The double centralizer theorem categorified is...?

Or: Two different and yet similar answers

Daniel Tubbenhauer

$$\mathrm{A} \cong \mathcal{E}\mathrm{nd}_{\mathcal{E}\mathrm{nd}_{\mathrm{A}}(\mathtt{M})}(\mathtt{M})$$

Joint with Marco Mackaay, Volodymyr Mazorchuk, Vanessa Miemietz and Xiaoting Zhang

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Self-injective $\Leftrightarrow$ projectives=injectives,
faithful $\Leftrightarrow$ only 0 acts as zero.

One version of the double centralizer theorem (DCT)

This is not the most general version, but I will stick to it for simplicity.

The DCT (Schur ∼1901+1927, Thrall ∼1947, Morita ∼1958).
Let A be a self-injective, finite-dimensional algebra, and $\mathtt{M}$ be a finite-dimensional, faithful A-module. Then there is a canonical algebra map

$$\mathrm{can}\colon \mathrm{A} \to \mathcal{E}\mathrm{nd}_{\mathcal{E}\mathrm{nd}_{\mathrm{A}}(\mathtt{M})}(\mathtt{M}),$$

$\mathtt{M}$ should be a A-B-bimodule, so $\mathcal{E}\mathrm{nd}_{\mathrm{A}}(\mathtt{M})$ means right operators, while $\mathcal{E}\mathrm{nd}_{\mathrm{B}}(\mathtt{M})$ are left operators. I will ignore this technicality.

which is an isomorphism.

- **Bad news.** We can not create many new algebras out of $(\mathrm{A},\mathtt{M})$. (Same for the categorified versions.)
- **Good news.** We can play A and $\mathrm{B}=\mathcal{E}\mathrm{nd}_{\mathrm{A}}(\mathtt{M})$ against each other.
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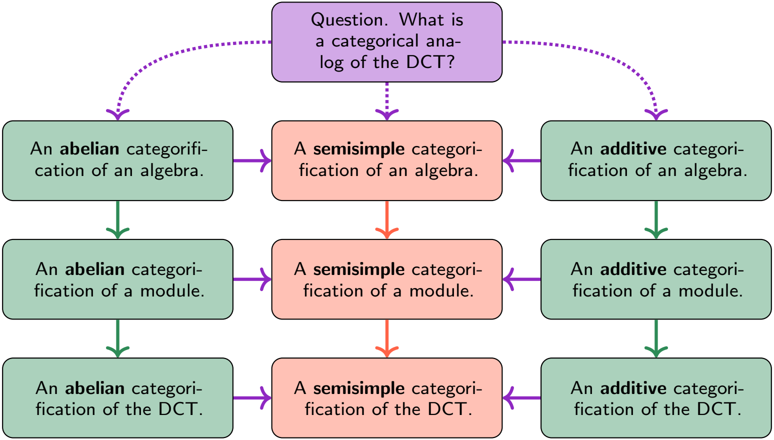

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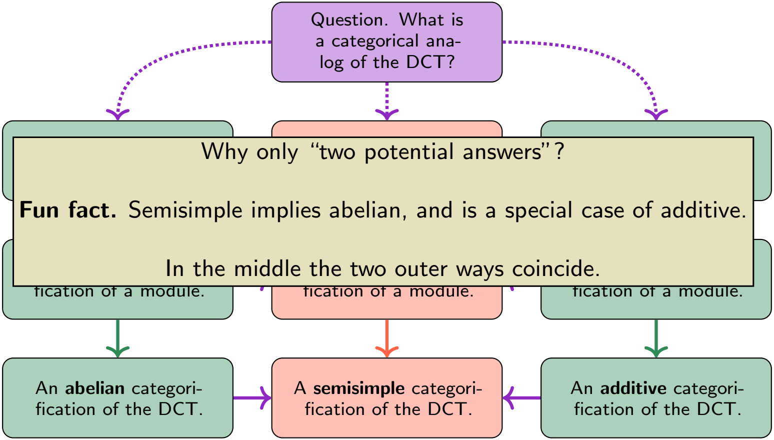

Why only "two potential answers"?

Fun fact. Semisimple implies abelian, and is a special case of additive.

In the middle the two outer ways coincide.

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Do not worry: I will explain all the words, at least in the abelian case! The second statement already sounds more complicated, right?

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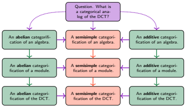

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A knows B, and B knows A, right?

$$\mathrm{A}\text{-}\mathcal{M}\mathrm{od} \simeq \mathrm{B}\text{-}\mathcal{M}\mathrm{od}$$
$$\Leftrightarrow$$
$$\exists \mathtt{M} \text{ progenerator such that } \mathrm{A} \cong \mathcal{E}\mathrm{nd}_{\mathrm{B}}(\mathtt{M})$$
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Back

Morita ∼1958.
The DCT goes hand-in-hand with classical Morita-theory.

Example $G = \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ **(Klein four group).**

If $\mathbb{K}$ is not of characteristic 2, $\mathbb{K}G$ is semisimple and additive=abelian. So let us have a look at characteristic 2, where we have $\mathbb{K}G \cong \mathbb{K}[X,Y]/(X^2,Y^2)$

First, abelian:
- X and Y have to act as zero on each simple, so $\mathbb{K}G$ has just $\mathbb{K}$ as a simple.
- $\mathbb{K}G$-$\mathcal{M}$od has just one element.

Then additive:
- Only X^2 and Y^2 have to act as zero on each indecomposable, and one can cook-up infinitely many, *e.g.*

$$\bullet \xleftarrow{X} \bullet \xrightarrow{Y} \bullet \xleftarrow{X} \bullet \xrightarrow{Y} \bullet \xleftarrow{X} \dots \xrightarrow{Y} \bullet \xleftarrow{X} \bullet$$

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Back

Example (G-$\mathcal{M}$od, **ground field** $\mathbb{C}$).

- Let $\mathscr{A} = G$-$\mathcal{M}$od, for G being a finite group. As $\mathscr{A}$ is semisimple, abelian=additive. Simples are simple G-modules.
- For any $\mathtt{M},\mathtt{N} \in \mathscr{A}$, we have $\mathtt{M}\otimes\mathtt{N} \in \mathscr{A}$:

$$g(m\otimes n) = gm\otimes gn$$

for all $g \in G$, $m \in \mathtt{M}$, $n \in \mathtt{N}$. There is a trivial module $\mathbb{C}$.
- The regular $\mathscr{A}$-module $\mathrm{M}\colon \mathscr{A} \to \mathcal{E}\mathrm{nd}_{\mathbb{C}}(\mathscr{A})$:

$$\begin{array}{ccc} \mathtt{M} & \longrightarrow & \mathtt{M}\otimes\_ \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\otimes\_} \\ \mathtt{N} & \longrightarrow & \mathtt{N}\otimes\_ \end{array}$$

- The decategorification is the regular $K_0(\mathscr{A})$-module.

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Semisimple example.
- $\mathscr{A} = \mathscr{V}\mathrm{ect}$, and fix $\mathrm{M} = \mathrm{Vect}^{\oplus n}$, which is faithful.
- $\mathscr{B} = \mathcal{E}\mathrm{nd}_{\mathscr{V}\mathrm{ect}}(\mathrm{Vect}^{\oplus n}) \cong \mathscr{M}\mathrm{at}_{n\times n}(\mathscr{V}\mathrm{ect})$ and $\mathcal{E}\mathrm{nd}_{\mathscr{M}\mathrm{at}_{n\times n}(\mathscr{V}\mathrm{ect})}(\mathrm{Vect}^{\oplus n}) \cong \mathscr{V}\mathrm{ect}$.

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An abelian example.
- $\mathscr{A} = \mathrm{H}$-$\mathcal{M}$od, and fix $\mathrm{M} = \mathrm{Vect}$, which is faithful.
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Back Upshot

Exact ⇔ the unit acts as an exact functor. If M is semisimple, then exactness is automatic.

$\mathscr{A}$ knows $\mathscr{B}$, and $\mathscr{B}$ knows $\mathscr{A}$, right?

Morita equivalence (Etingof–Ostrik ∼2003).
Let $\mathscr{B} = \mathcal{E}\mathrm{nd}_{\mathscr{A}}(\mathrm{M})$ for M a faithful, exact $\mathscr{A}$-module. Then

$$\mathscr{A}\text{-mod} \simeq \mathscr{B}\text{-mod}.$$

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Sorry, this example is not self-contained. But just to explain all the ingredients carefully is another talk.

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Additive example (∼2020).
$\mathscr{S} = \mathscr{S}(W,\mathbb{C})$ Soergel bimodules for W finite, the coinvariant algebra and over $\mathbb{C}$, $\mathcal{J}$ a two-sided cell and $\mathrm{C}_{\mathcal{J}}$ the cell $\mathscr{S}_{\mathcal{J}}$-module.
- Additive DCT. We have

$$\mathrm{can}\colon \mathscr{S}_{\mathcal{J}} \to \mathcal{E}\mathrm{nd}_{\mathcal{E}\mathrm{nd}_{\mathscr{S}_{\mathcal{J}}}(\mathrm{C}_{\mathcal{J}})}(\mathrm{C}_{\mathcal{J}}),$$

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This looks weaker than the abelian DCT, but this is what we can prove right now. Anyway, let explain why it is weaker, which finally explains all words in the additive DCT.

There is still much to do...

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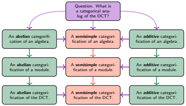

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Morita ∼1958.

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If $\mathrm{A} \subset \mathcal{E}\mathrm{nd}_{\mathbb{K}}(\mathtt{M})$, $\mathrm{B} = \mathcal{E}\mathrm{nd}_{\mathrm{A}}(\mathtt{M})$ and A is semisimple, then:

- $\mathrm{A} = \mathcal{E}\mathrm{nd}_{\mathrm{B}}(\mathtt{M})$;
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- As a $\mathrm{A} \otimes \mathrm{B}^{\mathrm{op}}$-module we have

$$\mathtt{M} \cong \bigoplus_{\text{simples of } \mathrm{A},\mathrm{B}} {}_{\mathrm{A}}\mathtt{L}^{i} \otimes \mathtt{L}^{i}{}_{\mathrm{B}}.$$

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Schur ∼1901+1927.

The DCT goes hand-in-hand with classical Schur–Weyl duality.

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If $\mathtt{M} = \mathrm{A}e$ for $e^2 = e$, $\mathtt{M}$ faithful and $\mathrm{B} = \mathcal{E}\mathrm{nd}_{\mathrm{A}}(\mathrm{A}e)$, then:

- $\mathrm{B} \cong e\mathrm{A}e$ and $\mathrm{A} \cong \mathcal{E}\mathrm{nd}_{e\mathrm{A}e}(\mathrm{A}e)$;
- The B-simples are in bijection with A-simples $\mathtt{N}$ such that $\mathtt{N}e \neq 0$;
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Green $\sim$1980.

The DCT applies for Schur–Weyl in the non-semisimple case.

Soergel $\sim$1990.

The DCT applies in category $\mathcal{O}$.

Example. (Looks silly, but is prototypical.)

- $\mathrm{A} = \mathbb{K}$, and fix $\mathtt{M} = \mathbb{K}^n$, which is faithful.
- $\mathrm{B} = \mathcal{E}\mathrm{nd}_{\mathbb{K}}(\mathbb{K}^n) \cong \mathrm{Mat}_{n\times n}(\mathbb{K})$ and $\mathcal{E}\mathrm{nd}_{\mathrm{Mat}_{n\times n}(\mathbb{K})}(\mathbb{K}^n) \cong \mathbb{K}$.
- $\mathtt{M} \cong \mathbb{K} \otimes \mathbb{K}^n$, perfect matching of isotypic components.

Non-example. (Faithfulness missing.)

- $\mathrm{A} = \mathbb{K}[X]/(X^3)$, and fix $\mathtt{M} = \mathbb{K}^2$, $X \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, which is not-faithful.
- $\mathrm{B} = \mathcal{E}\mathrm{nd}_{\mathbb{K}[X]/(X^3)}(\mathbb{K}^2) \cong \mathbb{K}[X]/(X^2)$ and $\mathcal{E}\mathrm{nd}_{\mathbb{K}[X]/(X^2)}(\mathbb{K}^2) \cong \mathbb{K}[X]/(X^2)$.
- $\mathtt{M} \cong \mathbb{K}^2 \otimes \mathbb{K}$ as a $\mathbb{K}[X]/(X^3)$-module, $\mathtt{M} \cong \mathbb{K} \otimes \mathbb{K}^2$ as a $\mathbb{K}[X]/(X^2)$-module.

Non-example. (Self-injectivity missing.)

- $\mathrm{A} = \left(\begin{smallmatrix} \mathbb{K} & \mathbb{K} \\ 0 & \mathbb{K} \end{smallmatrix}\right)$, and fix $\mathtt{M} = \mathbb{K}^2$, which is faithful.
- $\mathrm{B} = \mathcal{E}\mathrm{nd}_{\left(\begin{smallmatrix} \mathbb{K} & \mathbb{K} \\ 0 & \mathbb{K} \end{smallmatrix}\right)}(\mathbb{K}^2) \cong \mathbb{K}$ and $\mathcal{E}\mathrm{nd}_{\mathbb{K}}(\mathbb{K}^2) \cong \mathrm{Mat}_{2\times 2}(\mathbb{K})$.
- $\mathtt{M} \cong \mathbb{K}^2 \otimes \mathbb{K}$ as a $\left(\begin{smallmatrix} \mathbb{K} & \mathbb{K} \\ 0 & \mathbb{K} \end{smallmatrix}\right)$-module, $\mathtt{M} \cong \mathbb{K} \otimes \mathbb{K}^2$ as a $\mathrm{Mat}_{2\times 2}(\mathbb{K})$-module.

◂ Back ▸ More sophisticated

Example (Schur ∼1901+1927, Green ∼1980).

- $\mathrm{A} = \mathbb{K}[S_d]$, and fix $\mathtt{M} = (\mathbb{K}^n)^{\otimes d}$ for $n \geq d$, which is faithful.
- $\mathrm{B} = \mathcal{E}\mathrm{nd}_{\mathbb{K}[S_d]}\big((\mathbb{K}^n)^{\otimes d}\big) \cong S(n,d)$ (Schur algebra) and $\mathcal{E}\mathrm{nd}_{S(n,d)}\big((\mathbb{K}^n)^{\otimes d}\big) \cong \mathbb{K}[S_d]$.
- $\mathbb{K}[S_d] \cong eS(n,d)e$ and the $\mathbb{K}[S_d]$-simples are in bijection with $S(n,d)$-simples $\mathtt{N}$ such that $\mathtt{N}e \neq 0$.

Example (Soergel's Struktursatz ∼1990).

- A a finite-dimensional algebra for $\mathcal{O}_0(\mathfrak{g}_{\mathbb{C}})$. Fix $\mathtt{M} = \mathrm{A}e$, which is faithful for the right choice of idempotent e_{w_0} (the big projective).
- $\mathrm{B} = \mathcal{E}\mathrm{nd}_{\mathrm{A}}(\mathrm{A}e_{w_0}) \cong e_{w_0}\mathrm{A}e_{w_0}$ (Soergel's Endomorphismensatz ∼1990: B=coinvariant algebra) and $\mathcal{E}\mathrm{nd}_{e_{w_0}\mathrm{A}e_{w_0}}(\mathrm{A}e_{w_0}) \cong \mathrm{A}$.
- A can be recovered from $e_{w_0}\mathrm{A}e_{w_0}$, although A is much more complicated. Explicitly, for $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_2$ one gets *e.g.*

$$\mathrm{A} = \; 1 \underset{b}{\overset{a}{\rightleftarrows}} s \;/(a|b=0), \quad \mathrm{B} \cong \mathbb{C}\{s,b|a\}, \quad As = \qquad \xleftarrow[b]{} \; s$$

Example $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (Klein four group).

If $\mathbb{K}$ is not of characteristic 2, $\mathbb{K}G$ is semisimple and additive=abelian. So let us have a look at characteristic 2, where we have $\mathbb{K}G \cong \mathbb{K}[X, Y]/(X^2, Y^2)$

First, abelian:

- X and Y have to act as zero on each simple, so $\mathbb{K}G$ has just $\mathbb{K}$ as a simple.
- $\mathbb{K}G\text{-}\mathscr{M}\mathrm{od}$ has just one element (in the periodic table sense).

Then additive:

- Only X^2 and Y^2 have to act as zero on each indecomposable, and one can cook-up infinitely many, *e.g.*

$$\bullet \xleftarrow{X} \bullet \xrightarrow{Y} \bullet \xleftarrow{X} \bullet \xrightarrow{Y} \bullet \xleftarrow{X} \dots \xrightarrow{Y} \bullet \xleftarrow{X} \bullet$$

- $\mathbb{K}G\text{-}\mathscr{M}\mathrm{od}$ has infinitely many elements (in the periodic table sense).

Back

Example $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (Klein four group).

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First, abelian:

- ple.
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The

Theorem (Higman ∼1953).

For $\mathrm{char}(\mathbb{K}) = p$, $\mathbb{K}G\text{-}\mathcal{M}\mathrm{od}$ is...

...always a finite, pivotal multitensor category.

... monoidal fiat if and only if ($p \nmid |G|$ or the p-Sylow subgroup of G is cyclic).

- an

cook-up infinitely many, *e.g.*

$$\bullet \xleftarrow{X} \bullet \xrightarrow{Y} \bullet \xleftarrow{X} \bullet \xrightarrow{Y} \bullet \xleftarrow{X} \dots \xrightarrow{Y} \bullet \xleftarrow{X} \bullet$$

- $\mathbb{K}G\text{-}\mathcal{M}\mathrm{od}$ has infinitely many elements (in the periodic table sense).

Back

Example (G-$\mathscr{M}\mathrm{od}$, ground field $\mathbb{C}$).

- Let $\mathscr{A} = G$-$\mathscr{M}\mathrm{od}$, for G being a finite group. As $\mathscr{A}$ is semisimple, abelian=additive. Simples are simple G-modules.
- For any $\mathtt{M}, \mathtt{N} \in \mathscr{A}$, we have $\mathtt{M} \otimes \mathtt{N} \in \mathscr{A}$:

$$g(m \otimes n) = gm \otimes gn$$

  for all $g \in G, m \in \mathtt{M}, n \in \mathtt{N}$. There is a trivial module $\mathbb{C}$.
- The regular $\mathscr{A}$-module $\mathrm{M} \colon \mathscr{A} \to \mathscr{E}\mathrm{nd}_{\mathbb{C}}(\mathscr{A})$:

$$\begin{array}{ccc} \mathtt{M} & \longrightarrow & \mathtt{M} \otimes \_ \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes \_} \\ \mathtt{N} & \longrightarrow & \mathtt{N} \otimes \_ \end{array} .$$

- The decategorification is the regular $K_0(\mathscr{A})$-module.

Back

Example (G-$\mathscr{M}\mathrm{od}$, ground field $\mathbb{C}$).

- Let $K \subset G$ be a subgroup.
- K-Mod is a $\mathscr{A}$-module, with action

$$\mathcal{R}es_K^G \otimes \_ \colon G\text{-}\mathscr{M}\mathrm{od} \to \mathscr{E}\mathrm{nd}_{\mathbb{C}}\big(K\text{-Mod}\big),$$

$$\begin{array}{ccc} \mathtt{M} & \longrightarrow & \mathcal{R}es_K^G(\mathtt{M}) \otimes \_ \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{R}es_K^G(f)\otimes\_} \\ \mathtt{N} & \longrightarrow & \mathcal{R}es_K^G(\mathtt{N}) \otimes \_ \end{array} .$$

  which is indeed an action because $\mathcal{R}es_K^G$ is a $\otimes$-functor.
- The decategorifications are $K_0(\mathscr{A})$-modules.

Back

Left partial preorder $\geq_L$ on indecomposable objects by

$$\mathrm{F} \geq_L \mathrm{G} \Leftrightarrow \text{there exists } \mathrm{H} \text{ such that } \mathrm{F} \text{ is isomorphic to a direct summand of } \mathrm{HG}.$$

Left cells $\mathcal{L}$ are the equivalence classes with respect to $\geq_L$, on which $\geq_L$ induces a partial order. Similarly, right and two-sided, denoted by $\mathcal{R}$ and $\mathcal{J}$ respectively. Cell $\mathscr{A}$-modules associated to $\mathcal{L}$ are:

$$\mathrm{add}\big(\{\mathrm{F} \mid \mathrm{F} \geq_L \mathcal{L}\}\big) / \text{“kill } \geq_L\text{-bigger stuff”}.$$

Examples.

- Cells in $\mathscr{A}$ give $\otimes$-ideals.
- If $\mathscr{A}$ is semisimple, then $\mathrm{FF}^{\star}$ and $\mathrm{F}^{\star}\mathrm{F}$ both contain the identity, so cell theory is trivial. The cell $\mathscr{A}$-module is the regular $\mathscr{A}$-module.
- For Soergel bimodules cells are Kazhdan–Lusztig cells and cell modules categorify Kazhdan–Lusztig cell modules.
- For categorified quantum groups you can push everything to cyclotomic KLR algebras, and cell modules categorify simple modules.

A finite, pivotal multitensor category $\mathscr{A}$:

- Basics. $\mathscr{A}$ is $\mathbb{K}$-linear and monoidal, $\otimes$ is $\mathbb{K}$-bilinear. Moreover, $\mathscr{A}$ is abelian (this implies idempotent complete).
- Involution. $\mathscr{A}$ is pivotal, *e.g.* $\mathrm{F}^{\star\star} \cong \mathrm{F}$.
- Finiteness. Hom-spaces are finite-dimensional, the number of simples is finite, finite length, enough projectives.
- Categorification. The abelian Grothendieck ring gives a finite-dimensional algebra with involution.

A monoidal fiat category $\mathscr{A}$:

- Basics. $\mathscr{A}$ is $\mathbb{K}$-linear and monoidal, $\otimes$ is $\mathbb{K}$-bilinear. Moreover, $\mathscr{A}$ is additive and idempotent complete.
- Involution. $\mathscr{A}$ is pivotal, *e.g.* $\mathrm{F}^{\star\star} \cong \mathrm{F}$.
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- Finiteness. Hom-spaces are finite-dimensional, the number of simples is

The crucial difference...

...is what we like to consider as "elements" of our theory:

Abelian prefers simples,
additive prefers indecomposables.

This is a huge difference – for example in the fiat case there is simply no Schur's lemma.

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Back Further

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- Fi... is fi...
- Ca... ...onal algebra with involution.

Abelian examples.

H-$\mathscr{M}$od for H a finite-dimensional Hopf algebra. (Think: $\mathbb{K}G$, G finite.)
Finite Serre quotients of G-$\mathscr{M}$od for G being a reductive group.

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Abelian and additive examples.

H-$\mathscr{M}$od for H a finite-dimensional, semisimple Hopf algebra. (Think: $\mathbb{C}G$, G finite.)
$\mathscr{V}\mathrm{ect}_G$ for G graded $\mathbb{K}$-vector spaces, *e.g.* $\mathscr{V}\mathrm{ect} = \mathscr{V}\mathrm{ect}_1$.

- Finiteness. Hom-spaces are finite-dimensional, the number of indecomposables is finite.
- Ca... ...onal al...

Additive examples.

H-$\mathscr{P}$roj for H a finite-dimensional Hopf algebra. (Think: $\mathbb{K}G$, G finite.)
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Why I like the additive case.

All the example I know from my youth are not abelian, but only additive:

Diagram categories, categorified quantum group
and their Schur quotients, Soergel bimodules,
tilting module categories *etc.*

And these only fit into the fiat and not the tensor framework.

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i
- Categorification. The additive Grothendieck ring gives a finite-dimensional algebra with involution.

Back Further

Faithful $\Leftrightarrow$ only 0 (the object) acts as zero (functor).
This already clarifies the abelian DCT.

Abelian. An $\mathscr{A}$-module M:

- Basics. M is $\mathbb{K}$-linear and abelian. The action is a monoidal functor $M\colon \mathscr{A} \to \mathscr{E}\mathrm{nd}_{\mathbb{K},lex}(M)$ ($\mathbb{K}$-linear, left exactness).
- Finiteness. Hom-spaces are finite-dimensional, the number of simples is finite, finite length, enough projectives.
- Categorification. The abelian Grothendieck group gives a finite-dimensional $G_0(\mathscr{A})$-module.

Additive. An $\mathscr{A}$-module M:

- Basics. M is $\mathbb{K}$-linear, additive and idempotent complete. The action is a monoidal functor $M\colon \mathscr{A} \to \mathscr{E}\mathrm{nd}_{\mathbb{K}}(M)$ ($\mathbb{K}$-linear).
- Finiteness. Hom-spaces are finite-dimensional, the number of indecomposables is finite.
- Categorification. The additive Grothendieck group gives a finite-dimensional $K_0(\mathscr{A})$-module.

Back Further

Abelian. An $\mathscr{A}$-module M:

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Additive

- Ba ... is a mo
- Fin ... indecomposables is finite.
- Categorification. The additive Grothendieck group gives a finite-dimensional $K_0(\mathscr{A})$-module.

Example.

Everything is constructed such that
the regular $\mathscr{A}$-module $\mathscr{A}$ exists.

Smarter version of the regular $\mathscr{A}$-module are cell $\mathscr{A}$-modules. What?

But of course there are many more examples.

Back Further

Semisimple example.

- $\mathscr{A} = \mathscr{V}\mathrm{ect}$, and fix $\mathrm{M} = \mathrm{Vect}^{\oplus n}$, which is faithful.
- $\mathscr{B} = \mathscr{E}\mathrm{nd}_{\mathscr{V}\mathrm{ect}}(\mathrm{Vect}^{\oplus n}) \cong \mathscr{M}\mathrm{at}_{n\times n}(\mathscr{V}\mathrm{ect})$ and $\mathscr{E}\mathrm{nd}_{\mathscr{M}\mathrm{at}_{n\times n}(\mathscr{V}\mathrm{ect})}(\mathrm{Vect}^{\oplus n}) \cong \mathscr{V}\mathrm{ect}$.

Another semisimple example.

- $\mathscr{A} = \mathscr{V}\mathrm{ect}_G$, and fix $\mathrm{M} = \mathrm{Vect}$, which is faithful.
- $\mathscr{B} = \mathscr{E}\mathrm{nd}_{\mathscr{V}\mathrm{ect}_G}(\mathrm{Vect}) \cong G\text{-}\mathscr{M}\mathrm{od}$ and $\mathscr{E}\mathrm{nd}_{G\text{-}\mathscr{M}\mathrm{od}}(\mathrm{Vect}) \cong \mathscr{V}\mathrm{ect}_G$.

An abelian example.

- $\mathscr{A} = \mathrm{H}\text{-}\mathscr{M}\mathrm{od}$, and fix $\mathrm{M} = \mathrm{Vect}$, which is faithful.
- $\mathscr{B} = \mathscr{E}\mathrm{nd}_{\mathrm{H}\text{-}\mathscr{M}\mathrm{od}}(\mathrm{Vect}) \cong \mathrm{H}^{\star}\text{-}\mathscr{M}\mathrm{od}$ and $\mathscr{E}\mathrm{nd}_{\mathrm{H}^{\star}\text{-}\mathscr{M}\mathrm{od}}(\mathrm{Vect}) \cong \mathrm{H}\text{-}\mathscr{M}\mathrm{od}$.

Back Upshot

$\mathscr{A}$ knows $\mathscr{B}$, and $\mathscr{B}$ knows $\mathscr{A}$, right?

Exact $\Leftrightarrow$ the unit acts as an exact functor.
If M is semisimple, then exactness is automatic.

Morita equivalence (Etingof–Ostrik ∼2003).
Let $\mathscr{B} = \mathscr{E}\mathrm{nd}_{\mathscr{A}}(\mathrm{M})$ for M a faithful, exact $\mathscr{A}$-module. Then

$$\mathscr{A}\text{-mod} \simeq \mathscr{B}\text{-mod}.$$

Example.
$\mathscr{A} = \mathscr{V}\mathrm{ect}_G$ and $\mathscr{B} = G\text{-}\mathscr{M}\mathrm{od}$ have the "same" module categories, which is a very non-trivial fact.

◂ Back ▸ An additive example

$\mathscr{A}$ knows $\mathscr{B}$, and $\mathscr{B}$ knows $\mathscr{A}$, right?

Sorry, this example is not self-contained.
But just to explain all the ingredients carefully is another talk.

Additive example (∼2020).
$\mathscr{S}=\mathscr{S}(W,\mathbb{C})$ Soergel bimodules for W finite, the coinvariant algebra and over $\mathbb{C}$, $\mathcal{J}$ a two-sided cell and $\mathsf{C}_{\mathcal{J}}$ the cell $\mathscr{S}_{\mathcal{J}}$-module.

- Additive DCT. We have

$$\mathrm{can}\colon \mathscr{S}_{\mathcal{J}}\to\mathscr{E}\mathrm{nd}_{\mathscr{E}\mathrm{nd}_{\mathscr{S}_{\mathcal{J}}}(\mathsf{C}_{\mathcal{J}})}(\mathsf{C}_{\mathcal{J}}),$$

  is an equivalence when restricted to $\mathrm{add}(\mathcal{J})$ and corestricted to $\mathscr{E}\mathrm{nd}^{\mathrm{inj}}_{\mathscr{E}\mathrm{nd}_{\mathscr{A}_{\mathcal{J}}}(\mathsf{C}_{\mathcal{J}})}(\mathsf{C}_{\mathcal{J}})$.
- "Endomorphismensatz". We have

$$\mathscr{E}\mathrm{nd}_{\mathscr{A}_{\mathcal{J}}}(\mathsf{C}_{\mathcal{J}})\simeq\mathscr{A}_{\mathcal{J}}$$

  where $\mathscr{A}_{\mathcal{J}}$ is the asymptotic category (semisimple!).
- Morita equivalence. We have

$$\mathscr{S}_{\mathcal{J}}\text{-stmod}\simeq\mathscr{A}_{\mathcal{J}}\text{-stmod}.$$

◂ Back

This looks weaker than the abelian DCT, but this is what we can prove right now.
Anyway, let explain why it is weaker, which finally explains all words in the additive DCT.

$\mathscr{A}$ knows $\mathscr{B}$, and $\mathscr{B}$ knows $\mathscr{A}$, right?

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$$\mathscr{E}\mathrm{nd}_{\mathscr{A}_{\mathcal{J}}}(\mathsf{C}_{\mathcal{J}}) \simeq \mathscr{A}_{\mathcal{J}}$$

  w

- Morita equivalence. We have

To make $\mathsf{C}_{\mathcal{J}}$ faithful,
quotient $\mathscr{S}$ by "bigger stuff"
and get $\mathscr{S}_{\mathcal{J}}$.

$\mathrm{add}(\mathcal{J})$: Since "lower stuff" still acts pretty much in an uncontrolable way,
restrict to only things in $\mathcal{J}$.

inj means injective endofunctors.
In this case you could also consider projective endofunctors.

Back

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  is an equivalence … tricted to $\mathscr{E}\mathrm{nd}^{\mathrm{inj}}_{\mathscr{E}\mathrm{nd}_{\mathscr{A}_{\mathcal{J}}}(\mathsf{C}_{\mathcal{J}})}(\mathsf{C}$ …

  $\mathscr{A}_{\mathcal{J}}$ is the "degree zero part" of $\mathscr{S}_{\mathcal{J}}$.
  "$\mathscr{A}_{\mathcal{J}}$ is the crystal associated to $\mathscr{S}_{\mathcal{J}}$."

- "Endomorphismensatz". We have

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  where $\mathscr{A}_{\mathcal{J}}$ is the asymptotic category (semisimple!).

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  stmod are simple transitive modules.
  The analogs of categories of simple modules downstairs.

  where $\mathscr{A}_{\mathcal{J}}$ is the asymptotic category (semisimple!).

- Morita equivalence. We have

$$\mathscr{S}_{\mathcal{J}}\text{-}\mathrm{stmod} \simeq \mathscr{A}_{\mathcal{J}}\text{-}\mathrm{stmod}.$$

Back