of the g-simples in the sense of Chuang–Rouquier are 2-simple.

Back Further

Example (G-$\mathscr{M}\mathrm{od}$, ground field $\mathbb{C}$).

- Let $\mathscr{S} = G$-$\mathscr{M}\mathrm{od}$, for G being a finite group. As $\mathscr{S}$ is semisimple, abelian=additive. Simples are simple G-modules.
- For any $\mathtt{M}, \mathtt{N} \in \mathscr{S}$, we have $\mathtt{M} \otimes \mathtt{N} \in \mathscr{S}$:

$$g(m \otimes n) = gm \otimes gn$$

  for all $g \in G, m \in \mathtt{M}, n \in \mathtt{N}$. There is a trivial module $\mathbb{C}$.
- The regular $\mathscr{S}$-module $\mathrm{M}\colon \mathscr{S} \to \mathscr{E}\mathrm{nd}_{\mathbb{C}}(\mathscr{S})$:

$$\begin{array}{ccc} \mathtt{M} & \longrightarrow & \mathtt{M} \otimes \_ \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes \_} \\ \mathtt{N} & \longrightarrow & \mathtt{N} \otimes \_ \end{array} .$$

- The decategorification is the regular $K_0(\mathscr{S})$-module.

Back

Example (G-$\mathscr{M}\mathrm{od}$, ground field $\mathbb{C}$).

- Let $K \subset G$ be a subgroup.
- K-Mod is a $\mathscr{S}$-module, with action

$$\mathcal{R}es^G_K \otimes \_\colon G\text{-}\mathscr{M}\mathrm{od} \to \mathscr{E}\mathrm{nd}_{\mathbb{C}}\big(K\text{-Mod}\big),$$

$$\begin{array}{ccc} \mathtt{M} & \longrightarrow & \mathcal{R}es^G_K(\mathtt{M}) \otimes \_ \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{R}es^G_K(f)\otimes\_} \\ \mathtt{N} & \longrightarrow & \mathcal{R}es^G_K(\mathtt{N}) \otimes \_ \end{array}.$$

  which is indeed an action because $\mathcal{R}es^G_K$ is a $\otimes$-functor.
- All of these are 2-simple.
- The decategorifications are $K_0(\mathscr{S})$-modules.

Back