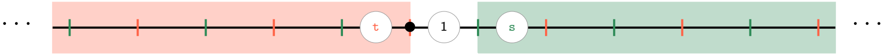
A tale of dihedral groups, $\mathrm{SL}(2)_q$, and beyond

Or: Who colored my Dynkin diagrams?

Daniel Tubbenhauer



Joint work with Marco Mackaay, Volodymyr Mazorchuk, Vanessa Miemietz and Xiaoting Zhang

March 2019

Let $A(\mathbf{\Gamma})$ be the adjacency matrix of a finite, connected, loopless graph $\mathbf{\Gamma}$. Let $\mathrm{U}_{e+1}(\mathtt{X})$ be the ▸ Chebyshev polynomial.

Classification problem (CP). Classify all $\mathbf{\Gamma}$ such that $\mathrm{U}_{e+1}(A(\mathbf{\Gamma}))=0$.

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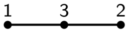
$$U_3(X) = (X - 2\cos(\tfrac{\pi}{4}))X(X - 2\cos(\tfrac{3\pi}{4}))$$

$A_3 =$ 

$$\rightsquigarrow \quad A(A_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \rightsquigarrow \quad S_{A_3} = \{2\cos(\tfrac{\pi}{4}), 0, 2\cos(\tfrac{3\pi}{4})\}$$

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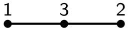
$\mathrm{A}_3 =$ 

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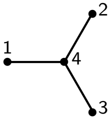
$$\mathrm{U}_5(\mathtt{X}) = (\mathtt{X} - 2\cos(\tfrac{\pi}{6}))(\mathtt{X} - 2\cos(\tfrac{2\pi}{6}))\mathtt{X}(\mathtt{X} - 2\cos(\tfrac{4\pi}{6}))(\mathtt{X} - 2\cos(\tfrac{5\pi}{6}))$$

$\mathrm{D}_4 =$ 

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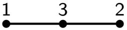
$A_3 =$ 

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✓ for $e = 2$

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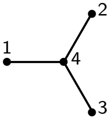
$D_4 =$ 

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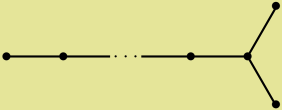
✓ for $e = 4$

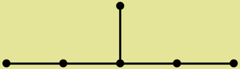
Let $A(\Gamma)$ be the adjacency matrix of a finite, connected, loopless graph Γ. Let $U_{e+1}(X)$ be the

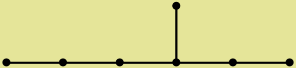
Clas … $= 0$.

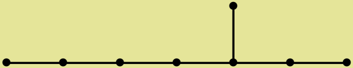
Smith ∼1969. The graphs solutions to (CP) are precisely ADE graphs for $e+2$ being (at most) the Coxeter number.

Type A_m:  ✓ for $e = m-1$

Type D_m:  ✓ for $e = 2m-4$

Type E_6:  ✓ for $e = 10$

Type E_7:  ✓ for $e = 16$

Type E_8:  ✓ for $e = 28$

$A_3 =$ 1 … $\cos(\frac{3\pi}{4})\}$

$D_4 =$ 1 … $\cos(\frac{5\pi}{6})\}$

1. **Dihedral representation theory**
   - Classical representation theory
   - $\mathbb{N}$-representation theory
   - Dihedral $\mathbb{N}$-representation theory

2. **Non-semisimple fusion rings**
   - The asymptotic limit
   - Cell modules
   - The dihedral example

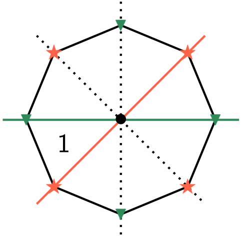
3. **Beyond**

The dihedral groups are of Coxeter type $I_2(e+2)$:

$$W_{e+2} = \langle s, t \mid s^2 = t^2 = 1,\ \overline{s}_{e+2} = \underbrace{\ldots sts}_{e+2} = w_0 = \underbrace{\ldots tst}_{e+2} = \overline{t}_{e+2} \rangle,$$

$$e.g.\ :\ W_4 = \langle s, t \mid s^2 = t^2 = 1,\ tsts = w_0 = stst \rangle$$

Example. These are the symmetry groups of regular $e+2$-gons, *e.g.* for $e=2$ the Coxeter complex is:



I will sneak in the Hecke case (*a.k.a.* quantum case) later on.

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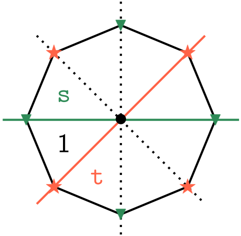

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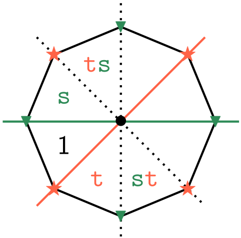

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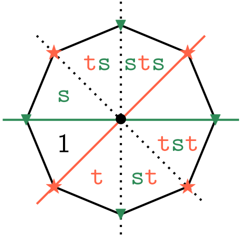

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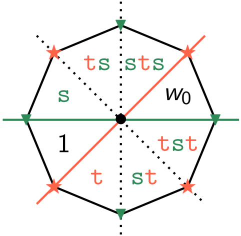

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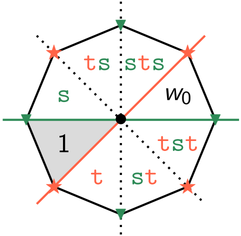

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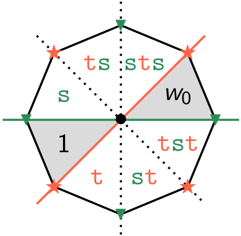

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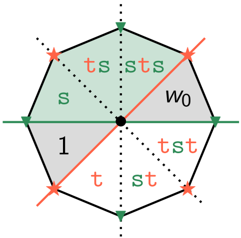

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s-cell.

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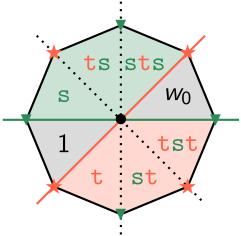

Lowest cell.

Biggest cell.

s-cell.

t-cell.

Dihedral representation theory on one slide.

The Bott–Samelson (BS) generators $\theta_{\mathtt{s}} = \mathtt{s} + 1, \theta_{\mathtt{t}} = \mathtt{t} + 1$.
There is also a Kazhdan–Lusztig (KL) bases. Explicit formulas do not matter today.

One-dimensional modules. $\mathrm{M}_{\lambda_{\mathtt{s}},\lambda_{\mathtt{t}}}, \lambda_{\mathtt{s}}, \lambda_{\mathtt{t}} \in \mathbb{C}, \theta_{\mathtt{s}} \mapsto \lambda_{\mathtt{s}}, \theta_{\mathtt{t}} \mapsto \lambda_{\mathtt{t}}$.

$e \equiv 0 \bmod 2$	$e \not\equiv 0 \bmod 2$
$\mathrm{M}_{0,0}, \mathrm{M}_{2,0}, \mathrm{M}_{0,2}, \mathrm{M}_{2,2}$	$\mathrm{M}_{0,0}, \mathrm{M}_{2,2}$

Two-dimensional modules. $\mathrm{M}_{z}, z \in \mathbb{C}, \theta_{\mathtt{s}} \mapsto \left(\begin{smallmatrix} 2 & z \\ 0 & 0 \end{smallmatrix}\right), \theta_{\mathtt{t}} \mapsto \left(\begin{smallmatrix} 0 & 0 \\ \bar{z} & 2 \end{smallmatrix}\right)$.

$e \equiv 0 \bmod 2$	$e \not\equiv 0 \bmod 2$
$\mathrm{M}_{z}, z \in \mathrm{V}_{e}^{\pm} - \{0\}$	$\mathrm{M}_{z}, z \in \mathrm{V}_{e}^{\pm}$

$\mathrm{V}_{e} = \mathrm{roots}(\mathsf{U}_{e+1}(\mathtt{X}))$ and $\mathrm{V}_{e}^{\pm}$ the $\mathbb{Z}/2\mathbb{Z}$-orbits under $z \mapsto -z$.

Dihedral representation theory on one slide.

One-dimension

Proposition (Lusztig?).

The list of one- and two-dimensional W_{e+2}-modules is a complete, irredundant list of simple modules.

$\mathrm{M}_{0,0}, \mathrm{M}_{2,0}, \mathrm{M}_{0,2}, \mathrm{M}_{2,2}$	$\mathrm{M}_{0,0}, \mathrm{M}_{2,2}$

Two-dimensiona

I learned this construction from Mackaay in 2017.

$e \equiv 0 \bmod 2$	$e \not\equiv 0 \bmod 2$
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T

Example.

$\mathrm{M}_{0,0}$ is the sign representation and $\mathrm{M}_{2,2}$ is the trivial representation.

In case e is odd, $\mathrm{U}_{e+1}(\mathtt{X})$ has a constant term, so $\mathrm{M}_{2,0}, \mathrm{M}_{0,2}$ are not representations.

$\mathrm{M}_z, z \in \mathrm{V}_e^{\pm} - \{0\}$	$\mathrm{M}_z, z \in \mathrm{V}_e^{\pm}$

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Example.

M_z for z being a root of the Chebyshev polynomial is a representation because the braid relation in terms of BS generators involves the coefficients of the Chebyshev polynomial.

Two-dim

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Dihed

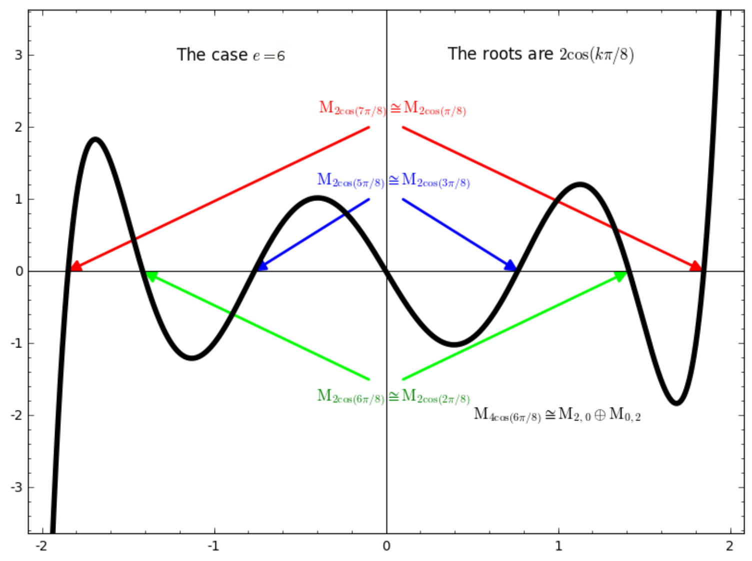
One-d

Two-d

$V_e =$

Example.

These representations are indexed by $\mathbb{Z}/2\mathbb{Z}$-orbits of the Chebyshev roots:



An algebra P with a fixed basis $\mathrm{B^P}$ is called a (multi) $\mathbb{N}$-algebra if

$$\mathrm{xy} \in \mathbb{N}\mathrm{B^P} \quad (\mathrm{x}, \mathrm{y} \in \mathrm{B^P}).$$

A P-module M with a fixed basis $\mathrm{B^M}$ is called a $\mathbb{N}$-module if

$$\mathrm{xm} \in \mathbb{N}\mathrm{B^M} \quad (\mathrm{x} \in \mathrm{B^P}, \mathrm{m} \in \mathrm{B^M}).$$

These are $\mathbb{N}$-equivalent if there is a $\mathbb{N}$-valued change of basis matrix.

Example. $\mathbb{N}$-algebras and $\mathbb{N}$-modules arise naturally as the decategorification of 2-categories and 2-modules, and $\mathbb{N}$-equivalence comes from 2-equivalence.

Example.

Group algebras of finite groups with basis given by group elements are $\mathbb{N}$-algebras.

The regular module is a $\mathbb{N}$-module.

An

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Fusion rings are with basis given by classes of simples elements are $\mathbb{N}$-algebras.

Key example: $K_0(\mathcal{R}\mathrm{ep}(G))$ (easy $\mathbb{N}$-representation theory).

Key example: $K_0(\mathcal{R}\mathrm{ep}_q^{ss}(\mathrm{U}_q(\mathfrak{g})) = \mathrm{G}_q)$ (intricate $\mathbb{N}$-representation theory).

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Example. $\mathbb{N}$-alg ... egorification of 2-categories and ... ivalence.

Example.

Hecke algebras of (finite) Coxeter groups with their KL basis are $\mathbb{N}$-algebras.

Their $\mathbb{N}$-representation theory is mostly widely open.

Clifford, Munn, Ponizovskiĭ ∼1942++, Kazhdan–Lusztig ∼1979. $\mathrm{x} \leq_{\mathsf{L}} \mathrm{y}$ if x appears in zy with non-zero coefficient for $\mathrm{z} \in \mathrm{B}^{\mathrm{P}}$. $\mathrm{x} \sim_{\mathsf{L}} \mathrm{y}$ if $\mathrm{x} \leq_{\mathsf{L}} \mathrm{y}$ and $\mathrm{y} \leq_{\mathsf{L}} \mathrm{x}$. $\sim_{\mathsf{L}}$ partitions P into left cells L. Similarly for right R, two-sided cells LR or $\mathbb{N}$-modules.

A $\mathbb{N}$-module M is transitive if all basis elements belong to the same $\sim_{\mathsf{L}}$ equivalence class. An apex of M is a maximal two-sided cell not killing it.

Fact. Each transitive $\mathbb{N}$-module has a unique apex.

Hence, one can study them cell-wise.

Example. Transitive $\mathbb{N}$-modules arise naturally as the decategorification of simple 2-modules.

Clifford, Munn ... **79.** $x \leq_L y$ if x appears in zy w ... y and $y \leq_L x$. $\sim_L$ partitions P ... lls LR or $\mathbb{N}$-modules.

A $\mathbb{N}$-module M ... e $\sim_L$ equivalence cla ... lling it.

Hence, one can ...

Example. Tran ... cation of simple 2-modules.

Philosophy.

Imagine a graph whose vertices are the x's or the m's.
$v_1 \to v_2$ if v_1 appears in zv_2.

cells = connected components
transitive = one connected component

"The atoms of $\mathbb{N}$-representation theory".

Question ($\mathbb{N}$-representation theory). Classify them!

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Group algebras with the group element basis have only one cell, G itself.

Transitive $\mathbb{N}$-modules are $\mathbb{C}[G/H]$ for $H \subset G$ subgroup/conjugacy. The apex is G.

A $\mathbb{N}$-module M is transitive if all basis elements belong to the same $\sim_{\mathsf{L}}$ equivalence class. An apex of M is a maximal two-sided cell not killing it.

Fact. Each transitive $\mathbb{N}$-module has a unique apex.

Hence, one can study them cell-wise.

Example. Transitive $\mathbb{N}$-modules arise naturally as the decategorification of simple 2-modules.

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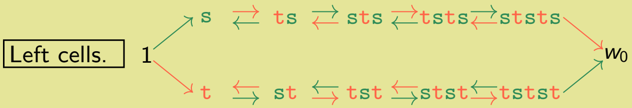

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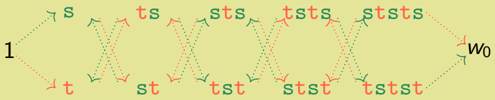

Right cells.

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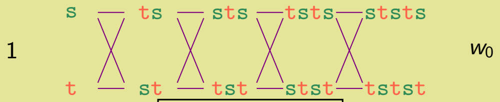

Two-sided cells.

We will see the transitive $\mathbb{N}$-modules in a second.

Clifford, Munn, Ponizovskiĭ ∼1942++, Kazhdan–Lusztig ∼1979. $x \leq_L y$ if x appears in zy with non-zero coefficient for $z \in B^P$. $x \sim_L y$ if $x \leq_L y$ and $y \leq_L x$. $\sim_L$ partitions P into left cells L. Similarly for right R, two-sided cells LR or $\mathbb{N}$-modules.

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Fact. Each transitive $\mathbb{N}$-module has a unique apex.

Morally.

The further away an $\mathbb{N}$-algebra is from being semisimple, the more useful and interesting is its cell structure.

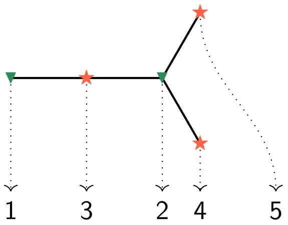
Hence, one ca

Example. Tra ation of simple 2-modules.

$\mathbb{N}$-modules via graphs.

Construct a W_∞-module M associated to a bipartite graph $\boldsymbol{\Gamma}$:

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3, 4, 5 \rangle$$



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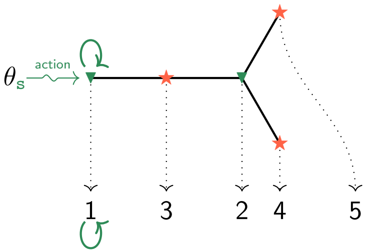

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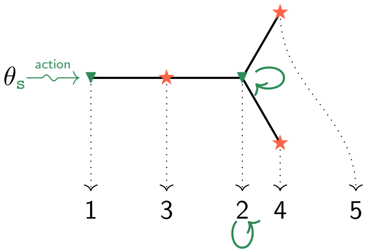

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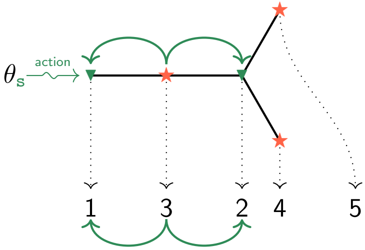

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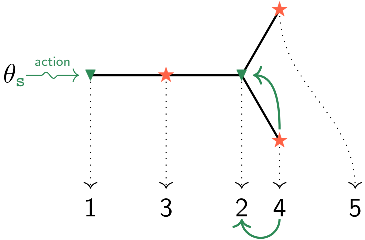

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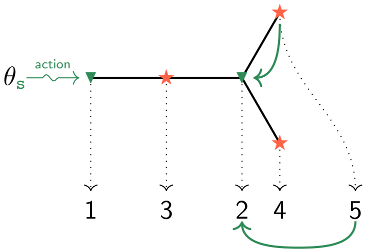

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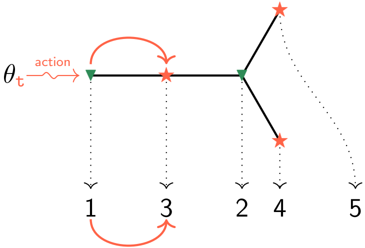

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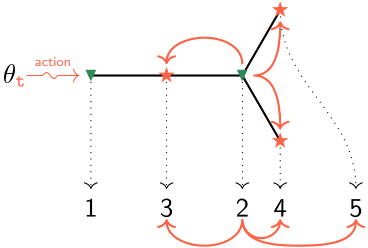

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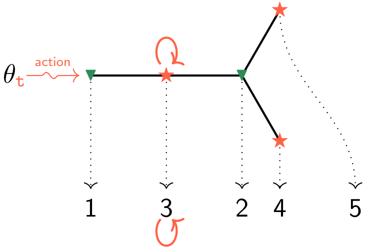

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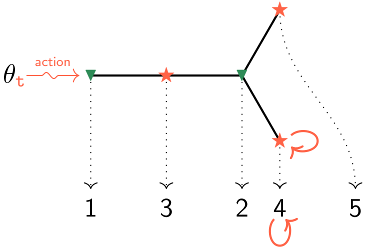

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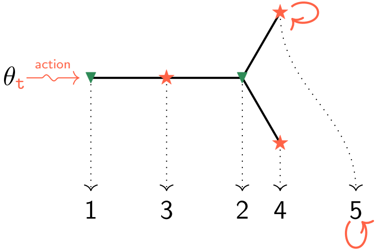

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Construct a W_∞-module M associated to a bipartite graph $\mathbf{\Gamma}$:

The adjacency matrix $A(\mathbf{\Gamma})$ of $\mathbf{\Gamma}$ is

$$A(\mathbf{\Gamma}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

These are W_{e+2}-modules for some e
only if $A(\mathbf{\Gamma})$ is killed by the Chebyshev polynomial $\mathrm{U}_{e+1}(\mathtt{X})$.

Morally speaking: These are constructed as the simples
but with integral matrices having the Chebyshev-roots as eigenvalues.

It is not hard to see that the Chebyshev–braid-like relation can not hold otherwise.

$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} & & \dots & & \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} & & \dots & & \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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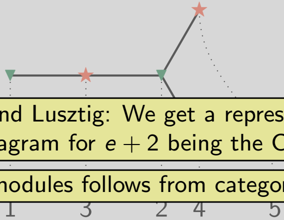

Hence, by Smith's (CP) and Lusztig: We get a representation of W_{e+2} if Γ is a ADE Dynkin diagram for $e+2$ being the Coxeter number.

That these are $\mathbb{N}$-modules follows from categorification.

'Smaller solutions' are never $\mathbb{N}$-modules.

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Construct a W_∞-module M associated to a bipartite graph $\mathbf{\Gamma}$:

Classification.

Complete, irredundant list of transitive $\mathbb{N}$-modules of W_{e+2}:

apex	1 cell	s – t cell	w_0 cell
$\mathbb{N}$-reps.	$\mathrm{M}_{0,0}$	$\mathrm{M}_{\mathrm{ADE+bicolering}}$ for $e+2=$ Cox. num.	$\mathrm{M}_{2,2}$

Further upshot: A cute classification of ADE graphs.
$\mathbf{\Gamma}$ is finite type ADE iff $\mathrm{U}_{e+1}(A(\mathbf{\Gamma}))$ gets periodic.
$\mathbf{\Gamma}$ is affine type ADE iff $\mathrm{U}_{e+1}(A(\mathbf{\Gamma}))$ grows linearly.
$\mathbf{\Gamma}$ is anything else iff $\mathrm{U}_{e+1}(A(\mathbf{\Gamma}))$ grows exponentially.

0	2	1	1	1
0	0	0	0	0

0	0	0	0	0
0	1	0	0	2

I learned this from/with Kildetoft–Mackaay–Mazorchuk–Zimmermann ∼2016.

Example ($e = 2$).

The Weyl group of type B_2. Number of elements: 8. Number of cells: 3, named 0 (trivial) to 2 (top).

Cell order:

$$0 — 1 — 2$$

Size of the cells:

cell	0	1	2
size	1	6	1

Cell structure:

1	
s, sts	ts
st	t, tst
w_0	

$\xrightarrow{\text{number of elements}}$

1	
2	1
1	2
1	

Example ($e = 2$).

The W … med 0
(trivia …

Example (SAGE).

$$1 \cdot 1 = \mathtt{v}^0 1.$$

($\mathtt{v}$ is the Hecke parameter deforming the reflection equations $\mathtt{s}^2 = \mathtt{t}^2 = 1$.)

Cell o…

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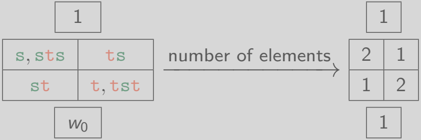

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$$\theta_{sts} \cdot \theta_{tst} = (\text{lower powers})\theta_{st} + \text{higher cell elements}.$$

s, sts	ts
st	t, tst

w_0

number of elements →

2	1
1	2

1

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s, sts	ts	number of elements	2	1
				2

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$$\theta_{\mathrm{sts}} \cdot \theta_{\mathrm{tst}} = (\text{lower powers})\theta_{\mathrm{st}} + \text{higher cell elements}.$$

Example (SAGE).

$$\theta_{w_0} \cdot \theta_{w_0} = (\mathrm{v}^4 + \text{lower powers})\theta_{w_0}.$$

Example ($e=2$).

Fact (Lusztig ∼1984++).

For any Coxeter group W there is a well-defined function

$$a\colon \mathrm{W} \to \mathbb{N}$$

which is constant on two-sided cells.

▸ Big example

The Weyl group of type … ber of cells: 3, named 0 (trivial) to 2 (top).

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Cell structure:

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Ignore everything except the leading coefficient a(two-sided cell).

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Ignore everything except the leading coefficient
a(two-sided cell).

Why isn't that stupid?
Because a is also turns up as the leading coefficients
of traces of standard generators acting on simple modules.

Upshot. One can associate an ▸ apex to simples,
and the simples should be uniquely determent by the leading coefficients.

Let $\mathrm{H_v(W)}$ be the Hecke algebra associated to W. The asymptotic limit $\mathrm{J_\infty(W)}$ of $\mathrm{H_v(W)}$ is defined as follows.

As a free $\mathbb{Z}$-module:

$$\mathrm{J_\infty(W)} = \bigoplus_{\mathsf{LR}} \mathbb{Z}\{t_w \mid w \in \mathsf{LR}\}. \quad \text{Compare: } \mathrm{H_v(W)} = \mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]\{\theta_w \mid \mathrm{W}\}.$$

Multiplication.

$$t_x t_y = \textstyle\sum_{z \in \mathsf{LR}} \gamma_{x,y}^z t_z. \quad \text{Compare: } \theta_x \theta_y = \textstyle\sum_{z \in \mathsf{LR}} h_{x,y}^z \theta_z + \text{bigger friends.}$$

where $\gamma_{x,y}^z \in \mathbb{N}$ is the leading coefficient of $h_{x,y}^z \in \mathbb{N}[\mathrm{v}, \mathrm{v}^{-1}]$.

Example ($e = 2$).

The multiplication tables (empty entries are 0 and $[2] = \mathrm{v} + \mathrm{v}^{-1}$) in 1:

	t_s	t_{sts}	t_{st}	t_t	t_{tst}	t_{ts}
t_s	t_s	t_{sts}	t_{st}			
t_{sts}	t_{sts}	t_s	t_{st}			
t_{ts}	t_{ts}	t_{ts}	$t_t + t_{tst}$			
t_t				t_t	t_{tst}	t_{ts}
t_{tst}				t_{tst}	t_t	t_{ts}
t_{st}				t_{st}	t_{st}	$t_s + t_{sts}$

	θ_s	θ_{sts}	θ_{st}	θ_t	θ_{tst}	θ_{ts}
θ_s	$[2]\theta_s$	$[2]\theta_{sts}$	$[2]\theta_{st}$	θ_{st}	$\theta_{st} + \theta_{w_0}$	$\theta_s + \theta_{sts}$
θ_{sts}	$[2]\theta_{sts}$	$[2]\theta_s + [2]^2\theta_{w_0}$	$[2]\theta_{st} + [2]\theta_{w_0}$	$\theta_s + \theta_{sts}$	$\theta_s + [2]^2\theta_{w_0}$	$\theta_s + \theta_{sts} + [2]\theta_{w_0}$
θ_{ts}	$[2]\theta_{ts}$	$[2]\theta_{ts} + [2]\theta_{w_0}$	$[2]\theta_t + [2]\theta_{tst}$	$\theta_t + \theta_{tst}$	$\theta_t + \theta_{tst} + [2]\theta_{w_0}$	$2\theta_{ts} + \theta_{w_0}$
θ_t	θ_{ts}	$\theta_{ts} + \theta_{w_0}$	$\theta_t + \theta_{tst}$	$[2]\theta_t$	$[2]\theta_{tst}$	$[2]\theta_{ts}$
θ_{tst}	$\theta_t + \theta_{tst}$	$\theta_t + [2]^2\theta_{w_0}$	$\theta_t + \theta_{tst} + [2]\theta_{w_0}$	$[2]\theta_{tst}$	$[2]\theta_t + [2]^2\theta_{w_0}$	$[2]\theta_{ts} + [2]\theta_{w_0}$
θ_{st}	$\theta_s + \theta_{sts}$	$\theta_s + \theta_{sts} + [2]\theta_{w_0}$	$2\theta_{st} + \theta_{w_0}$	$[2]\theta_{st}$	$[2]\theta_{st} + [2]\theta_{w_0}$	$[2]\theta_s + [2]\theta_{sts}$

(Note the "subalgebras".)

The asymptotic algebra is much simpler!

▸ Big example

Fact (Lusztig ∼1984++).

$J_\infty(W) = \bigoplus_{LR} J_\infty^{LR}(W)$ with the t_w basis
and all its summands $J_\infty^{LR}(W) = \mathbb{Z}\{t_w \mid w \in LR\}$
are multifusion algebras.

(Meaning semisimple $\mathbb{N}$-algebras with a certain nice trace form.)

Let $H_v(W)$ ... it $J_\infty(W)$
of $H_v(W)$

As a free $\mathbb{Z}$

$J_\infty(W) = \bigoplus_{LR} \mathbb{Z}\{t_w \mid w \in LR\}$. Compare: $H_v(W) = \mathbb{Z}[v, v^{-1}]\{\theta_w \mid W\}$.

Multiplication.

$t_x t_y = \sum_{z\in LR} \gamma_{x,y}^z t_z$. Compare: $\theta_x\theta_y = \sum_{z\in LR} h_{x,y}^z \theta_z$ + bigger friends.

where $\gamma_{x,y}^z \in \mathbb{N}$ is the leading coefficient of $h_{x,y}^z \in \mathbb{N}[v, v^{-1}]$.

Let $\mathrm{H_v(W)}$ ... it $\mathrm{J}_\infty(\mathrm{W})$
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As a free $\mathbb{Z}$ ...

(Meaning semisimple $\mathbb{N}$-algebras with a certain nice trace form.)

$\mathrm{J}_\infty(\mathrm{W}) = \bigoplus$ ... $^{-1}]\{\theta_w \mid \mathrm{W}\}$.

Surprising fact 1 (Lusztig ∼1984++).
It seems one throws almost away everything, but:

There is an explicit embedding

$$\mathrm{H_v(W)} \hookrightarrow \mathrm{J}_\infty(\mathrm{W}) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]$$

which is an isomorphism after scalar extension to $\mathbb{Q}(\mathrm{v})$.

Multiplication.

$t_x t_y = \sum_z$ ... ger friends.

where $\gamma_{x,y}^z \in \mathbb{N}$ is the leading coefficient of $h_{x,y}^z \in \mathbb{N}[\mathrm{v}, \mathrm{v}^{-1}]$.

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Multiplication.

$t_x t_y = \sum_z$ … ger friends.

where $\gamma_{x,y}^{z} \in$ …

Surprising fact 2 (Lusztig ∼1984++).

There is an explicit 1:1 correspondence

$$\{\text{simples of } \mathrm{H}_{\mathrm{v}}(\mathrm{W}) \text{ with apex LR}\} \xleftrightarrow{1:1} \{\text{simples of } \mathrm{J}_{\infty}^{\mathrm{LR}}(\mathrm{W})\}.$$

▸ Time flies

"Induced" transitive $\mathbb{N}$-algebras and -modules.

Fix a left cell L. Let $\mathrm{M}(\geq_{\mathsf{L}})$, respectively $\mathrm{M}(>_{\mathsf{L}})$, be the $\mathbb{N}$-modules spanned by all $\mathrm{x} \in \mathrm{B}^{\mathrm{P}}$ in the union $\mathsf{L}' \geq_{\mathsf{L}} \mathsf{L}$, respectively $\mathsf{L}' >_{\mathsf{L}} \mathsf{L}$. Similarly for right R, two-sided LR and diagonal $\mathsf{H} = \mathsf{L} \cap \mathsf{R}$ cells.

Left cell module $\mathrm{C}_{\mathsf{L}} = \mathrm{M}(\geq_{\mathsf{L}})/\mathrm{M}(>_{\mathsf{L}})$. (Left $\mathbb{N}$-module.)

Right cell module $\mathrm{C}_{\mathsf{R}} = \mathrm{M}(\geq_{\mathsf{R}})/\mathrm{M}(>_{\mathsf{R}})$. (Right $\mathbb{N}$-module.)

Two-sided cell module $\mathrm{C}_{\mathsf{LR}} = \mathrm{M}(\geq_{\mathsf{LR}})/\mathrm{M}(>_{\mathsf{LR}})$. ($\mathbb{N}$-bimodule.)

The diagonal cell $\mathrm{C}_{\mathsf{H}} = \mathrm{J}^{\mathsf{H}}_{\infty}(\mathrm{W}) = (\mathrm{M}(\geq_{\mathsf{LR}})/\mathrm{M}(>_{\mathsf{LR}})) \cap \mathbb{K}\mathrm{B}^{\mathrm{P}}(\mathsf{H})$. ($\mathbb{N}$-subalgebra.)

Big example

“Induced” transitive ℕ-algebras and -modules.

Fix a left cell L. Let $\mathrm{M}(\geq_{\mathsf{L}})$, respectively $\mathrm{M}(>_{\mathsf{L}})$, be the ℕ-modules spanned by all $\mathrm{x} \in \mathrm{B}^{\mathrm{P}}$ in the union $\mathsf{L}' \geq_{\mathsf{L}} \mathsf{L}$, respectively $\mathsf{L}' >_{\mathsf{L}} \mathsf{L}$. Similarly for right R, two-sided LR and diagonal $\mathsf{H} = \mathsf{L} \cap \mathsf{R}$ cells.

Left cell

Right cel

Two-sided cell module $\mathrm{C}_{\mathsf{LR}} = \mathrm{M}(\geq_{\mathsf{LR}})/\mathrm{M}(>_{\mathsf{LR}})$. (ℕ-bimodule.)

Analogy: “Hierarchy of structures”.

$$\mathrm{C}_{\mathsf{H}} \subset \mathrm{C}_{\mathsf{L}}, \mathrm{C}_{\mathsf{R}} \subset \mathrm{C}_{\mathsf{LR}} \quad \leftrightsquigarrow \quad \text{subalgebras} \subset \text{left induced}, \text{right induced} \subset \text{biinduced}$$

The diagonal cell $\mathrm{C}_{\mathsf{H}} = \mathrm{J}^{\mathsf{H}}_{\infty}(\mathrm{W}) = (\mathrm{M}(\geq_{\mathsf{LR}})/\mathrm{M}(>_{\mathsf{LR}})) \cap \mathbb{K}\mathrm{B}^{\mathrm{P}}(\mathsf{H})$. (ℕ-subalgebra.)

Big example

Example.

$\mathbb{C}[G]$ with the group element basis has only one cell module, the regular module.

Similarly for any fusion algebra.

"I

Fix ... y

all ...

two-sided LR and diagonal $H = L \cap R$ cells.

Left cell module $C_L = M(\geq_L)/M(>_L)$. (Left $\mathbb{N}$-module.)

Right cell module $C_R = M(\geq_R)/M(>_R)$. (Right $\mathbb{N}$-module.)

Two-sided cell module $C_{LR} = M(\geq_{LR})/M(>_{LR})$. ($\mathbb{N}$-bimodule.)

The diagonal cell $C_H = J_\infty^H(W) = (M(\geq_{LR})/M(>_{LR})) \cap \mathbb{K}B^P(H)$. ($\mathbb{N}$-subalgebra.)

▸ Big example

"I

Fix ... y

all

two-sided LR and diagonal $\mathrm{H} = \mathrm{L} \cap \mathrm{R}$ cells.

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$\mathbb{C}[G]$ with the group element basis has only one cell module, the regular module.

Similarly for any fusion algebra.

Left c

Right

Example (Kazhdan–Lusztig ∼1979, Lusztig ∼1983++).

For Hecke algebras of the symmetric group with KL basis
the cell modules are Lusztig's
cell modules studied in connection with reductive groups in characteristic p.

Two-sided cell module $\mathrm{C}_{\mathsf{LR}} = \mathrm{M}(\geq_{\mathsf{LR}})/\mathrm{M}(>_{\mathsf{LR}})$. ($\mathbb{N}$-bimodule.)

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Right

Two-sided cell module $\mathrm{C}_{\mathrm{LR}} = \mathrm{M}(\geq_{\mathrm{LR}})/\mathrm{M}(>_{\mathrm{LR}})$. ($\mathbb{N}$-bimodule.)

Example (dihedral case).

Cells:

cell	0	1	2
size	1	$2n-2$	1
a	0	1	n

1 for n even:

$\frac{n}{2}$	$\frac{n-2}{2}$
$\frac{n-2}{2}$	$\frac{n}{2}$

1 for n odd:

$\frac{n-1}{2}$	$\frac{n-1}{2}$
$\frac{n-1}{2}$	$\frac{n-1}{2}$

n **even.** Two left cell modules ⇝ Two bicolorings of the type A graph.
n **odd.** One left cell module ⇝ One bicoloring of the type A graph.

▸ Big example

Example ($e = 2$).

The fusion ring $K_0(\mathrm{SL}(2)_q)$ for $q^{2e} = 1$ has simple objects $[L_0], [L_1], [L_2]$. The fusion ring $\mathrm{J}^{\mathsf{LR}}_{\infty}(\mathrm{W})$ has simple objects $t_{\mathtt{s}}, t_{\mathtt{sts}}, t_{\mathtt{st}}, t_{\mathtt{t}}, t_{\mathtt{tst}}, t_{\mathtt{ts}}$.

Comparison of multiplication tables:

	$[L_0]$	$[L_2]$	$[L_1]$
$[L_0]$	$[L_0]$	$[L_2]$	$[L_1]$
$[L_2]$	$[L_2]$	$[L_0]$	$[L_1]$
$[L_1]$	$[L_1]$	$[L_1]$	$[L_0] + [L_2]$

&

	$t_{\mathtt{s}}$	$t_{\mathtt{sts}}$	$t_{\mathtt{st}}$	$t_{\mathtt{t}}$	$t_{\mathtt{tst}}$	$t_{\mathtt{ts}}$
$t_{\mathtt{s}}$	$t_{\mathtt{s}}$	$t_{\mathtt{sts}}$	$t_{\mathtt{st}}$			
$t_{\mathtt{sts}}$	$t_{\mathtt{sts}}$	$t_{\mathtt{s}}$	$t_{\mathtt{st}}$			
$t_{\mathtt{ts}}$	$t_{\mathtt{ts}}$	$t_{\mathtt{ts}}$	$t_{\mathtt{t}} + t_{\mathtt{tst}}$			
$t_{\mathtt{t}}$				$t_{\mathtt{t}}$	$t_{\mathtt{tst}}$	$t_{\mathtt{ts}}$
$t_{\mathtt{tst}}$				$t_{\mathtt{tst}}$	$t_{\mathtt{t}}$	$t_{\mathtt{ts}}$
$t_{\mathtt{st}}$				$t_{\mathtt{st}}$	$t_{\mathtt{st}}$	$t_{\mathtt{s}} + t_{\mathtt{sts}}$

$\mathrm{J}^{\mathsf{LR}}_{\infty}(\mathrm{W})$ is a bicolored version of $K_0(\mathrm{SL}(2)_q)$:

$$t_{\mathtt{s}} \& t_{\mathtt{t}} \leftrightsquigarrow [L_0], \quad t_{\mathtt{sts}} \& t_{\mathtt{tst}} \leftrightsquigarrow [L_2], \quad t_{\mathtt{st}} \& t_{\mathtt{ts}} \leftrightsquigarrow [L_1].$$

Example ($e = 2$).

The fusion ring $K_0(\mathrm{SO}(3)_q)$ for $q^{2e} = 1$ has simple objects $[L_0], [L_2]$. The fusion ring $\mathrm{J}^{\mathsf{H}}_{\infty}(\mathrm{W})$ ($\mathsf{H} = \mathsf{L}_{\mathtt{s}} \cap \mathsf{R}_{\mathtt{s}}$) has simple objects $t_{\mathtt{s}}, t_{\mathtt{sts}}$.

Comparison of multiplication tables:

	$[L_0]$	$[L_2]$
$[L_0]$	$[L_0]$	$[L_2]$
$[L_2]$	$[L_2]$	$[L_0]$

&

	$t_{\mathtt{s}}$	$t_{\mathtt{sts}}$
$t_{\mathtt{s}}$	$t_{\mathtt{s}}$	$t_{\mathtt{sts}}$
$t_{\mathtt{sts}}$	$t_{\mathtt{sts}}$	$t_{\mathtt{s}}$

$\mathrm{J}^{\mathsf{H}}_{\infty}(\mathrm{W})$ is $K_0(\mathrm{SO}(3)_q)$:

$$t_{\mathtt{s}} \leftrightsquigarrow [L_0], \quad t_{\mathtt{sts}} \leftrightsquigarrow [L_2].$$

This is the slightly nicer statement.

Example ($e = 2$)

Fact.

Both connections are always true (*i.e.* for any e).

The fusion ring K … $[L_2]$. The fusion ring $\mathrm{J}^{\mathrm{H}}_{\infty}(\mathrm{W})$ ($\mathrm{H} = \mathrm{L}_{\mathrm{s}} \sqcup \mathrm{R}_{\mathrm{s}}$) has simple objects $t_{\mathrm{s}}, t_{\mathrm{sts}}$.

Comparison of multiplication tables:

	$[L_0]$	$[L_2]$
$[L_0]$	$[L_0]$	$[L_2]$
$[L_2]$	$[L_2]$	$[L_0]$

&

	t_{s}	t_{sts}
t_{s}	t_{s}	t_{sts}
t_{sts}	t_{sts}	t_{s}

$\mathrm{J}^{\mathrm{H}}_{\infty}(\mathrm{W})$ is $K_0(\mathrm{SO}(3)_q)$:

$$t_{\mathrm{s}} \leftrightsquigarrow [L_0], \quad t_{\mathrm{sts}} \leftrightsquigarrow [L_2].$$

Example $(e = 2)$

The fusion ring K ... $[L_2]$. The fusion ring $\mathrm{J}^{\mathsf{H}}_{\infty}(\mathrm{W})$ ($\mathsf{H} = \mathsf{L_s} \cap \mathsf{R_s}$) has simple objects $t_{\mathtt{s}}, t_{\mathtt{sts}}$.

Fact.

Both connections are always true (*i.e.* for any e).

H-**cell-theorem.**

There are 1:1 correspondences

$$\{\text{transitives of } \mathrm{H_v}(\mathrm{W}) \text{ with apex LR}\} \xleftrightarrow{1:1} \{\text{transitives of } \mathrm{J}^{\mathsf{LR}}_{\mathrm{v}}(\mathrm{W})\} \xleftrightarrow{1:1} \{\text{transitives of } \mathrm{J}^{\mathsf{H}}_{\mathrm{v}}(\mathrm{W})\},$$

$$\{\text{transitives of } \mathrm{H_v}(\mathrm{W}) \text{ with apex LR}\} \xleftrightarrow{1:1} \{\text{transitives of } K_0(\mathrm{SL}(2)_q^{\mathtt{s},\mathtt{t}})\} \xleftrightarrow{1:1} \{\text{transitives of } K_0(\mathrm{SO}(3)_q)\}.$$

Example

$$t_{\mathtt{s}} \leftrightsquigarrow [L_0], \quad t_{\mathtt{sts}} \leftrightsquigarrow [L_2].$$

Example ($e = 2$)

The fusion ring K ... $[L_2]$. The fusion ring $J_\infty^H(W)$ ($H = L_s \cap R_s$) has simple objects t_s, t_{sts}.

Fact.

Both connections are always true (*i.e.* for any e).

H-**cell-theorem.**

There are 1:1 correspondences

$$\{\text{transitives of } H_v(W) \text{ with apex LR}\} \xleftrightarrow{1:1} \{\text{transitives of } J_v^{LR}(W)\} \xleftrightarrow{1:1} \{\text{transitives of } J_v^{H}(W)\},$$

$$\{\text{transitives of } H_v(W) \text{ with apex LR}\} \xleftrightarrow{1:1} \{\text{transitives of } K_0(SL(2)_q^{s,t})\} \xleftrightarrow{1:1} \{\text{transitives of } K_0(SO(3)_q)\}.$$

▸ Example

C ...

J ...

$$t_s \leftrightsquigarrow [L_0], \quad t_{sts} \leftrightsquigarrow [L_2].$$

Upshot.

$H_v(W)$ is a non-semisimple version of $K_0(SL(2)_q)$,

$H_v^H(W)$ is a non-semisimple version of $K_0(SO(3)_q)$.

In particular, the Hecke algebras have a v parameter.

Example ($e = 2$).

The fusion ring $K_0(\mathrm{SO}(3)_q)$ for $q^{2e} = 1$ has simple objects $[L_0], [L_2]$. The fusion ring $\mathrm{J}^{\mathsf{H}}_{\infty}(\mathrm{W})$ ($\mathsf{H} = \mathsf{L}_{\mathtt{s}} \cap \mathsf{R}_{\mathtt{s}}$) has simple objects $t_{\mathtt{s}}, t_{\mathtt{sts}}$.

Comparison of

$\mathrm{J}^{\mathsf{H}}_{\infty}(\mathrm{W})$ is $K_0(\mathrm{S}$

Fact.

With a bit more care (with the H-cell-theorem) all the above generalizes to any Coxeter group W.

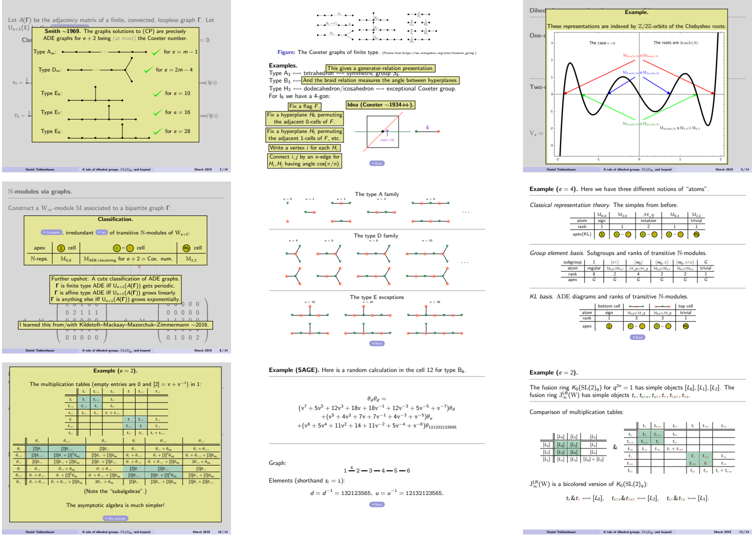
Thus, Hecke algebras are non-semisimple fusion rings.

In general $\mathrm{J}_{\infty}(\mathrm{W})$ is not understood, but for W being a finite Weyl group $\mathrm{J}^{\mathsf{H}}_{\infty}(\mathrm{W})$ is very ▸ nice.

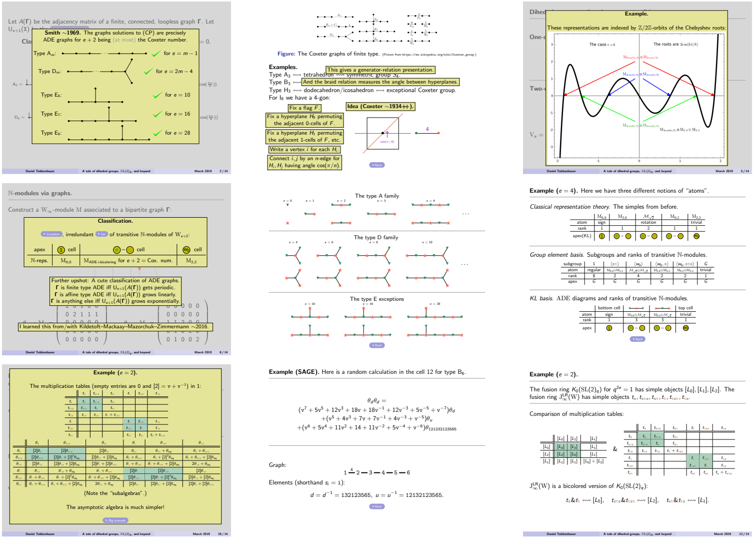
▸ Please stop!

Beyond?

- Categorification?
  - Non-semisimple: Replace Hecke algebra by Soergel bimodules. ✓
  - Non-semisimple: Categorical $\mathbb{N}$-modules for dihedral groups. ✓ Zigzag algebras appear.
  - Fusion: Replace asymptotic Hecke algebra by asymptotic Soergel bimodules. ✓
  - Fusion: Categorical $\mathbb{N}$-modules for $\mathrm{SL}(2)_q$. ✓ Algebras are trivial.
  - H: Asymptotic Soergel bimodules are very nice, just remove K_0 everywhere. ✓
  - H-cell-theorem ? . Work in progress! Click
- $\mathrm{SL}(n)_q$?
  - Non-semisimple: Nhedral; leaves the realm of groups. ✓
  - Non-semisimple: Categorical $\mathbb{N}$-modules for Nhedral algebras have a Ncolored ADE-type classification. ✓ Generalized zigzag algebras and Chebyshev polynomials appear.
  - Fusion: One gets $\mathrm{SL}(N)_q$. ✓
  - Fusion: Categorical $\mathbb{N}$-modules of $\mathrm{SL}(N)_q$ have an ADE-type classification. ✓ Algebras are trivial. Click



There is still much to do...



Thanks for your attention!

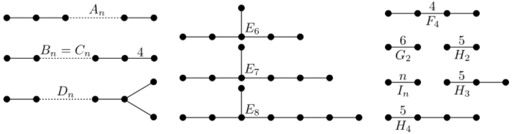
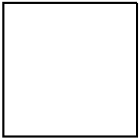


Figure: The Coxeter graphs of finite type. (Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)

Examples.
Type A_3 $\leftrightsquigarrow$ tetrahedron $\leftrightsquigarrow$ symmetric group S_4.
Type B_3 $\leftrightsquigarrow$ cube/octahedron $\leftrightsquigarrow$ Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \ltimes S_3$.
Type H_3 $\leftrightsquigarrow$ dodecahedron/icosahedron $\leftrightsquigarrow$ exceptional Coxeter group.
For I_8 we have a 4-gon:

Idea (Coxeter ∼1934++).



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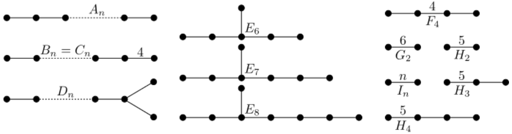


Figure: The Coxeter graphs of finite type. (Picture from https://en.wikipedia.org/wiki/Coxeter_group.)

Examples.

Type A_3 ⇝ tetrahedron ⇝ symmetric group S_4.

Fact. The symmetries are given by exchanging flags.

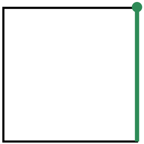
Type B_3 ⇝ cube/octahedron ⇝ Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \rtimes S_3$.

Type H_3 ⇝ dodecahedron/icosahedron ⇝ exceptional Coxeter group.

For I_8 we have a 4-gon:

Fix a flag F.

Idea (Coxeter ∼1934++).



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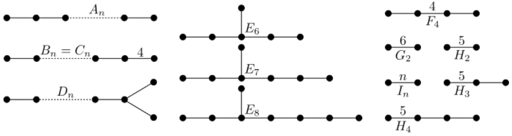

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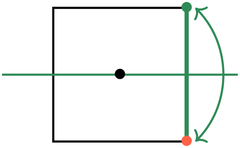
For I_8 we have a 4-gon:

Fix a flag F.

Fix a hyperplane H_0 permuting the adjacent 0-cells of F.

Idea (Coxeter ∼1934++).



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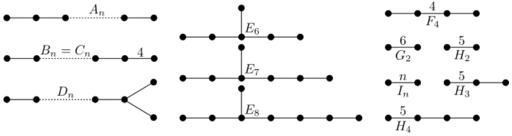

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Examples.

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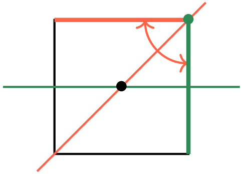
Type H_3 $\leftrightsquigarrow$ dodecahedron/icosahedron $\leftrightsquigarrow$ exceptional Coxeter group.
For I_8 we have a 4-gon:

Fix a flag F.

Fix a hyperplane H_0 permuting the adjacent 0-cells of F.

Fix a hyperplane H_1 permuting the adjacent 1-cells of F, etc.

Idea (Coxeter ∼1934++).



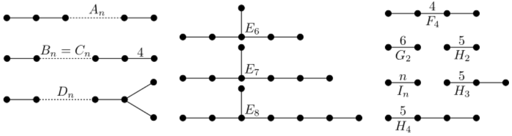


Figure: The Coxeter graphs of finite type. (Picture from https://en.wikipedia.org/wiki/Coxeter_group.)

Examples.

Type A_3 ⇝ tetrahedron ⇝ symmetric group S_4.

Type B_3 ⇝ cube/octahedron ⇝ Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \ltimes S_3$.

Type H_3 ⇝ dodecahedron/icosahedron ⇝ exceptional Coxeter group.

For I_8 we have a 4-gon:

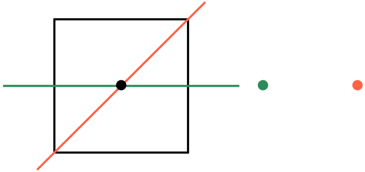
Fix a flag F.

Fix a hyperplane H_0 permuting the adjacent 0-cells of F.

Fix a hyperplane H_1 permuting the adjacent 1-cells of F, etc.

Write a vertex i for each H_i.

Idea (Coxeter ∼1934++).



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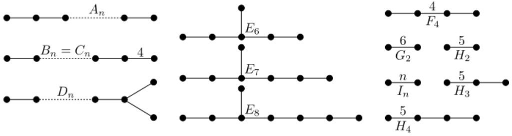


Figure: The Coxeter graphs of finite type. (Picture from https://en.wikipedia.org/wiki/Coxeter_group.)

Examples.

This gives a generator-relation presentation.

Type A_3 ⇝ tetrahedron ⇝ symmetric group S_4.

Type B_3 ⇝

And the braid relation measures the angle between hyperplanes.

Type H_3 ⇝ dodecahedron/icosahedron ⇝ exceptional Coxeter group.

For I_8 we have a 4-gon:

Fix a flag F.

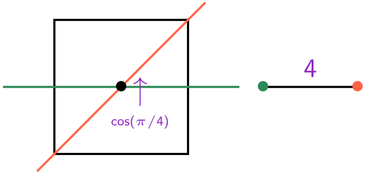
Fix a hyperplane H_0 permuting the adjacent 0-cells of F.

Fix a hyperplane H_1 permuting the adjacent 1-cells of F, etc.

Write a vertex i for each H_i.

Connect i, j by an n-edge for H_i, H_j having angle $\cos(\pi/n)$.

Idea (Coxeter ∼1934++).



Back

$$\mathrm{U}_0(\mathtt{X}) = 1, \quad \mathrm{U}_1(\mathtt{X}) = \mathtt{X}, \quad \mathtt{X}\,\mathrm{U}_{e+1}(\mathtt{X}) = \mathrm{U}_{e+2}(\mathtt{X}) + \mathrm{U}_e(\mathtt{X})$$

$$\mathrm{U}_0(\mathtt{X}) = 1, \quad \mathrm{U}_1(\mathtt{X}) = 2\mathtt{X}, \quad 2\mathtt{X}\,\mathrm{U}_{e+1}(\mathtt{X}) = \mathrm{U}_{e+2}(\mathtt{X}) + \mathrm{U}_e(\mathtt{X})$$

Kronecker ∼1857. Any complete set of conjugate algebraic integers in $]-2,2[$ is a subset of $\mathrm{roots}(\mathrm{U}_{e+1}(\mathtt{X}))$ for some e.

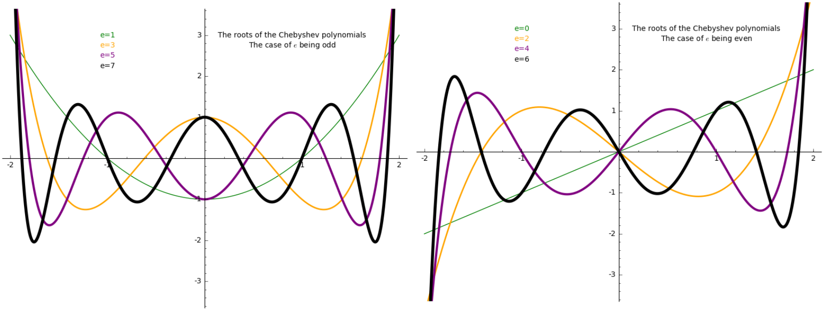


Figure: The roots of the Chebyshev polynomials (of the second kind).

Back

In case you are wondering why this is supposed to be true, here is the main observation of **Smith ∼1969**:

$$\mathsf{U}_{e+1}(\mathtt{X},\mathtt{Y}) = \pm\det(\mathtt{X}\mathrm{Id} - A(\mathsf{A}_{e+1}))$$

Chebyshev poly. = char. poly. of the type A_{e+1} graph

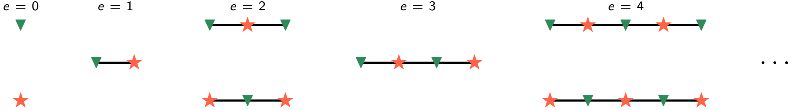
and

$$\mathtt{X}\mathsf{T}_{n-1}(\mathtt{X}) = \pm\det(\mathtt{X}\mathrm{Id} - A(\mathsf{D}_n)) \pm (-1)^{n \bmod 4}$$

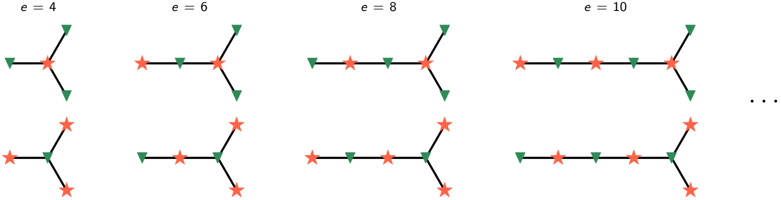
first kind Chebyshev poly. '=' char. poly. of the type D_n graph ($n = \frac{e+4}{2}$).

◂ Back

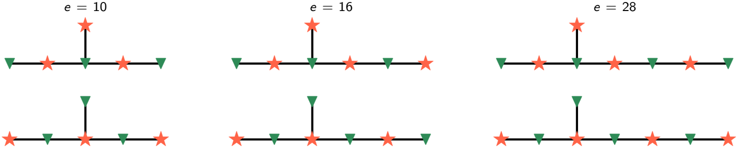
The type A family



The type D family

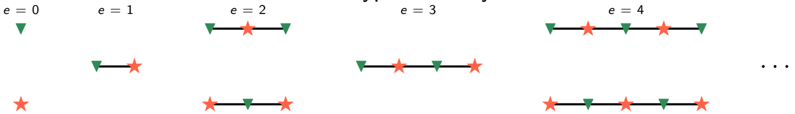


The type E exceptions

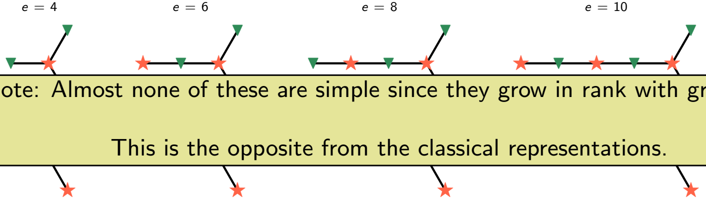


Back

The type A family



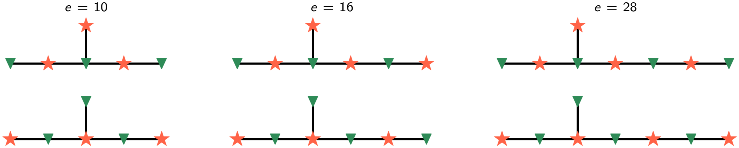
The type D family



Note: Almost none of these are simple since they grow in rank with growing e.

This is the opposite from the classical representations.

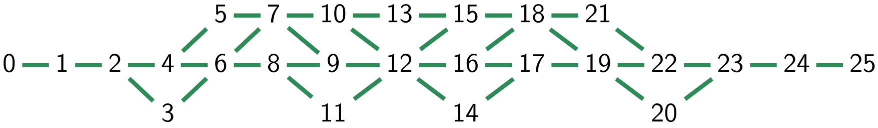
The type E exceptions



Back

Example (SAGE). The Weyl group of type B_6. Number of elements: 46080. Number of cells: 26, named 0 (trivial) to 25 (top).

Cell order:



Size of the cells and whether the cells are strongly regular (sr):

cell	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
size	1	62	342	576	650	3150	350	1600	2432	3402	900	2025	14500	600	2025	900	3402	2432	1600	350	576	3150	650	342	62	1
a	0	1	2	3	3	4	4	5	5	6	6	6	7	9	10	10	10	15	11	16	17	12	15	25	25	36

Back

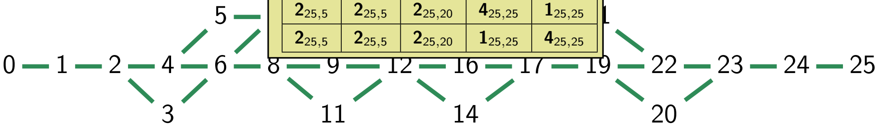
Example (cell 12).

Cell 12 is a bit scary:

$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{5,5}$	$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{20,5}$	$\mathbf{1}_{20,5}$	$\mathbf{4}_{20,20}$	$\mathbf{2}_{20,25}$	$\mathbf{2}_{20,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{4}_{25,25}$	$\mathbf{1}_{25,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{1}_{25,25}$	$\mathbf{4}_{25,25}$

Example (SAGE). The W… …er of elements: 46080.
Number of cells: 26, nam…

Cell order:



Size of the cells and whether the cells are strongly regular (sr):

cell	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
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a	0	1	2	3	3	4	4	5	5	6	6	6	7	9	10	10	10	15	11	16	17	12	15	25	25	36

Back

Example ($e = 4$). Here we have three different notions of "atoms".

Classical representation theory. The simples from before.

	$\mathrm{M}_{0,0}$	$\mathrm{M}_{2,0}$	$\mathcal{M}_{\sqrt{2}}$	$\mathrm{M}_{0,2}$	$\mathrm{M}_{2,2}$
atom	sign		rotation		trivial
rank	1	1	2	1	1
apex(KL)	1	s – t	s – t	s – t	w_0

Group element basis. Subgroups and ranks of transitive $\mathbb{N}$-modules.

subgroup	1	$\langle st \rangle$	$\langle w_0 \rangle$	$\langle w_0, s \rangle$	$\langle w_0, sts \rangle$	G
atom	regular	$\mathrm{M}_{0,0} \oplus \mathrm{M}_{2,2}$	$\mathcal{M}_{\sqrt{2}} \oplus \mathcal{M}_{\sqrt{2}}$	$\mathrm{M}_{2,0} \oplus \mathrm{M}_{2,2}$	$\mathrm{M}_{0,2} \oplus \mathrm{M}_{2,2}$	trivial
rank	8	2	4	2	2	1
apex	G	G	G	G	G	G

KL basis. ADE diagrams and ranks of transitive $\mathbb{N}$-modules.

	bottom cell			top cell
atom	sign	$\mathrm{M}_{2,0} \oplus \mathcal{M}_{\sqrt{2}}$	$\mathrm{M}_{0,2} \oplus \mathcal{M}_{\sqrt{2}}$	trivial
rank	1	3	3	1
apex	1	s – t	s – t	w_0

Back

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atom	sign		rotation		trivial
rank	1	1	2	1	1
apex(KL)	1	s – t	s – t	s – t	w_0

Fun fact.

Choose your favorite field and perform the Jordan decomposition. Then you will see all simples appearing!

Group ele

atom	regular	$M_{0,0} \oplus M_{2,2}$	$\mathcal{M}_{\sqrt{2}} \oplus \mathcal{M}_{\sqrt{2}}$	$M_{2,0} \oplus M_{2,2}$	$M_{0,2} \oplus M_{2,2}$	trivial
rank	8	2	4	2	2	1
apex	G	G	G	G	G	G

KL basis. ADE diagrams and ranks of transitive $\mathbb{N}$-modules.

	bottom cell			top cell
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Fun fact.

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Group ele

atom	regular	$\mathrm{M}_{0,0}\oplus\mathrm{M}_{2,2}$	$\mathcal{M}_{\sqrt{2}}\oplus\mathcal{M}_{\sqrt{2}}$	$\mathrm{M}_{2,0}\oplus\mathrm{M}_{2,2}$	$\mathrm{M}_{0,2}\oplus\mathrm{M}_{2,2}$	trivial
rank	8	2	4	2	2	1
apex	G	G	G	G	G	G

"Knowing the transitive $\mathbb{N}$-modules
$\Leftrightarrow$
knowing the simples for all primes $p \geq 0$."

KL basis. ADE diag

	bottom cell			top cell
atom	sign	$\mathrm{M}_{2,0}\oplus\mathcal{M}_{\sqrt{2}}$	$\mathrm{M}_{0,2}\oplus\mathcal{M}_{\sqrt{2}}$	trivial
rank	1	3	3	1
apex	1	s – t	s – t	w_0

Back

Example (SAGE). Here is a random calculation in the cell 12 for type B_6.

Graph:

$$1 \overset{4}{—} 2 — 3 — 4 — 5 — 6$$

Elements (shorthand $s_i = \mathtt{i}$):

$$d = d^{-1} = \mathtt{132123565}, \; u = u^{-1} = \mathtt{12132123565}.$$

Back

Example (SAGE). Here is a random calculation in the cell 12 for type B_6.

$$\theta_d \theta_d =$$
$$(\mathtt{v}^7 + 5\mathtt{v}^5 + 12\mathtt{v}^3 + 18\mathtt{v} + 18\mathtt{v}^{-1} + 12\mathtt{v}^{-3} + 5\mathtt{v}^{-5} + \mathtt{v}^{-7})\theta_d$$
$$+(\mathtt{v}^5 + 4\mathtt{v}^3 + 7\mathtt{v} + 7\mathtt{v}^{-1} + 4\mathtt{v}^{-3} + \mathtt{v}^{-5})\theta_u$$
$$+(\mathtt{v}^6 + 5\mathtt{v}^4 + 11\mathtt{v}^2 + 14 + 11\mathtt{v}^{-2} + 5\mathtt{v}^{-4} + \mathtt{v}^{-6})\theta_{121232123565}$$

Graph:

$$1 \overset{4}{\text{---}} 2 \text{---} 3 \text{---} 4 \text{---} 5 \text{---} 6$$

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Back

Example (SAGE). Here is a random calculation in the cell 12 for type B_6.

$$
\begin{gathered}
t_d t_d = \\
(\mathtt{v}^7 + 5\mathtt{v}^5 + 12\mathtt{v}^3 + 18\mathtt{v} + 18\mathtt{v}^{-1} + 12\mathtt{v}^{-3} + 5\mathtt{v}^{-5} + \mathtt{v}^{-7})\theta_d \\
+(\mathtt{v}^5 + 4\mathtt{v}^3 + 7\mathtt{v} + 7\mathtt{v}^{-1} + 4\mathtt{v}^{-3} + \mathtt{v}^{-5})\theta_u \\
+(\mathtt{v}^6 + 5\mathtt{v}^4 + 11\mathtt{v}^2 + 14 + 11\mathtt{v}^{-2} + 5\mathtt{v}^{-4} + \mathtt{v}^{-6})\theta_{121232123565}
\end{gathered}
$$

Graph:

$$1 \overset{4}{\text{—}} 2 \text{—} 3 \text{—} 4 \text{—} 5 \text{—} 6$$

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Back

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$$+(\mathrm{v}^5 + 4\mathrm{v}^3 + 7\mathrm{v} + 7\mathrm{v}^{-1} + 4\mathrm{v}^{-3} + \mathrm{v}^{-5})\theta_u$$
$$+(\mathrm{v}^6 + 5\mathrm{v}^4 + 11\mathrm{v}^2 + 14 + 11\mathrm{v}^{-2} + 5\mathrm{v}^{-4} + \mathrm{v}^{-6})\theta_{121232123565}$$

Bigger friends.

Graph:

$$1 \overset{4}{—} 2 — 3 — 4 — 5 — 6$$

Elements (shorthand $s_i = \mathtt{i}$):

$$d = d^{-1} = 132123565, \; u = u^{-1} = 12132123565.$$

Back

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$$
\begin{gathered}
t_d t_d = \\
(\mathtt{v}^7 + 5\mathtt{v}^5 + 12\mathtt{v}^3 + 18\mathtt{v} + 18\mathtt{v}^{-1} + 12\mathtt{v}^{-3} + 5\mathtt{v}^{-5} + \mathtt{v}^{-7})\theta_d \\
+(\mathtt{v}^5 + 4\mathtt{v}^3 + 7\mathtt{v} + 7\mathtt{v}^{-1} + 4\mathtt{v}^{-3} + \mathtt{v}^{-5})\theta_u
\end{gathered}
$$

Graph:

$$1 \overset{4}{—} 2 — 3 — 4 — 5 — 6$$

Elements (shorthand $s_i = \mathtt{i}$):

$$d = d^{-1} = \mathtt{132123565}, \; u = u^{-1} = \mathtt{12132123565}.$$

Back

Example (SAGE). Here is a random calculation in the cell 12 for type B_6.

$$t_d t_d =$$
$$(v^7 + 5v^5 + 12v^3 + 18v + 18v^{-1} + 12v^{-3} + 5v^{-5} + v^{-7})\theta_d$$
$$+(v^5 + 4v^3 + 7v + 7v^{-1} + 4v^{-3} + v^{-5})\theta_u$$

Killed in the limit $v \to \infty$.

Graph:

$$1 \overset{4}{—} 2 — 3 — 4 — 5 — 6$$

Elements (shorthand $s_i = \mathtt{i}$):

$$d = d^{-1} = 132123565, \; u = u^{-1} = 12132123565.$$

Back

Example (SAGE). Here is a random calculation in the cell 12 for type B_6.

$$t_d t_d =$$
$$t_d$$

Looks much simpler.

Graph:

$$1 \overset{4}{—} 2 — 3 — 4 — 5 — 6$$

Elements (shorthand $s_i = \mathtt{i}$):

$$d = d^{-1} = 132123565, \ u = u^{-1} = 12132123565.$$

Back

Example (SAGE; Type B_6).

Up to $\mathbb{N}$-equivalence: five left cell modules, five right cell modules, one two-sided cell bimodule, three H subalgebras:

L =

$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{5,5}$	$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{20,5}$	$\mathbf{1}_{20,5}$	$\mathbf{4}_{20,20}$	$\mathbf{2}_{20,25}$	$\mathbf{2}_{20,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{4}_{25,25}$	$\mathbf{1}_{25,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{1}_{25,25}$	$\mathbf{4}_{25,25}$

R =

$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{5,5}$	$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{20,5}$	$\mathbf{1}_{20,5}$	$\mathbf{4}_{20,20}$	$\mathbf{2}_{20,25}$	$\mathbf{2}_{20,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{4}_{25,25}$	$\mathbf{1}_{25,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{1}_{25,25}$	$\mathbf{4}_{25,25}$

LR =

$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{5,5}$	$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{20,5}$	$\mathbf{1}_{20,5}$	$\mathbf{4}_{20,20}$	$\mathbf{2}_{20,25}$	$\mathbf{2}_{20,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{4}_{25,25}$	$\mathbf{1}_{25,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{1}_{25,25}$	$\mathbf{4}_{25,25}$

H =

$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{5,5}$	$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{20,5}$	$\mathbf{1}_{20,5}$	$\mathbf{4}_{20,20}$	$\mathbf{2}_{20,25}$	$\mathbf{2}_{20,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{4}_{25,25}$	$\mathbf{1}_{25,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{1}_{25,25}$	$\mathbf{4}_{25,25}$

Fact. The three $\mathbb{N}$-algebras $J_{\infty}^{H}(W)$ are all "categorical Morita equivalent". (They have the same number of transitive $\mathbb{N}$-modules.)

Example ($e = 2$).

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3 \rangle$$

$$\theta_{\mathtt{s}} \rightsquigarrow \begin{pmatrix} \mathtt{v}+\mathtt{v}^{-1} & 0 & 1 \\ 0 & \mathtt{v}+\mathtt{v}^{-1} & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \theta_{\mathtt{t}} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & \mathtt{v}+\mathtt{v}^{-1} \end{pmatrix}$$

$$\theta_{\mathtt{sts}} \rightsquigarrow \begin{pmatrix} 0 & \mathtt{v}+\mathtt{v}^{-1} & 1 \\ \mathtt{v}+\mathtt{v}^{-1} & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \theta_{\mathtt{tst}} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & \mathtt{v}+\mathtt{v}^{-1} \end{pmatrix}$$

$$\theta_{\mathtt{ts}} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathtt{v}+\mathtt{v}^{-1} & \mathtt{v}+\mathtt{v}^{-1} & 1 \end{pmatrix} \qquad \theta_{\mathtt{st}} \rightsquigarrow \begin{pmatrix} 1 & 1 & \mathtt{v}+\mathtt{v}^{-1} \\ 1 & 1 & \mathtt{v}+\mathtt{v}^{-1} \\ 0 & 0 & 0 \end{pmatrix}$$

Back

Example ($e = 2$).

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3\rangle$$

$$\theta_{s} \rightsquigarrow \begin{pmatrix} v+v^{-1} & 0 & 1 \\ 0 & v+v^{-1} & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \theta_{t} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & v+v^{-1} \end{pmatrix}$$

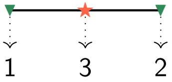
$$\theta_{sts} \rightsquigarrow \begin{pmatrix} 0 & v+v^{-1} & 1 \\ v+v^{-1} & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \theta_{tst} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & v+v^{-1} \end{pmatrix}$$

$$\theta_{ts} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ v+v^{-1} & v+v^{-1} & 1 \end{pmatrix} \qquad \theta_{st} \rightsquigarrow \begin{pmatrix} 1 & 1 & v+v^{-1} \\ 1 & 1 & v+v^{-1} \\ 0 & 0 & 0 \end{pmatrix}$$

Back

Example ($e = 2$).

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3 \rangle$$



$$t_{s} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad t_{t} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$t_{sts} \rightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad t_{tst} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$t_{ts} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \qquad t_{st} \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

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Example $(e = 2)$.

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3 \rangle$$

Example.

$$t_{st} t_{ts} = t_s + t_{sts}$$

$$\leftrightsquigarrow$$

$$[L_1][L_1] = [L_0] + [L_2]$$

$$\leftrightsquigarrow$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$t_{ts} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \qquad t_{st} \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Back

Example ($e = 2$).

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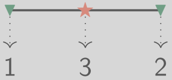

This works in general and recovers the transitive $\mathbb{N}$-modules of $K_0(\mathrm{SL}(2)_q)$ found by Etingof–Khovanov ∼1995 and Kirillov–Ostrik ∼2001, which are also ADE classified.

(For the experts: the bicoloring kills the tadpole solutions.)

$$t_{ts} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \qquad t_{st} \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

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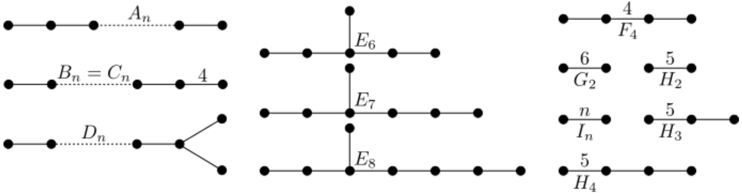

Figure: The connected Coxeter diagrams of finite type. The finite Weyl groups are of type A, B = C, D, E, F and G.

Example: Hecke algebras as non-semisimple fusion rings (Lusztig ∼1984).

type	A	B = C	D	E_6
worst case	$J^H_\infty \cong 1$	$J^H_\infty \cong K_0(\mathcal{V}\mathrm{ec}_{(\mathbb{Z}/2\mathbb{Z})^d})$	$J^H_\infty \cong K_0(\mathcal{V}\mathrm{ec}_{(\mathbb{Z}/2\mathbb{Z})^d})$	$J^H_\infty \cong K_0(\mathcal{R}\mathrm{ep}(S_3))$

type	E_7	E_8	F_4	G_2
worst case	$J^H_\infty \cong K_0(\mathcal{R}\mathrm{ep}(S_3))$	$J^H_\infty \cong K_0(\mathcal{R}\mathrm{ep}(S_5))$	$J^H_\infty \cong K_0(\mathcal{R}\mathrm{ep}(S_4))$	$J^H_\infty \cong K_0(\mathcal{R}\mathrm{ep}(S_2))$

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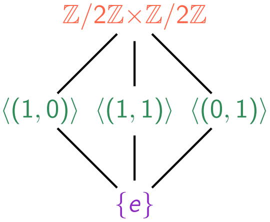
(Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)

Example ($G = \mathbb{Z}/2\times\mathbb{Z}/2$).

Subgroups, Schur multipliers and 2-simples.

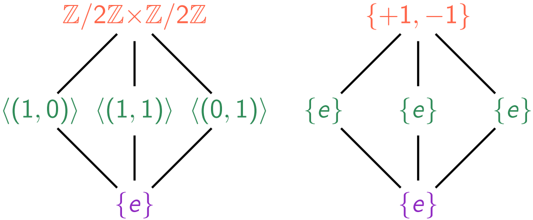


In particular, there are two categorifications of the trivial module, and the rank sequences read

$$\text{decat: } 1, 2, 2, 2, 4, \qquad \text{cat: } 1, 1, 2, 2, 2, 4.$$

Example ($G = \mathbb{Z}/2 \times \mathbb{Z}/2$).

Subgroups, Schur multipliers and 2-simples.



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$$\text{decat: } 1, 2, 2, 2, 4, \qquad \text{cat: } 1, 1, 2, 2, 2, 4.$$

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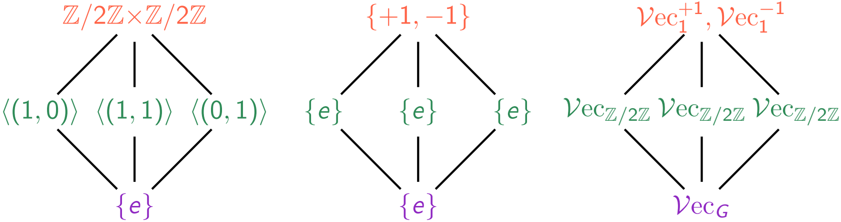
Subgroups, Schur multipliers and 2-simples.



In particular, there are two categorifications of the trivial module, and the rank sequences read

$$\text{decat: } 1,2,2,2,4, \qquad \text{cat: } 1,1,2,2,2,4.$$

Example (SAGE; Type B_6).

Reducing from 46080 to 14500 to 4:

LR =

$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{5,5}$	$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{20,5}$	$\mathbf{1}_{20,5}$	$\mathbf{4}_{20,20}$	$\mathbf{2}_{20,25}$	$\mathbf{2}_{20,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{4}_{25,25}$	$\mathbf{1}_{25,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{1}_{25,25}$	$\mathbf{4}_{25,25}$

$\rightsquigarrow$ H =

$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{5,5}$	$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{20,5}$	$\mathbf{1}_{20,5}$	$\mathbf{4}_{20,20}$	$\mathbf{2}_{20,25}$	$\mathbf{2}_{20,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{4}_{25,25}$	$\mathbf{1}_{25,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{1}_{25,25}$	$\mathbf{4}_{25,25}$

$$\mathscr{J}_{\infty}^{\mathsf{H}} = \mathcal{V}\mathrm{ec}_{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}},$$ rank sequence: $1, 1, 2, 2, 2, 4$.

In particular, there is one non-cell 2-simple: one 2 is missing.

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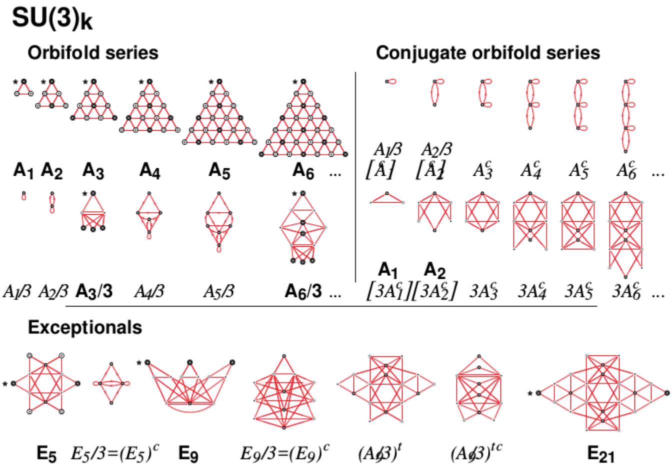

Figure: "Subgroups" of $\mathrm{SU}(3)_q$.

(Picture from "The classification of subgroups of quantum $\mathrm{SU}(N)$", **Ocneanu ∼2000**.)

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